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On Cantrell–Rosalsky's strong laws of large numbers

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ABSTRACT

Starting from a result of almost sure (a.s.) convergence for nonnegative random variables, a simplified proof of a strong law of large numbers (SLLN) established in ([Cantrell, A., Rosalsky, A., 2003. Some strong law of large numbers for Banach space valued summands irrespective of their joint distributions. Stochastic Anal. Appl. 21, 79–95], Theorem 1) for random elements in a real separable Banach space is presented, and some other results of a.s. convergence related to SLLNs in ([Cantrell, A., Rosalsky, A., 2004. A strong law for compactly uniformly integrable sequences of independent random elements in Banach spaces. Bull. Inst. Math. Acad. Sinica 32, 15–33], Th. 3.1) and ([Cantrell, A., Rosalsky, A., 2003. Some strong law of large numbers for Banach space valued summands irrespective of their joint distributions. Stochastic Anal. Appl. 21, 79–95], Theorem 2) are derived. No conditions of independence or on the joint distribution of random elements are required. Likewise, no geometric condition on the Banach space where random elements take values is imposed. Some applications to weighted (for an array of constants) sums of random elements and to the case of random sets are also considered.

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1. Introduction and preliminaries

Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a probability space, and let $\{V_n, n \geq 1\}$ be a sequence of random elements taking values in a real separable Banach space $(X, \|.\|)$, i.e., a sequence of functions from Ω into X which are measurable with respect to the Borel subsets of X. The sequence $\{V_n, n \geq 1\}$ is said to obey the strong law of large numbers (SLLN) with centering elements $\{c_n, n \geq 1\} \subset X$ and norming constants $0 < b_n \to \infty$ if $\frac{1}{b_n}\sum_{i=1}^{n}(V_i - c_i) \to 0$ almost sure (a.s.).

In many cases the centering elements are related to the expectation of random elements. We define the expectation or expected value of a random element V, denoted by EV, to be the Pettis integral of V, provided it exists. Thus, a random element V in X has expected value if there exists an element $EV \in X$ such that $Ef(V) = f(EV)$ for each $f \in X^*$, where X^* denotes the dual space of X, i.e., the space of all continuous linear functionals on X. A sufficient condition for the existence of EV is that $E\|V\| < \infty$ (e.g. Taylor (1978)).

Cantrell and Rosalsky (2003) established SLLNs for normed and centered sums of random elements $\{V_n, n \geq 1\}$ irrespective of their joint distributions and without any condition of independence or pairwise independence on them. No geometric condition was imposed on the Banach space X and conditions on the random elements involved only their marginal distributions.

Starting from a simple result of convergence a.s. for nonnegative random variables (Theorem 1), we establish the main result in Cantrell and Rosalsky (2003) as a corollary, and we show that some formal simplifications are even possible in the statement of hypothesis.

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Cantrell and Rosalsky (2004), Theorem 3.1) established a SLLN for a compactly uniformly integrable sequence of independent random elements, by using a result in (Cantrell and Rosalsky, 2002) which was stated for a sequence of random elements $\{V_n, n \geq 1\}$ in a real separable Rademacher type p $(1 \leq p \leq 2)$ Banach space. We also obtain a result (Corollary 3) very close to the one from Cantrell and Rosalsky (2004) without requiring the condition of compactly uniform integrability and without any condition of independence among the random elements.

We also extend our results in a direct manner to obtain two SLLNs for weighted sums of random elements (Corollaries 5 and 6) and a result of a.s. convergence in the setting of random sets (Corollary 7).

2. Strong laws of large numbers for random elements

Our starting point is the following result:

Theorem 1. *Let $\{X_n, n \geq 1\}$ be a sequence of nonnegative random variables and $\{b_n, n \geq 1\}$ be an increasing to infinity sequence of positive numbers. If*

$$\sum_{n=1}^{\infty} \frac{EX_n^{\alpha}}{b_n^{\alpha}} < \infty \tag{1}$$

for some $\alpha \in (0, 1]$, then

$$\sum_{n=1}^{\infty} \frac{X_n}{b_n} < \infty \quad a.s. \tag{2}$$

and by Kronecker lemma

$$\frac{1}{b_n} \sum_{k=1}^{n} X_k \to 0 \quad a.s.$$

Proof. By monotone convergence theorem it is possible to exchange signs of summation and expectation in (1)

$$E \sum_{n=1}^{\infty} \frac{X_n^{\alpha}}{b_n^{\alpha}} < \infty.$$

Therefore

$$\sum_{n=1}^{\infty} \frac{X_n^{\alpha}}{b_n^{\alpha}} < \infty \quad \text{a.s.}$$

Since a term of an a.s. convergent series converges to zero a.s., then there exists $n_0 \in \mathbf{N}$ such that $\frac{X_n^{\alpha}}{b_n^{\alpha}} < 1$ for all $n \geq n_0$ and hence $\frac{X_n}{b_n} \leq \frac{X_n^{\alpha}}{b_n^{\alpha}}$ for all $n \geq n_0$. This implies (2). □

Now, we can state, as a corollary, the following Cantrell–Rosalsky's SLLN (e.g. Cantrell and Rosalsky (2003), Theorem 1):

Corollary 1. *Let $\{V_n, n \geq 1\}$ be a sequence of random elements in a real separable Banach space, and let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be sequences of positive constants with $b_n \uparrow \infty$, such that*

$$\sum_{i=1}^{n} a_i = \mathcal{O}(b_n).$$

Suppose that for some $\lambda > 0$ and all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P[\|V_n\| > \lambda b_n] < \infty$$

and

$$\sum_{n=1}^{\infty} \frac{1}{b_n} E\|V_n I[\varepsilon a_n < \|V_n\| \leq \lambda b_n] - EV_n I[\varepsilon a_n < \|V_n\| \leq \lambda b_n]\| < \infty.$$

Then the SLLN

$$\frac{1}{b_n} \sum_{i=1}^{n} (V_i - EV_i I[\|V_i\| \leq \lambda b_i]) \to 0 \quad a.s.$$

obtains irrespective of the joint distribution of the $\{V_n, n \geq 1\}$.

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Proof. Set, for each $n \in \mathbf{N}$:

$$X_n = \|V_n I[\varepsilon a_n < \|V_n\| \le \lambda b_n] - EV_n I[\varepsilon a_n < \|V_n\| \le \lambda b_n]\|$$
$$Y_n = \|V_n I[\|V_n\| > \lambda b_n]\|.$$

By applying Theorem 1 to the sequence of positive random variables $\{X_n, n \ge 1\}$, we obtain

$$\frac{1}{b_n}\sum_{i=1}^{n} \|V_i I[\varepsilon a_i < \|V_i\| \le \lambda b_i] - EV_i I[\varepsilon a_i < \|V_i\| \le \lambda b_i]\| \to 0 \quad \text{a.s.}$$

By the assumptions, $\varepsilon > 0$ is an arbitrary positive number and there exists $c > 0$ such that $0 \le \sum_{1=1}^{n} a_i \le cb_n$ for all $n \in \mathbf{N}$. Therefore

$$\limsup_{n\to\infty} \frac{1}{b_n}\sum_{i=1}^{n} \|V_i\| I[\|V_i\| \le \varepsilon a_i] \le c\varepsilon \quad \text{a.s.}$$

and

$$\lim_{n\to\infty} \frac{1}{b_n}\sum_{i=1}^{n} E\|V_i\| I[\|V_i\| \le \varepsilon a_i] \le c\varepsilon.$$

From here we obtain

$$\frac{1}{b_n}\sum_{i=1}^{n} \|V_i I[\|V_i\| \le \lambda b_i] - EV_i I[\|V_k\| \le \lambda b_k]\| \to 0 \quad \text{a.s.}$$

From the assumption

$$\sum_{n=1}^{\infty} P[\|V_n\| > \lambda b_n] = P[Y_n \ne 0] < \infty$$

we have, via Borel–Cantelli lemma

$$P[\liminf_{n\to\infty}[Y_n = 0]] = P\left[\bigcup_{n=1}^{\infty}\bigcap_{i=n}^{\infty}[Y_i = 0]\right] = 1.$$

So, we can assert, with probability 1, that there exists $n_0 \in \mathbf{N}$ such that $Y_n = 0$ for every $n \ge n_0$. Then:

$$\frac{1}{b_n}\sum_{i=1}^{n} Y_i \to 0 \quad \text{a.s.}$$

Therefore:

$$\frac{1}{b_n}\sum_{i=1}^{n} \|V_i - EV_i I[\|V_i\| \le b_i]\| \to 0 \quad \text{a.s.} \quad \square$$

An attentive reading of this Cantrell–Rosalsky's SLLN shows that two sequences of positive constants $\{a_n, n \ge 1\}$ and $\{b_n, n \ge 1\}$ and two positive parameters λ and ε appear in the hypothesis, whereas in the thesis only the sequence $\{b_n, n \ge 1\}$ is considered, since the parameter λ can be obviated without any loss of generality. In this sense, we can obtain the following two corollaries with the referred simplifications in the hypothesis.

Corollary 2. *Let $\{V_n, n \ge 1\}$ be a sequence of random elements in a real separable Banach space, and let $\{b_n, n \ge 1\}$ be a sequence of positive constants with $b_n \uparrow \infty$, such that for some $\alpha \in (0, 1]$:*

(a) $\sum_{n=1}^{\infty} P[\|V_n\| > b_n] < \infty$

(b) $\sum_{n=1}^{\infty} \frac{1}{b_n^{\alpha}} E \parallel V_n I[\|V_n\| \le b_n] - EV_n I[\|V_n\| \le b_n] \parallel^{\alpha} < \infty$.

Then $\frac{1}{b_n}\sum_{i=1}^{n} \|V_i - EV_i I[\|V_i\| \le b_i]\| \to 0$ a.s. and so

$$\frac{1}{b_n}\sum_{i=1}^{n} (V_i - EV_i I[\|V_i\| \le b_i]) \to 0 \quad \textit{a.s.}$$

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Proof. Set, for each $n \in \mathbf{N}$:

$$X_n = \|V_n I[\|V_n\| \le b_n] - EV_n I[\|V_n\| \le b_n]\|$$
$$Y_n = \|V_n I[\|V_n\| > b_n]\|$$

and use Theorem 1 and Borel–Cantelli lemma as in the proof of Corollary 1. □

Corollary 3. *Let $\{V_n, n \ge 1\}$ be a sequence of random elements in a real separable Banach space, and let $\{b_n, n \ge 1\}$ be a sequence of positive constants with $b_n \uparrow \infty$, such that for some $\alpha \in (0, 1]$:*

(a) $\sum_{n=1}^{\infty} P[\|V_n\| > b_n] < \infty$

(b) $\sum_{n=1}^{\infty} \frac{1}{b_n^{\alpha}} E \parallel V_n I[\|V_n\| \le b_n] - EV_n I[\|V_n\| \le b_n] \parallel^{\alpha} < \infty$

(c) $\frac{1}{b_n} \sum_{i=1}^{n} E\|V_i I[\|V_i\| > b_i]\| \to 0.$

Then $\frac{1}{b_n} \sum_{i=1}^{n} (V_i - EV_i) \to 0$ *a.s.*

Proof. It is immediate, because of

$$\frac{1}{b_n}\sum_{i=1}^{n}\|V_i - EV_i\| \le \frac{1}{b_n}\sum_{1=1}^{n}\|V_i - EV_i I[\|V_i\| \le b_i]\| + \frac{1}{b_n}\sum_{i=1}^{n} E[\|V_i I\|V_i\| > b_i]\|,$$

and applying Corollary 2 and assumption (c). □

Cantrell and Rosalsky (2004) established the following SLLN:

Theorem 2 (*Cantrell and Rosalsky, 2004, Theorem 3.1*). *Let $\{V_n, n \ge 1\}$ be a sequence of independent random elements in a real separable Banach space, let $1 \le p \le 2$, and let $\{a_n, n \ge 1\}$ and $\{b_n, n \ge 1\}$ be sequences of positive constants with $b_n \uparrow \infty$ and $n = \mathcal{O}(b_n)$, such that either*

$$\sum_{n=1}^{\infty} \frac{a_n^p}{b_n^p} < \infty \quad \text{or} \quad \sum_{i=1}^{n} a_i = \mathcal{O}(b_n).$$

Suppose that for some $\lambda > 0$ and for all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P[\|V_n\| > \lambda b_n] < \infty$$

and

$$\sum_{n=1}^{\infty} \frac{1}{b_n^p} E\ \|\|V_n\| I[\varepsilon a_n < \|V_n\| \le \lambda b_n] - E\|V_n\| I[\varepsilon a_n < \|V_n\| \le \lambda b_n]\ \|^p < \infty.$$

Then if $\{V_n, n \ge 1\}$ is compactly uniformly integrable:

$$\frac{1}{b_n}\sum_{i=1}^{n}(V_i - EV_i) \to 0 \quad \text{a.s.}$$

In the proof of their theorem, they show that compactly uniform integrability of $\{V_n, n \ge 1\}$ and condition $n = \mathcal{O}(b_n)$ are sufficient to obtain condition (c) in our Corollary 3. So, Corollary 3 can be considered *almost* a very simplified version of Cantrell–Rosalsky's SLLN; the only fault is the fact that $\alpha \in (0, 1]$ in our condition (b) is a requirement stronger than the consideration of $p \in [1, 2]$ in Cantrell–Rosalsky's theorem, but, anyway, the assumption of independence of $\{V_n, n \ge 1\}$ is not required.

Next result is in the spirit of Theorem 2 in Cantrell and Rosalsky (2003), with some modifications in hypothesis: we suppress condition $\sum_{i=1}^{n} b_i = \mathcal{O}(b_n)$ for the norming sequence $\{b_n, n \ge 1\}$ in Cantrell–Rosalsky's theorem and we require condition (a) for a sequence $\{\varepsilon_n, n \ge 1\}$ with $\sum_{n=1}^{\infty} \varepsilon_n < \infty$ instead of for all $\varepsilon > 0$.

Corollary 4. *Let $\{V_n, n \ge 1\}$ be a sequence of random elements in a real separable Banach space, and let $\{b_n, n \ge 1\}$ and $\{\varepsilon_n, n \ge 1\}$ be two sequences of positive constants with $b_n \uparrow \infty$ and $\sum_{n=1}^{\infty} \varepsilon_n < \infty$ such that:*

(a) $\sum_{n=1}^{\infty} P[\|V_n\| > \varepsilon_n b_n] < \infty.$

Then $\frac{1}{b_n} \sum_{i=1}^{n} \|V_i\| \to 0$ *a.s.*

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Proof. According to Borel–Cantelli lemma, there exists $n_0 \in \mathbf{N}$ such that $\|V_n\| \le \varepsilon_n b_n$ for every $n \ge n_0$, with probability one.

Then $\sum_{n=n_0}^{\infty} \frac{\|V_n\|}{b_n} \le \sum_{n=n_0}^{\infty} \varepsilon_n$ a.s., and so $\sum_{n=1}^{\infty} \frac{\|V_n\|}{b_n} < \infty$ a.s., which implies, via Kronecker lemma:

$$\frac{1}{b_n} \sum_{i=1}^{n} \|V_i\| \to 0 \quad \text{a.s.} \quad \square$$

Now, we can extend our last two results in a direct manner to the case of weighted sums of random elements when the weights are an array of constants, not a sequence. We omit the proofs.

Corollary 5. *Let $\{V_n, n \ge 1\}$ be a sequence of random elements in a real separable Banach space, and let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be a triangular array of constants with $\max_{1\le i\le n} |a_{ni}| \downarrow 0$.*

Let $\{b_n, n \ge 1\}$ be the sequence $b_n = (\max_{1\le i\le n} |a_{ni}|)^{-1}$, and suppose that conditions in Corollary 3 hold.

Then $\sum_{i=1}^{n} a_{ni}(V_i - EV_i) \to 0$ a.s.

Corollary 6. *Let $\{V_n, n \ge 1\}$ be a sequence of random elements in a real separable Banach space, and let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be a triangular array of constants with $b_n = (\max_{1\le i\le n} |a_{ni}|)^{-1} \uparrow \infty$.*

Let $\{\varepsilon_n, n \ge 1\}$ be a sequence of positive constants verifying the hypothesis in Corollary 4.

Then $\sum_{i=1}^{n} a_{ni} V_i \to 0$ a.s.

3. An application to random sets

Recently, Terán (2005) proved an extension to random sets of Theorem 1 in Cantrell and Rosalsky (2003). We are going to prove that it is also possible to extend our Corollary 2 to the setting of random sets in the same sense.

We need to state previously some basic definitions and notations.

We will denote by $\mathcal{C}(X)$ the space of nonempty closed bounded subsets of X, endowed with the Hausdorff metric

$$d_H(A, C) = \max\{\sup_{x\in A} \inf_{y\in C} \|x - y\|, \sup_{y\in C} \inf_{x\in A} \|x - y\|\}$$

and the Minkowski addition, $A + C = \{x + y : x \in A, y \in C\}$ and the product by a scalar, $\lambda A = \{\lambda x : x \in A\}$, $\lambda \in \mathbf{R}$.

We will denote by co$A$ the convex hull of $A \in \mathcal{C}(X)$. The norm of $A \in \mathcal{C}(X)$ is $\|A\| = \sup\{\|x\| : x \in A\} = d_H(A, \{0\})$.

A random closed bounded set is a measurable mapping V from $(\Omega, \mathcal{A}, \mathcal{P})$ into $\mathcal{C}(X)$. We denote by EV the set $\{Ev\}$ where v is a selection of V, i.e., $v(\omega) \in V(\omega)$ a.e.. EV is called the Aumann expectation of V. If $\|V\|$ is integrable, then EV is well defined, i.e., $EV \in \mathcal{C}(X)$.

We recommend to the interested reader the reference Artstein and Vitale (1975) and the references in Terán (2005).

Having into account definitions and notations before and the following lemma, in Terán (2005):

Lemma 1. *Let V, W be random closed bounded sets and let $A \subset X$ be closed. Then, the event $\{V \subset A\}$ is measurable, and $d_H(V, W)$ is a random variable.*

A formal application of Theorem 1 and Corollary 2 allows us to obtain the following SLLN for random closed bounded sets:

Corollary 7. *Let $\{V_n, n \ge 1\}$ be a sequence of random closed bounded sets in a real separable Banach space X. Let $\{B_n, n \ge 1\}$ be a sequence of closed subsets of X. Let $\{b_n, n \ge 1\}$ be a sequence of positive constants such that $b_n \uparrow \infty$. Suppose that for some $\alpha \in (0, 1]$:*

(a) $\sum_{n=1}^{\infty} P[V_n \not\subset B_n] < \infty$

(b) $\sum_{n=1}^{\infty} \frac{1}{b_n^{\alpha}} E\left(d_H(V_n I[V_n \subset B_n], E(\mathrm{co}V_n I[V_n \subset B_n]))\right)^{\alpha} < \infty$.

Then

$$\frac{1}{b_n} \sum_{i=1}^{n} d_H(V_i, E(\mathrm{co}V_i I[V_i \subset B_i])) \to 0 \quad \text{a.s.}$$

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