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LIMITING BEHAVIOUR OF MOVING AVERAGE PROCESSES UNDER NEGATIVE ASSOCIATION ASSUMPTION

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Abstract. Let $\{Y_i, -\infty < i < \infty\}$ be a doubly infinite sequence of identically distributed negatively associated random variables, $\{a_i, -\infty < i < \infty\}$ an absolutely summable sequence of real numbers. In this paper, we prove the complete convergence and complete moment convergence of the maximal partial sums of moving average processes $\left\{\sum_{i=-\infty}^{\infty} a_i Y_{i+n}, n \geq 1\right\}$. In the paper we improve the results of Baek *et al.* [2] and Li and Zhang [12].

1. Introduction

We assume that $\{Y_i, -\infty < i < +\infty\}$ is a doubly infinite sequence of identically distributed random variables. Let $\{a_i, -\infty < i < +\infty\}$ be an absolutely summable sequence of real numbers and

$$X_n = \sum_{i=-\infty}^{\infty} a_i Y_{i+n}, \qquad n \geq 1,$$

be the *moving average process based on the sequence* $\{Y_i, -\infty < i < +\infty\}$.

For the moving average process $\{X_n, n \geq 1\}$, many limiting results have been obtained. For example, under the independence assumption of the base sequence

$$\{Y_i, -\infty < i < +\infty\},$$

Burton and Dehling [3] obtained a large deviation principle, and Li *et al.* [11] obtained the complete convergence result. Under different dependence assumptions of the base sequence $\{Y_i, -\infty < i < +\infty\}$, Zhang [17] obtained the complete convergence result when the base sequence $\{Y_i, -\infty < i < +\infty\}$ consists of φ-mixing random variables, and Baek *et al.* [2] and Liang *et al.* [13] obtained the complete convergence result when the base sequence $\{Y_i, -\infty < i < +\infty\}$ consists of negatively associated random variables. For the Banach space generalizations we refer to the papers Ahmed *et al.* [1], Chen *et al.* [5, 6].

Recall that a finite family of random variables $\{Y_i, 1 \leq i \leq n\}$ is said to be *negatively associated* (abbreviated to NA) if for any disjoint subsets A and B of $\{1, 2, \ldots, n\}$ and any real coordinatewise nondecreasing functions f on R^A and g on R^B,

$$\operatorname{Cov}\left(f(Y_i, i \in A), g(Y_j, j \in B)\right) \leq 0$$

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whenever the covariance exists. An infinite family of random variables

$$\{Y_i, < -\infty \nleq i < \infty\}$$ <

is NA if every finite subfamily is NA. This concept was introduced by Joag-Dev and Proschan [10].

In the following we let $S_n = \sum_{k=1}^{n} X_k$, $n \geq 1$, be the partial sums of the sequence $\{X_i, i \geq 1\}$ and $\{a_i, -\infty < i < \infty\}$ be an absolute summable sequence of real numbers, that is $\sum_{i=-\infty}^{\infty} |a_i| < \infty$. Next, for a real number x, let $x_+ = \max\{0, x\}$ and for any number q, we define $x_+^q = (x_+)^q$. As usual, C represents a positive constant although its value may change from one appearance to the next.

First we discuss the previous results connected with complete convergence. The following was proved in Hsu and Robbins [9] and Erdös [8].

Theorem A. *Suppose that $\{X_n, n \geq 1\}$ is a sequence of independent identically distributed random variables. Then $\mathsf{E}\, X_1 = 0$, $\mathsf{E}\,|X_1|^2 < \infty$ if and only if*

$$\sum_{n=1}^{\infty} \mathsf{P}\{|S_n| \geq \varepsilon n\} < \infty$$

for all $\varepsilon > 0$.

Hsu–Robbins–Erdös result was generalized by Li *et al.* [11] for a moving average process based on a sequence of i.i.d. random variables $\{Y_i, -\infty < i < +\infty\}$.

Theorem B. *Suppose $\{X_n, n \geq 1\}$ is the moving average process based on a sequence of independent identically distributed random variables $\{Y_i, -\infty < i < \infty\}$ with $\mathsf{E}\, Y_1 = 0$, $\mathsf{E}\,|Y_1|^{2t} < \infty$, $1 \leq t < 2$. Then $\sum_{n=1}^{\infty} \mathsf{P}\{|S_n| \geq \varepsilon n^{1/t}\} < \infty$ for all $\varepsilon > 0$.*

The result of Li *et al.* was generalized for a moving average process based on a sequence of NA random variables $\{Y_i, -\infty < i < +\infty\}$ by Baek *et al* [2] and Liang *et al.* [13]. If we omit some insignificant details connected with slowly varying functions and stochastic domination condition, their result could be formulated in the following way.

Theorem C. *Suppose that $\{X_n, n \geq 1\}$ is the moving average process based on a sequence of NA identically distributed random variables $\{Y_i, -\infty < i < \infty\}$ with $\mathsf{E}\, Y_1 = 0$, $\mathsf{E}\,|Y_1|^{2t} < \infty$, $1 \leq t < 2$. Then $\sum_{n=1}^{\infty} \mathsf{P}\{|S_n| \geq \varepsilon n^{1/t}\} < \infty$ for all $\varepsilon > 0$.*

Next, we discuss the previous results connected with complete moment convergence. The notion was introduced in Chow [7], where the following result was also proved.

Theorem D. *Suppose that $\{X_n, n \geq 1\}$ is a sequence of independent and identically distributed random variables with $\mathsf{E}\, X_1 = 0$ and $1 \leq p < 2$, $r > p$. If $\mathsf{E}\{|X_1|^{rp} + |X_1| \log(1 + |X_1|)\} < \infty$, then*

$$\sum_{n=1}^{\infty} n^{r-2-1/p}\, \mathsf{E}\{|S_n| - \varepsilon n^{1/p}\}_+ < \infty \quad \textit{for all } \varepsilon > 0.$$

We refer the interested reader to the papers Chen [4] and Rosalsky *et al.* [14] for the generalizations of Theorem D on the Banach space setting.

Li and Zhang [12] extended Theorem D for the case of moving average process based on NA random variables. Their result can be formulated in the following way, where we again omit some insignificant details connected with slowly varying functions.

Theorem E. *Suppose $\{X_n, n \geq 1\}$ is the moving average process based on a sequence of independent identically distributed random variables with $\mathsf{E}\, Y_1 = 0$, $\mathsf{E}\, Y_i^2 < \infty$. Let $1 \leq p < 2$, $r > 1 + p/2$. Then $\mathsf{E}\,|Y_1|^{rp} < \infty$ implies that*

$$\sum_{n=1}^{\infty} n^{r-2-1/p}\, \mathsf{E}\{|S_n| - \varepsilon n^{1/p}\}_+ < \infty \quad \textit{for all } \varepsilon > 0.$$

2. Formulation of the main results

The purpose of this paper is to improve the results of Baek *et al.* [2] (presented above as Theorem C) to the maximal partial sums, and extend the results of Li and Zhang [12] (presented above as Theorem E) to the maximal partial sums of a moving average process based on a sequence of NA random variables $\{Y_i, -\infty < i < +\infty\}$ under more optimal moment conditions.

Our main results are as follows.

Theorem 1. *Let $1 \le p < 2$, $r \ge 1$, $rp \ne 1$. Suppose that $\{X_n, n \ge 1\}$ is the moving average process based on a sequence of ~~independent~~ identically distributed random variables* NA *$\{Y_i, -\infty < i < \infty\}$. If $\mathsf{E}\, Y_1 = 0$ and $\mathsf{E}\,|Y_1|^{rp} < \infty$, then*

$$\sum_{n=1}^{\infty} n^{r-2}\, \mathsf{P}\left\{\max_{1\le k\le n} |S_k| > \varepsilon n^{1/p}\right\} < \infty \quad \textit{for all } \varepsilon > 0.$$

Theorem 2. *Let $q > 0$, $1 \le p < 2$, $r \ge 1$, $rp \ne 1$. Suppose that $\{X_n, n \ge 1\}$ is the moving average process based on a sequence of ~~independent~~ identically distributed random* NA *variables $\{Y_i, -\infty < i < \infty\}$. If $\mathsf{E}\, Y_1 = 0$ and*

$$\mathsf{E}\,|Y_1|^{rp} < \infty, \quad \textit{if } q < rp,$$
$$\mathsf{E}\,|Y_1|^{rp} \log(1 + |Y_1|) < \infty, \quad \textit{if } q = rp,$$
$$\mathsf{E}\,|Y_1|^{q} < \infty, \quad \textit{if } q > rp,$$

then

$$\sum_{n=1}^{\infty} n^{r-2-q/p}\, \mathsf{E}\left\{\max_{1\le k\le n} |S_k| - \varepsilon n^{1/p}\right\}_+^q < \infty \quad \textit{for all } \varepsilon > 0.$$

3. Two technical lemmas

The following lemma plays a crucial role in our proofs, see Theorem 2 of Shao [16].

Lemma 1. *Let $q \ge 2$, $\{X_j, 1 \le i \le n\}$ be a sequence of NA random variables with mean zero and $\mathsf{E}\,|X_j|^q < \infty$ for every $1 \le j \le n$. Then*

$$\mathsf{E} \max_{1\le k\le n} \left|\sum_{j=1}^{k} X_i\right|^q \le C\left\{\left(\sum_{j=1}^{n} \mathsf{E}\,|X_j|^2\right)^{q/2} + \sum_{j=1}^{n} \mathsf{E}\,|X_j|^q\right\}.$$

The next lemma is pure technical.

Lemma 2. *Let Y be a random variable with $\mathsf{E}\,|Y|^{rp}\psi(1+|Y|) < \infty$, where $r \ge 1$, $p \ge 1$, and $\psi(x) = 1$ or $\psi(x) = \log(x)$, $x \ge 1$.*

(i) $\sum_{n=1}^{\infty} n^{r-1}\, \mathsf{P}\{|Y| > n^{1/p}\} \le C\, \mathsf{E}\,|Y|^{rp}$.

(ii) *If $q > rp$, then*

$$\sum_{n=1}^{\infty} n^{r-1-\ q/p}\psi(n)\, \mathsf{E}\,|Y|^q I\left\{|Y| \le n^{1/p}\right\} \le C\, \mathsf{E}\,|Y|^{rp}\psi(|Y|).$$

(iii) *If $s \ge 1$, $v > 0$, $sp > v$ and $\mathsf{E}\,|Y|^{sp}\psi(1+|Y|) < \infty$, then*

$$\sum_{n=1}^{\infty} n^{s-1-\ v/p}\psi(n)\, \mathsf{E}\,|Y|^v I\left\{|Y| > n^{1/p}\right\} \le C\, \mathsf{E}\,|Y|^{sp}\psi(1+|Y|).$$

Proof. First of all, we mention that statement (i) is well known, so we will prove only (ii) and (iii). Note that the function ψ has the following properties:

(a) for any $m \geq 1$

$$\sum_{n=1}^{m} n^{u-1}\psi(n) \leq Cm^u\psi(m), \quad \text{if } u > 0$$

and

$$\sum_{n=m}^{\infty} n^{u-1}\psi(n) \leq Cm^u\psi(m), \quad \text{if } u < 0$$

(b) for any $p > 0$, $\psi(|x|^p) = C\psi(|x|) \leq C\psi(1+|x|)$.

(ii) Since $r - q/p < 0$, we have that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-1-q/p}\psi(n)\,\mathsf{E}\,|Y|^q I\left\{|Y| \leq n^{1/p}\right\} \\
&\quad = C\sum_{n=1}^{\infty} n^{r-1-q/p}\psi(n) \sum_{m=1}^{n} \mathsf{E}\,|Y|^q I\{m-1 < |Y|^p \leq m\} \\
&\quad = C\sum_{m=1}^{\infty} \mathsf{E}\,|Y|^q I\{m-1 < |Y|^p \leq m\} \sum_{n=m}^{\infty} n^{r-1-q/p}\psi(n) \\
&\quad \leq C\sum_{m=1}^{\infty} m^{r-q/p}\psi(m)\,\mathsf{E}\,|Y|^q I\{m-1 < |Y|^p \leq m\} \qquad \text{(by (a))} \\
&\quad \leq C\sum_{m=1}^{\infty} \mathsf{E}\, m^{r-q/p}\psi(m)|Y|^q I\{m-1 < |Y|^p \leq m\} \\
&\quad \leq C\sum_{m=1}^{\infty} \mathsf{E}(|Y|^p)^{r-q/p}\psi(1+|Y|)|Y|^q I\{m-1 < |Y|^p \leq m\} \\
&\quad \leq C\sum_{m=1}^{\infty} \mathsf{E}\,|Y|^{rp}\psi(1+|Y|) I\{m-1 < |Y|^p \leq m\} \\
&\quad \leq C\,\mathsf{E}\,|Y|^{rp}\psi(1+|Y|).
\end{aligned}$$

(iii) We have that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{s-1-v/p}\psi(n)\,\mathsf{E}\,|Y|^v I\left\{|Y| > n^{1/p}\right\} \\
&\quad = C\sum_{n=1}^{\infty} n^{s-1-v/p}\psi(n) \sum_{m=n}^{\infty} \mathsf{E}\,|Y|^v I\{m < |Y|^p \leq m+1\} \\
&\quad = C\sum_{m=1}^{\infty} \mathsf{E}\,|Y|^v I\{m < |Y|^p \leq m+1\} \sum_{n=1}^{m} n^{s-1-v/p}\psi(n) \\
&\quad \leq C\sum_{m=1}^{\infty} m^{s-v/p}\psi(m)\,\mathsf{E}\,|Y|^v I\{m < |Y|^p \leq m+1\} \qquad \text{(by (a))} \\
&\quad = C\sum_{m=1}^{\infty} \mathsf{E}\, m^{s-v/p}\psi(m)|Y|^v I\{m < |Y|^p \leq m+1\} \\
&\quad \leq C\sum_{m=1}^{\infty} \mathsf{E}(|Y|^p)^{s-v/p}\psi(|Y|^p)|Y|^v I\{m < |Y|^p \leq m+1\} \\
&\quad \leq C\,\mathsf{E}\,|Y|^{sp}\psi(1+|Y|) \qquad \text{(by (b))}. \quad \square
\end{aligned}$$

4. Proof of the main results

First we prove Theorem 1.

Proof. Let

$$Y_{nj}^{(1)} = -n^{1/p} I\left\{Y_j < -n^{1/p}\right\} + Y_j I\left\{|Y_j| \le n^{1/p}\right\} + n^{1/p} I\left\{Y_j > n^{1/p}\right\},$$

and $Y_{nj}^{(2)} = Y_j - Y_{nj}^{(1)}$ be the *monotone truncations* of $Y_j, -\infty < j < \infty$. Then by Joag-Dev and Proschan [10], for any $n \ge 1$

$$\left\{Y_{nj}^{(1)} - \mathsf{E}\, Y_{nj}^{(1)}, -\infty < j < \infty\right\} \quad \text{and} \quad \left\{Y_{nj}^{(2)}, -\infty < j < \infty\right\}$$

are two sequences of NA random variables. Note that

$$\sum_{k=1}^{n} X_k = \sum_{k=1}^{n} \sum_{i=-\infty}^{\infty} a_i Y_{i+k} = \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_j$$

and

$$\begin{aligned}
n^{-1/p} \max_{1\le k\le n} \left| \mathsf{E} \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_{nj}^{(1)} \right| &\le n^{-1/p} \sum_{i=-\infty}^{\infty} |a_i| \max_{1\le k\le n} \left| \mathsf{E} \sum_{j=i+1}^{i+k} Y_{nj}^{(1)} \right| \\
&\le C n^{-1/p} n \left(\left| \mathsf{E}\, Y_1 I\left\{|Y_1| \le n^{1/p}\right\} \right| + n^{1/p}\, \mathsf{P}\left\{|Y_1| > n^{1/p}\right\} \right) \\
&\le C n^{1-1/p}\, \mathsf{E}\, |Y_1| I\left\{|Y_1| > n^{1/p}\right\} + C n\, \mathsf{P}\left\{|Y_1| > n^{1/p}\right\} \\
&\le C\, \mathsf{E}\, |Y_1|^p I\left\{|Y_1| > n^{1/p}\right\} + C n\, \mathsf{P}\left\{|Y_1| > n^{1/p}\right\} \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Hence, for any $\varepsilon > 0$ where exists n large enough such that

$$n^{-1/p} \max_{1\le k\le n} \left| \mathsf{E} \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_{nj}^{(1)} \right| < \varepsilon/4.$$

Therefore, in order to prove Theorem 1, it is enough to prove that

$$I := \sum_{n=1}^{\infty} n^{r-2}\, \mathsf{P}\left\{ \max_{1\le k\le n} \left| \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} \left(Y_{nj}^{(1)} - \mathsf{E}\, Y_{nj}^{(1)} \right) \right| > \varepsilon n^{1/p} \big/ 4 \right\} < \infty$$

and

$$J := \sum_{n=1}^{\infty} n^{r-2}\, \mathsf{P}\left\{ \max_{1\le k\le n} \left| \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_{nj}^{(2)} \right| > \varepsilon n^{1/p} \big/ 2 \right\} < \infty.$$

For J by Markov inequality we have

$$\begin{aligned}
J &\le C \sum_{n=1}^{\infty} n^{r-2} n^{-1/p}\, \mathsf{E} \max_{1\le k\le n} \left| \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_{nj}^{(2)} \right| \\
&\le C \sum_{n=1}^{\infty} n^{r-1-1/p} \left(\mathsf{E}\, |Y_1| I\left\{|Y_1| > n^{1/p}\right\} + n^{1/p}\, \mathsf{P}\left\{|Y_1| > n^{1/p}\right\} \right) \\
&\le C \sum_{n=1}^{\infty} n^{r-1-1/p}\, \mathsf{E}\, |Y_1| I\left\{|Y_1| > n^{1/p}\right\} + C\, \mathsf{E}\, |Y_1|^{rp} \qquad \text{(by Lemma 2(i))} \\
&\le C\, \mathsf{E}\, |Y_1|^{rp} < \infty \qquad \text{(by Lemma 2(iii) with } \psi(x) = 1, v = 1, \text{ and } s = r).
\end{aligned}$$

For I, fix any $q \geq 2$ (to be specified later). Then

$$I \leq C \sum_{n=1}^{\infty} n^{r-2-q/p} \, \mathsf{E} \max_{1\leq k\leq n} \left| \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} \left(Y_{nj}^{(1)} - \mathsf{E}\, Y_{nj}^{(1)} \right) \right|^q$$

(by Markov inequality)

$$\leq C \sum_{n=1}^{\infty} n^{r-2-q/p} \, \mathsf{E} \left(\sum_{i=-\infty}^{\infty} |a_i| \max_{1\leq k\leq n} \left| \sum_{j=i+1}^{i+k} \left(Y_{nj}^{(1)} - \mathsf{E}\, Y_{nj}^{(1)} \right) \right| \right)^q$$

$$\leq C \sum_{n=1}^{\infty} n^{r-2-q/p} \, \mathsf{E} \left(\sum_{i=-\infty}^{\infty} \left(|a_i|^{1-1/q} \right) \left(|a_i|^{1/q} \max_{1\leq k\leq n} \left| \sum_{j=i+1}^{i+k} \left(Y_{nj}^{(1)} - \mathsf{E}\, Y_{nj}^{(1)} \right) \right| \right) \right)^q$$

$$\leq C \sum_{n=1}^{\infty} n^{r-2-q/p} \left(\sum_{i=-\infty}^{\infty} |a_i| \right)^{q-1} \sum_{i=-\infty}^{\infty} |a_i| \, \mathsf{E} \max_{1\leq k\leq n} \left| \sum_{j=i+1}^{i+k} \left(Y_{nj}^{(1)} - \mathsf{E}\, Y_{nj}^{(1)} \right) \right|^q$$

(by Hölder inequality)

$$\leq C \sum_{n=1}^{\infty} n^{r-2-q/p} \sum_{i=-\infty}^{\infty} |a_i| \, \mathsf{E} \max_{1\leq k\leq n} \left| \sum_{j=i+1}^{i+k} \left(Y_{nj}^{(1)} - \mathsf{E}\, Y_{nj}^{(1)} \right) \right|^q$$

(since $\sum_{i=-\infty}^{\infty} |a_i| < \infty$)

$$\leq C \sum_{n=1}^{\infty} n^{r-2-q/p} \sum_{i=-\infty}^{\infty} |a_i| \left\{ \sum_{j=i+1}^{i+n} \left(\mathsf{E} \left(Y_{nj}^{(1)} - \mathsf{E}\, Y_{nj}^{(1)} \right)^2 \right)^{q/2} + \sum_{j=i+1}^{i+n} \mathsf{E} \left| Y_{nj}^{(1)} - \mathsf{E}\, Y_{nj}^{(1)} \right|^q \right\}$$

(by Lemma 1)

$$\leq C \sum_{n=1}^{\infty} n^{r-2-q/p} \sum_{i=-\infty}^{\infty} |a_i| \left\{ \left(n \left(\mathsf{E}\, Y_1^2 I \left\{ |Y_1| \leq n^{1/p} \right\} + n^{2/p} \, \mathsf{P} \left\{ |Y_1| > n^{1/p} \right\} \right) \right)^{q/2} + n \left(\mathsf{E}\, |Y_1|^q I \left\{ |Y_1| \leq n^{1/p} \right\} + n^{q/p} \, \mathsf{P} \left\{ |Y_1| > n^{1/p} \right\} \right) \right\}$$

$$\leq C \sum_{n=1}^{\infty} n^{r-2-q/p} \left\{ \left(n \left(\mathsf{E}\, Y_1^2 I \left\{ |Y_1| \leq n^{1/p} \right\} + n^{2/p} \, \mathsf{P} \left\{ |Y_1| > n^{1/p} \right\} \right) \right)^{q/2} + n \left(\mathsf{E}\, |Y_1|^q I \left\{ |Y_1| \leq n^{1/p} \right\} + n^{q/p} \, \mathsf{P} \left\{ |Y_1| > n^{1/p} \right\} \right) \right\}$$

(since $\sum_{i=-\infty}^{\infty} |a_i| < \infty$).

We consider two separate cases. If $rp < 2$, let $q = 2$. We have

$$I \leq C \sum_{n=1}^{\infty} n^{r-1-2/p} \left(\mathsf{E}\, |Y_1|^2 I \left\{ |Y_1| \leq n^{1/p} \right\} + n^{2/p} \, \mathsf{P} \left\{ |Y_1| > n^{1/p} \right\} \right)$$

$$\leq C \sum_{n=1}^{\infty} n^{r-1-2/p} \, \mathsf{E}\, |Y_1|^2 I \left\{ |Y_1| \leq n^{1/p} \right\} + C \, \mathsf{E}\, |Y_1|^{rp} \qquad \text{(by Lemma 2(i))}$$

$$\leq C \, \mathsf{E}\, |Y_1|^{rp} < \infty \qquad \text{(by Lemma 2(ii) with } q = 2 \text{ and } \psi(x) = 1\text{)}.$$

If $rp \geq 2$, let $q > 2p(r-1)/(2-p) \geq 2$. Note that in this case

$$\mathsf{E}\, |Y_1|^2 I \left\{ |Y_1| \leq n^{1/p} \right\} \leq \mathsf{E}\, |Y_1|^2 \leq \left(\mathsf{E}\, |Y_1|^{rp} \right)^{2/(rp)} < \infty$$

and by Markov inequality

$$n^{2/p}\,\mathsf{P}\left\{|Y_1| > n^{1/p}\right\} \le \mathsf{E}\,|Y_1|^2 < \infty.$$

Hence

$$\begin{aligned}
I &\le C\sum_{n=1}^{\infty} n^{r-2-q/p}\Big\{\Big(n\left(\mathsf{E}\,|Y_1|^2 I\left\{|Y_1|\le n^{1/p}\right\} + n^{2/p}\,\mathsf{P}\left\{|Y_1|>n^{1/p}\right\}\right)\Big)^{q/2} \\
&\qquad + n\left(\mathsf{E}\,|Y_1|^q I\left\{|Y_1|\le n^{1/p}\right\} + n^{q/p}\,\mathsf{P}\left\{|Y_1|>n^{1/p}\right\}\right)\Big\} \\
&\le C\sum_{n=1}^{\infty} n^{r-2-q/p+q/2} + C\sum_{n=1}^{\infty} n^{r-1-q/p}\,\mathsf{E}\,|Y_1|^q I\left\{|Y_1|\le n^{1/p}\right\} \\
&\quad + C\sum_{n=1}^{\infty} n^{r-1}\,\mathsf{P}\left\{|Y_1|>n^{1/p}\right\} \\
&\le C + C\,\mathsf{E}\,|Y_1|^{rp} < \infty
\end{aligned}$$

by Lemma 2(i) and (ii) with $\psi(x)=1$. □

Next, we prove Theorem 2.

Proof. For every $\varepsilon > 0$

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2-q/p}\,\mathsf{E}\left\{\max_{1\le k\le n}|S_k| - \varepsilon n^{1/p}\right\}_+^q \\
&\quad = \sum_{n=1}^{\infty} n^{r-2-q/p}\int_0^{\infty}\mathsf{P}\left\{\max_{1\le k\le n}|S_k| - \varepsilon n^{1/p} > t^{1/q}\right\}dt \\
&\quad = \sum_{n=1}^{\infty} n^{r-2-q/p}\int_0^{n^{q/p}}\mathsf{P}\left\{\max_{1\le k\le n}|S_k| - \varepsilon n^{1/p} > t^{1/q}\right\}dt \\
&\qquad + \sum_{n=1}^{\infty} n^{r-2-q/p}\int_{n^{q/p}}^{\infty}\mathsf{P}\left\{\max_{1\le k\le n}|S_k| - \varepsilon n^{1/p} > t^{1/q}\right\}dt \\
&\quad \le \sum_{n=1}^{\infty} n^{r-2}\,\mathsf{P}\left\{\max_{1\le k\le n}|S_k| > \varepsilon n^{1/p}\right\} + \sum_{n=1}^{\infty} n^{r-2-q/p}\int_{n^{q/p}}^{\infty}\mathsf{P}\left\{\max_{1\le k\le n}|S_k| > t^{1/q}\right\}dt.
\end{aligned}$$

Hence by Theorem 1, in order to prove Theorem 2, it is enough to show that

$$\sum_{n=1}^{\infty} n^{r-2-q/p}\int_{n^{q/p}}^{\infty}\mathsf{P}\left\{\max_{1\le k\le n}|S_k| > t^{1/q}\right\}dt < \infty.$$

Let

$$Y_j^{(t,1)} = -t^{1/q} I\left\{Y_j < -t^{1/q}\right\} + Y_i I\left\{|Y_j| \le t^{1/q}\right\} + t^{1/q} I\left\{Y_j > t^{1/q}\right\},$$

and $Y_j^{(t,2)} = Y_j - Y_j^{(t,1)}$ be the monotone truncations of Y_j, $-\infty < j < \infty$. Then by Joag-Dev and Proschan [10], for any $t > 0$, $\left\{Y_j^{(t,1)} - \mathsf{E}\,Y_j^{(t,1)}, -\infty < j < \infty\right\}$ and $\left\{Y_j^{(t,2)}, -\infty < j < \infty\right\}$ are two sequences of NA random variables. Note that

$$\sum_{k=1}^{n} X_k = \sum_{k=1}^{n}\sum_{i=-\infty}^{\infty} a_i Y_{i+k} = \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_j$$

and

$$\begin{aligned}
&\sup_{t\ge n^{q/p}} t^{-1/q} \max_{1\le k\le n}\left|\mathsf{E}\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_j^{(t,1)}\right| \\
&\quad\le \sup_{t\ge n^{q/p}} t^{-1/q} \sum_{i=-\infty}^{\infty} |a_i| \max_{1\le k\le n}\left|\mathsf{E}\sum_{j=i+1}^{i+k} Y_j^{(t,1)}\right| \\
&\quad\le C \sup_{t\ge n^{q/p}} t^{-1/q} n\left(\left|\mathsf{E}\, Y_1 I\left\{|Y_1|\le t^{1/q}\right\}\right| + t^{1/q}\,\mathsf{P}\left\{|Y_1|>t^{1/q}\right\}\right) \\
&\quad\le C \sup_{t\ge n^{q/p}} t^{-1/q} n\left(\mathsf{E}\,|Y_1| I\left\{|Y_1|> t^{1/q}\right\} + t^{1/q}\,\mathsf{P}\left\{|Y_1|>t^{1/q}\right\}\right) \\
&\quad\le C\big(n^{q/p}\big)^{-1/q} n\,\mathsf{E}\,|Y_1| I\left\{|Y_1|>\big(n^{q/p}\big)^{1/q}\right\} + Cn\,\mathsf{P}\left\{|Y_1|>\big(n^{q/p}\big)^{1/q}\right\} \\
&\quad\le C\,\mathsf{E}\, n^{1-\;1/p} |Y_1| I\left\{|Y_1|>n^{1/p}\right\} + Cn\,\mathsf{P}\left\{|Y_1|>n^{1/p}\right\} \\
&\quad\le C\,\mathsf{E}\,|Y_1|^p I\left\{|Y_1|>n^{1/p}\right\} + Cn\,\mathsf{P}\left\{|Y_1|>n^{1/p}\right\} \to 0
\end{aligned}$$

as $n\to\infty$. Hence for n large enough we have that

$$\sup_{t\ge n^{q/p}} t^{-1/q} \max_{1\le k\le n}\left|\mathsf{E}\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_j^{(t,1)}\right| < 1/4.$$

Therefore, in order to prove Theorem 2, it is enough to show that

$$I := \sum_{n=1}^{\infty} n^{r-2-q/p} \int_{n^{q/p}}^{\infty} \mathsf{P}\left\{\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k}\left(Y_j^{(t,1)} - \mathsf{E}\, Y_j^{(t,1)}\right)\right| > t^{1/q}/4\right\} dt < \infty$$

and

$$J := \sum_{n=1}^{\infty} n^{r-2-q/p} \int_{n^{q/p}}^{\infty} \mathsf{P}\left\{\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_j^{(t,2)}\right| > t^{1/q}/2\right\} dt < \infty.$$

We first show that $J<\infty$. By Markov inequality

$$\begin{aligned}
J &\le C \sum_{n=1}^{\infty} n^{r-2-q/p} \int_{n^{q/p}}^{\infty} t^{-1/q}\,\mathsf{E}\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_j^{(t,2)}\right| dt \\
&\le C \sum_{n=1}^{\infty} n^{r-1-q/p} \int_{n^{q/p}}^{\infty} t^{-1/q}\left(\mathsf{E}\,|Y_1| I\left\{|Y_1|>t^{1/q}\right\} + t^{1/q}\,\mathsf{P}\left\{|Y_1|>t^{1/q}\right\}\right) dt \\
&\le C \sum_{n=1}^{\infty} n^{r-1-q/p} \int_{n^{q/p}}^{\infty} t^{-1/q}\,\mathsf{E}\,|Y_1| I\left\{|Y_1|>t^{1/q}\right\} dt \\
&= C \sum_{n=1}^{\infty} n^{r-1-q/p} \sum_{m=n}^{\infty} \int_{m^{q/p}}^{(m+1)^{q/p}} t^{-1/q}\,\mathsf{E}\,|Y_1| I\left\{|Y_1|>t^{1/q}\right\} dt \\
&\le C \sum_{n=1}^{\infty} n^{r-1-q/p} \sum_{m=n}^{\infty} m^{q/p-1-1/p}\,\mathsf{E}\,|Y_1| I\left\{|Y_1|>m^{1/p}\right\} \\
&= C \sum_{m=1}^{\infty} m^{q/p-1-1/p}\,\mathsf{E}\,|Y_1| I\left\{|Y_1|>m^{1/p}\right\} \sum_{n=1}^{m} n^{r-1-q/p}
\end{aligned}$$

$$\leq \begin{cases} C\sum_{m=1}^{\infty} m^{r-1-1/p}\,\mathsf{E}\,|Y_1| I\{|Y_1| > m^{1/p}\}, & \text{if } q < rp, \\ C\sum_{m=1}^{\infty} m^{r-1-1/p}\log(1+m)\,\mathsf{E}\,|Y_1| I\{|Y_1| > m^{1/p}\}, & \text{if } q = rp, \\ C\sum_{m=1}^{\infty} m^{q/p-1-1/p}\,\mathsf{E}\,|Y_1| I\{|Y_1| > m^{1/p}\}, & \text{if } q > rp \end{cases}$$

$$\leq \begin{cases} C\,\mathsf{E}\,|Y_1|^{rp}, & \text{if } q < rp, \\ C\,\mathsf{E}\,|Y_1|^{rp}\log(1+|Y_1|), & \text{if } q = rp, \\ C\,\mathsf{E}\,|Y_1|^{q}, & \text{if } q > rp \end{cases}$$

(by Lemma 2(iii) with $v = 1$)

$< \infty$ (by the assumptions of Theorem 2).

We now prove that $I < \infty$. Fix any $s \geq 2$ (to be specified later). By Markov inequality,

$$I \leq C\sum_{n=1}^{\infty} n^{r-2-q/p}\int_{n^{q/p}}^{\infty} t^{-s/q}\,\mathsf{E}\max_{1\leq k\leq n}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k}\Big(Y_j^{(t,1)} - \mathsf{E}\,Y_j^{(t,1)}\Big)\right|^s dt$$

$$\leq C\sum_{n=1}^{\infty} n^{r-2-q/p}\int_{n^{q/p}}^{\infty} t^{-s/q}\,\mathsf{E}\left(\sum_{i=-\infty}^{\infty} |a_i| \max_{1\leq k\leq n}\left|\sum_{j=i+1}^{i+k}\Big(Y_j^{(t,1)} - \mathsf{E}\,Y_j^{(t,1)}\Big)\right|\right)^s dt$$

$$\leq C\sum_{n=1}^{\infty} n^{r-2-q/p}$$

$$\times\int_{n^{q/p}}^{\infty} t^{-s/q}\,\mathsf{E}\Bigg(\sum_{i=-\infty}^{\infty}\Big(|a_i|^{1-1/s}\Big)$$

$$\times\left(|a_i|^{1/s}\max_{1\leq k\leq n}\left|\sum_{j=i+1}^{i+k}\Big(Y_j^{(t,1)} - \mathsf{E}\,Y_j^{(t,1)}\Big)\right|\right)\Bigg)^s dt$$

$$\leq C\sum_{n=1}^{\infty} n^{r-2-q/p}$$

$$\times\int_{n^{q/p}}^{\infty} t^{-s/q}\left(\sum_{i=-\infty}^{\infty}|a_i|\right)^{s-1}\sum_{i=-\infty}^{\infty}|a_i|\,\mathsf{E}\max_{1\leq k\leq n}\left|\sum_{j=i+1}^{i+k}\Big(Y_j^{(t,1)} - \mathsf{E}\,Y_j^{(t,1)}\Big)\right|^s dt$$

(by Hölder inequality)

$$\leq C\sum_{n=1}^{\infty} n^{r-2-q/p}\int_{n^{q/p}}^{\infty} t^{-s/q}\sum_{i=-\infty}^{\infty}|a_i|\,\mathsf{E}\max_{1\leq k\leq n}\left|\sum_{j=i+1}^{i+k}\Big(Y_j^{(t,1)} - \mathsf{E}\,Y_j^{(t,1)}\Big)\right|^s dt$$

(since $\sum_{i=-\infty}^{\infty}|a_i| < \infty$)

$$\leq C\sum_{n=1}^{\infty} n^{r-2-q/p}\int_{n^{q/p}}^{\infty} t^{-s/q}\sum_{i=-\infty}^{\infty}|a_i|\Bigg\{\left(\sum_{j=i+1}^{i+n}\mathsf{E}\Big(Y_j^{(t,1)} - \mathsf{E}\,Y_j^{(t,1)}\Big)^2\right)^{s/2}$$

$$+\sum_{j=i+1}^{i+n}\mathsf{E}\left|Y_j^{(t,1)} - \mathsf{E}\,Y_j^{(t,1)}\right|^s\Bigg\}\,dt$$

(by Lemma 1)

$$\le C \sum_{n=1}^{\infty} n^{r-2-q/p}$$
$$\times \int_{n^{q/p}}^{\infty} t^{-s/q} \sum_{i=-\infty}^{\infty} |a_i| \Big\{ n^{s/2} \left(\mathsf{E}\,|Y_1|^2 I\left\{|Y_1| \le t^{1/q}\right\} + t^{2/q}\,\mathsf{P}\left\{|Y_1| > t^{1/q}\right\}\right)^{s/2}$$
$$+ n \left(\mathsf{E}\,|Y_1|^s I\left\{|Y_1| \le t^{1/q}\right\} + t^{s/q}\,\mathsf{P}\left\{|Y_1| > t^{1/q}\right\}\right)\Big\}\, dt$$
$$\le C \sum_{n=1}^{\infty} n^{r-2-q/p} \int_{n^{q/p}}^{\infty} t^{-s/q} \Big\{ n^{s/2} \left(\mathsf{E}\,|Y_1|^2 I\left\{|Y_1| \le t^{1/q}\right\} + t^{2/q}\left\{|Y_1| > t^{1/q}\right\}\right)^{s/2}$$
$$+ n \left(\mathsf{E}\,|Y_1|^s I\left\{|Y_1| \le t^{1/q}\right\} + t^{s/q}\,\mathsf{P}\left\{|Y_1| > t^{1/q}\right\}\right)\Big\}\, dt$$

(since $\sum_{i=-\infty}^{\infty} |a_i| < \infty$).

Consider two separate cases. If $\max\{q, rp\} < 2$, let $s = 2$. We have

$$
\begin{aligned}
I &\le C\sum_{n=1}^{\infty} n^{r-1-q/p}\int_{n^{q/p}}^{\infty} t^{-2/q}\left(\mathsf{E}\,|Y_1|^2 I\left\{|Y_1|\le t^{1/q}\right\} + t^{2/q}\,\mathsf{P}\left\{|Y_1|>t^{1/q}\right\}\right) dt \\
&\le C\sum_{n=1}^{\infty} n^{r-1-q/p}\int_{n^{q/p}}^{\infty} t^{-2/q}\,\mathsf{E}\,|Y_1|^2 I\left\{|Y_1|\le t^{1/q}\right\} dt \\
&\quad + C\sum_{n=1}^{\infty} n^{r-1-q/p}\int_{n^{q/p}}^{\infty} \mathsf{P}\left\{|Y_1|>t^{1/q}\right\} dt \\
&= C\sum_{n=1}^{\infty} n^{r-1-q/p}\sum_{m=n+1}^{\infty}\int_{(m-1)^{q/p}}^{m^{q/p}} t^{-2/q}\,\mathsf{E}\,|Y_1|^2 I\left\{|Y_1|\le t^{1/q}\right\} dt \\
&\quad + C\sum_{n=1}^{\infty} n^{r-1-q/p}\,\mathsf{E}\,|Y_1|^q I\left\{|Y_1|>n^{1/p}\right\} \\
&\le C\sum_{n=1}^{\infty} n^{r-1-q/p}\sum_{m=n+1}^{\infty}\mathsf{E}\,|Y_1|^2 I\left\{|Y_1|\le m^{1/q}\right\}\int_{(m-1)^{q/p}}^{m^{q/p}} t^{-2/q}\,dt + C\,\mathsf{E}\,|Y_1|^{rp} \\
&\quad \text{(by Lemma 2(iii) with } s=r \text{ and } v=q) \\
&\le C\sum_{n=1}^{\infty} n^{r-1-q/p}\sum_{m=n+1}^{\infty}(m-1)^{q/p-1-2/p}\,\mathsf{E}\,|Y_1|^2 I\left\{|Y_1|\le m^{1/p}\right\} + C\,\mathsf{E}\,|Y_1|^{rp} \\
&\le C\sum_{n=1}^{\infty} n^{r-1-q/p}\sum_{m=n}^{\infty} m^{q/p-1-2/p}\,\mathsf{E}\,|Y_1|^2 I\left\{|Y_1|\le m^{1/p}\right\} + C\,\mathsf{E}\,|Y_1|^{rp} \\
&= C\sum_{m=1}^{\infty} m^{q/p-1-2/p}\,\mathsf{E}\,|Y_1|^2 I\left\{|Y_1|\le m^{1/p})\right\}\sum_{n=1}^{m} n^{r-1-q/p} + C\,\mathsf{E}\,|Y_1|^{rp} \\
&\le \begin{cases}
C\sum_{m=1}^{\infty} m^{r-1-2/p}\,\mathsf{E}\,|Y_1|^2 I\{|Y_1|\le m^{1/p}\} + C\,\mathsf{E}\,|Y_1|^{rp}, & \text{if } q<rp,\\
C\sum_{m=1}^{\infty} m^{r-1-2/p}\log(1+m)\,\mathsf{E}\,|Y_1|^2 I\{|Y_1|\le m^{1/p}\} + C\,\mathsf{E}\,|Y_1|^{rp}, & \text{if } q=rp,\\
C\sum_{m=1}^{\infty} m^{q/p-1-2/p}\,\mathsf{E}\,|Y_1|^2 I\{|Y_1|\le m^{1/p}\} + C\,\mathsf{E}\,|Y_1|^{rp}, & \text{if } q>rp
\end{cases} \\
&\le \begin{cases}
C\,\mathsf{E}\,|Y_1|^{rp}, & \text{if } q<rp,\\
C\,\mathsf{E}\,|Y_1|^{rp}\log(1+|Y_1|) + C\,\mathsf{E}\,|Y_1|^{rp}, & \text{if } q=rp,\\
C\,\mathsf{E}\,|Y_1|^{q} + C\,\mathsf{E}\,|Y_1|^{rp}, & \text{if } q>rp
\end{cases} \\
&\quad \text{(by Lemma 2(ii))} \\
&< \infty.
\end{aligned}
$$

If $\max\{q, rp\} \ge 2$, let $s > \max\{q, 2p(r-1)/(2-p)\}$. Note that in this case

$$\mathsf{E}\,|Y_1|^2 I\left\{|Y_1|\le t^{1/q}\right\} \le \mathsf{E}\,|Y_1|^2 < \infty$$

and by Markov inequality

$$t^{2/q}\,\mathsf{P}\left\{|Y_1|>t^{1/q}\right\} \le \mathsf{E}\,|Y_1|^2 < \infty.$$

Hence

$$
\begin{aligned}
I &\le C \sum_{n=1}^{\infty} n^{r-2-q/p+s/2} \int_{n^{q/p}}^{\infty} t^{-s/q}\, dt \\
&\quad + C \sum_{n=1}^{\infty} n^{r-1-q/p} \int_{n^{q/p}}^{\infty} t^{-s/q}\, \mathsf{E}\,|Y_1|^s I\left\{|Y_1| \le t^{1/q}\right\} dt \\
&\quad + C \sum_{n=1}^{\infty} n^{r-1-q/p} \int_{n^{q/p}}^{\infty} \mathsf{P}\left\{|Y_1| > t^{1/q}\right\} dt \\
&\le C \sum_{n=1}^{\infty} n^{r-2-s/p+s/2} + C \sum_{n=1}^{\infty} n^{r-1-q/p} \int_{n^{q/p}}^{\infty} t^{-s/q}\, \mathsf{E}\,|Y_1|^s I\left\{|Y_1| \le t^{1/q}\right\} dt \\
&\quad + C\, \mathsf{E}\,|Y_1|^{rp} \\
&\le C + C\, \mathsf{E}\,|Y_1|^{rp} + C \sum_{n=1}^{\infty} n^{r-1-q/p} \int_{n^{q/p}}^{\infty} t^{-s/q}\, \mathsf{E}\,|Y_1|^s I\left\{|Y_1| \le t^{1/q}\right\} dt \\
&< \infty. \quad \square
\end{aligned}
$$

Remark. The key point of the proofs of Theorems 1 and 2 is the application of Hölder and the Rosenthal-type inequalities for maximum partial sums of NA sequence presented in Lemma 1. Note that the Rosenthal-type inequality for maximum partial sums also holds for ρ and ρ^*-mixing random variables (cf., for example, Shao [15]). Hence Theorems 1 and 2 remain true for ρ and ρ^*-mixing random variables.

References

1. S. E. Ahmed, R. Giuliano Antonini, and A. Volodin, *On the rate of complete convergence for weighted sums of arrays of Banach space valued random elements with application to moving average processes*, Statist. Probab. Lett. **58** (2002), 185–194.
2. J.-Il. Baek, T. S. Kim, and H. Y. Liang, *On the complete convergence of moving average processes under dependent conditions*, Aust. N.Z.J. Statis. **45** (2003), 331–342.
3. R. M. Burton and H. Dehling, *Large deviations for some weakly dependent random processes*, Statist. Probab. Lett. **9** (1990), 397–401.
4. P. Chen, *Complete moment convergence for sequence of independent random elements in Banach spaces*, Stoch. Anal. Appl. (2006) (to appear).
5. P. Chen, T.-C. Hu, and A. Volodin, *A note on the rate of complete convergence for maximums of partial sums for moving average processes in Rademacher type Banach spaces*, Lobachevskii J. Math. **21** (2006), 45–55 (electronic).
6. P. Chen, S. H. Sung, and A. Volodin, *Rate of complete convergence for arrays of B-valued random elements*, Siberian Adv. Math. (2006) (to appear).
7. Y. S. Chow, *On the rate of moment complete convergence of sample sums and extremes*, Bull. Inst. Math. Acad. Sinica **16** (1988), 177–201.
8. P. Erdös, *On a theorem of Hsu and Robbins*, Ann. Math. Statist. **20** (1949), 286–291.
9. P. L. Hsu and H. Robbins, *Complete convergence and the law of large numbers*, Proc. Nat. Acad. Sci. U.S.A. **33** (1947), 25–31.
10. K. Joag-Dev and F. Proschan, *Negative association of random variables with applications*, Ann. Statist. **11** (1983), 286–295.
11. D. Li, M. B. Rao, and X. Wang, *Complete convergence of moving average processes*, Statist. Probab. Lett. **14** (1992), 111–114.
12. Y. X. Li and L. X. Zhang, *Complete moment convergence of moving-average processes under dependence assumptions*, Statist. Probab. Lett. **70** (2005), 191–197.
13. H.-Y. Liang, T.-S. Kim, and J.-Il. Baek, *On the convergence of moving average processes under negatively associated random variables*, Indian J. Pure Appl. Math. **34** (2003), 461–476.
14. A. Rosalsky, L. V. Thanh, and A. Volodin, *On complete convergence in mean of normed sums of independent random elements in Banach spaces*, Stoch. Anal. Appl. **24** (2006), 23–35.
15. Q. M. Shao, *Maximal inequalities for partial sums of ρ-mixing sequences*, Ann. Probab. **23** (1995), 948–965.

16. Q. M. Shao, *A comparison theorem on inequalities between negatively associated and independent random variables*, J. Theor. Probab. **13** (2000), 343–356.
17. L. X. Zhang, *Complete convergence of moving average processes under dependence assumptions*, Statist. Probab. Lett. **30** (1996), 165–170.

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