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On the Relationship Between the Baum–Katz–Spitzer Complete Convergence Theorem and the Law of the Iterated Logarithm

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Abstract For a sequence of i.i.d. Banach space-valued random variables $\{X_n;\ n \geq 1\}$ and a sequence of positive constants $\{a_n;\ n \geq 1\}$, the relationship between the Baum–Katz–Spitzer complete convergence theorem and the law of the iterated logarithm is investigated. Sets of conditions are provided under which

(i) $\limsup_{n\to\infty} \frac{\|S_n\|}{a_n} < \infty$ a.s. and

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\frac{\|S_n\|}{a_n} \geq \varepsilon\right) < \infty \text{ for all } \varepsilon > \lambda \text{ for some constant } \lambda \in [0,\infty)$$

are equivalent;

(ii) For all constants $\lambda \in [0,\infty)$,

$$\limsup_{n\to\infty} \frac{\|S_n\|}{a_n} = \lambda \text{ a.s.}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\frac{\|S_n\|}{a_n} \geq \varepsilon\right) \begin{cases} < \infty, & \text{if } \varepsilon > \lambda \\ = \infty, & \text{if } \varepsilon < \lambda \end{cases}$$

are equivalent. In general, no geometric conditions are imposed on the underlying Banach space. Corollaries are presented and new results are obtained even in the case of real-valued random variables.

Keywords partial sums of i.i.d. Banach space-valued random variables, Baum–Katz–Spitzer complete convergence theorem, law of the iterated logarithm, almost sure convergence

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1 Introduction

Let $(\mathbf{B}, \|\cdot\|)$ be a real separable Banach space with topological dual $\mathbf{B}^*$ and let $\{X,\ X_n;\ n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) $\mathbf{B}$-valued random variables. As usual, let $S_n = \sum_{i=1}^{n} X_i,\ n \geq 1$ denote their partial sums. If $0 < p < 2$ and X is a real-valued random variable, then the following two statements, related to the Kolmogorov–Marcinkiewicz–Zygmund strong law of large numbers, are known to be equivalent:

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\frac{|S_n|}{n^{1/p}} \geq \varepsilon\right) < \infty \text{ for all } \ \varepsilon > 0 \tag{1.1}$$

and

$$E|X|^p < \infty, \quad \text{where} \ \ EX = 0 \ \ \text{whenever} \ \ p \geq 1. \tag{1.2}$$

One can label this remarkable result as the Baum–Katz–Spitzer complete convergence theorem. By using a combinatorial lemma, Spitzer [1] established this result for the particularly important case of $p = 1$. The equivalence of (1.1) and (1.2) in the general case, $0 < p < 2$, is due to Baum and Katz [2].

Versions of the Baum–Katz–Spitzer complete convergence theorem in a Banach space setting were obtained by Jain [3] for the case of $p = 1$, Azlarov and Volodin [4] for the case of $1 \leq p < 2$ under an appropriate geometric condition, and Yang and Wang [5] for the general case of $0 < p < 2$ without any geometric conditions. In fact, by using de Acosta's [6] inequality, Yang and Wang [5] proved that the following three statements are equivalent:

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\frac{\|S_n\|}{n^{1/p}} \geq \varepsilon\right) < \infty \text{ for all } \ \varepsilon > 0,$$

$$\lim_{n\to\infty} \frac{S_n}{n^{1/p}} = 0 \ \text{ almost surely (a.s.)},$$

$$E\|X\|^p < \infty \ \text{ and } \ \frac{S_n}{n^{1/p}} \to_P 0. \tag{1.3}$$

Let $\{a_n;\ n \geq 1\}$ be a sequence of positive constants such that

$$a_n \uparrow \quad \text{and} \quad 1 < \liminf_{n\to\infty} \frac{a_{2n}}{a_n} \leq \limsup_{n\to\infty} \frac{a_{2n}}{a_n} < \infty. \tag{1.4}$$

Since the condition (1.3) is, of course, equivalent to

$$\sum_{n=1}^{\infty} P(\|X\| \geq n^{1/p}) < \infty \text{ and } \ \frac{S_n}{n^{1/p}} \to_P 0,$$

it is natural to ask whether the following three statements are equivalent:

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\frac{\|S_n\|}{a_n} \geq \varepsilon\right) < \infty \text{ for all } \ \varepsilon > 0, \tag{1.5}$$

$$\lim_{n\to\infty} \frac{S_n}{a_n} = 0 \ \text{ a.s.}, \tag{1.6}$$

$$\sum_{n=1}^{\infty} P\left(\|X\| \geq a_n\right) < \infty \text{ and } \ \frac{S_n}{a_n} \to_P 0. \tag{1.7}$$

The answer to this question turns out to be negative if $\{X, X_n;\ n \geq 1\}$ is a sequence of real-valued random variables with $EX = 0$ and $EX^2 = 1$, and we choose $a_n = \sqrt{2nLLn}, n \geq 1$, where $Lx = \log\max\{e, x\}, x \geq 0$. Then (1.6) fails by the classical Hartman–Wintner–Strassen law of the iterated logarithm, but (1.7) holds by $EX^2 < \infty$ and Chebyshev's inequality. However, it is clear that (1.6) always implies (1.7). Recently, Li, Zhang, and Rosalsky [7] have shown that (1.5) and (1.6) are equivalent.

The main purpose of the present paper is to exhibit the relationship between the Baum–Katz–Spitzer complete convergence theorem and the law of the iterated logarithm.

Theorem 1 *Let $\{X,\ X_n;\ n \geq 1\}$ be a sequence of i.i.d.* **B***-valued random variables and let $\{a_n;\ n \geq 1\}$ be a sequence of positive constants such that* (1.4) *holds. Then*

$$\limsup_{n\to\infty} \frac{\|S_n\|}{a_n} < \infty \quad a.s., \tag{1.8}$$

if and only if there exists a constant $0 \leq \lambda < \infty$ such that

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\frac{\|S_n\|}{a_n} \geq \varepsilon\right) < \infty \quad \text{for all} \quad \varepsilon > \lambda. \tag{1.9}$$

Combining Theorem 1 above and Theorem 1 of Li, Zhang, and Rosalsky [7], we obtain the following result:

Theorem 2 *Suppose that all conditions for Theorem* 1 *are satisfied. Then we have*

(i) $\lim_{n\to\infty} \frac{S_n}{a_n} = 0$ *a.s., if and only if*

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\frac{\|S_n\|}{a_n} \geq \varepsilon\right) < \infty \quad \text{for all} \quad \varepsilon > 0;$$

(ii) *There exists a constant $0 < \lambda_1 < \infty$ such that*

$$\limsup_{n\to\infty} \frac{\|S_n\|}{a_n} = \lambda_1 \quad a.s.,$$

if and only if there exists a constant $0 < \lambda_2 < \infty$ such that

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\frac{\|S_n\|}{a_n} \geq \varepsilon\right) \begin{cases} < \infty, & \text{if} \ \ \varepsilon > \lambda_2, \\ = \infty, & \text{if} \ \ 0 < \varepsilon < \lambda_2; \end{cases}$$

(iii) $\limsup_{n\to\infty} \frac{\|S_n\|}{a_n} = \infty$ *a.s., if and only if*

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\frac{\|S_n\|}{a_n} \geq \varepsilon\right) = \infty \quad \text{for all} \quad \varepsilon > 0.$$

We conjecture that, in general, $\lambda_1 = \lambda_2$ in Part (ii). In fact, our conjecture is true under some mild additional conditions according to the next theorem.

Theorem 3 *Let $\{X,\ X_n;\ n \geq 1\}$ be a sequence of i.i.d.* **B***-valued random variables. Let $\{a_n;\ n \geq 1\}$ be a sequence of positive constants satisfying*

$$\lim_{\beta\downarrow 1} \limsup_{n\to\infty} \frac{a_{[\beta n]}}{a_n} = 1. \tag{1.10}$$

If

$$\liminf_{n\to\infty} P\left(\frac{\|S_n\|}{a_n} \leq \varepsilon\right) > 0 \quad \text{for all}\ \varepsilon > 0, \tag{1.11}$$

then for all constants $0 \leq \lambda < \infty$,

$$\limsup_{n\to\infty} \frac{\|S_n\|}{a_n} = \lambda \quad a.s., \tag{1.12}$$

if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\frac{\|S_n\|}{a_n} \geq \varepsilon\right) \begin{cases} < \infty, & \text{if} \ \ \varepsilon > \lambda, \\ = \infty, & \text{if} \ \ \varepsilon < \lambda. \end{cases} \tag{1.13}$$

Although in the formulation of the statement of Theorem 3 we used the phrase "for all constants $0 \leq \lambda < \infty$", it should be noted that there cannot be more than one value of λ satisfying (1.12) and (1.13). As an application of Theorem 3, under (1.10) and (1.11), theoretically one can find the value of λ in (1.12). In fact

$$\lambda = \sup\left\{\varepsilon \geq 0;\ \sum_{n=1}^{\infty} \frac{1}{n} P\left(\frac{\|S_n\|}{a_n} \geq \varepsilon\right) = \infty\right\}.$$

A similar observation pertains to Theorem 4 below.

In the case of real-valued random variables, it is natural to ask about the relationship between the one-sided Baum–Katz–Spitzer complete convergence theorem and the one-sided law of the iterated logarithm. The following theorem answers this question:

Theorem 4 *Let $\{X,\ X_n;\ n \geq 1\}$ be a sequence of i.i.d. real-valued random variables and let $\{a_n;\ n \geq 1\}$ be a sequence of positive constants such that* (1.10) *holds. If*

$$\liminf_{n\to\infty} P\left(\frac{S_n}{a_n} \geq -\varepsilon\right) > 0 \quad \text{for all } \varepsilon > 0, \tag{1.14}$$

then for all constants $0 \leq \lambda < \infty$,

$$\limsup_{n\to\infty} \frac{S_n}{a_n} = \lambda \quad a.s.$$

if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\frac{S_n}{a_n} \geq \varepsilon\right) \begin{cases} < \infty, & \text{if } \varepsilon > \lambda, \\ = \infty, & \text{if } \varepsilon < \lambda. \end{cases}$$

Clearly, condition (1.11) (resp., condition (1.14)) is satisfied if

$$\frac{S_n}{a_n} \to_P 0. \tag{1.15}$$

The condition (1.16) of the first corollary is the analytic condition that X lies outside of the domain of partial attraction of the normal law.

Corollary 1 *Let $\{X,\ X_n;\ n \geq 1\}$ be a sequence of i.i.d. real-valued symmetric random variables and let $\{a_n;\ n \geq 1\}$ be a sequence of positive constants such that* (1.4) *holds. If*

$$\liminf_{n\to\infty} \frac{x^2 P(|X| \geq x)}{E\left(X^2 I(|X| < x)\right)} > 0, \tag{1.16}$$

then either

$$\lim_{n\to\infty} \frac{S_n}{a_n} = 0 \quad a.s. \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n} P\left(\frac{|S_n|}{a_n} \geq \varepsilon\right) < \infty \quad \text{for all } \varepsilon > 0$$

or

$$\limsup_{n\to\infty} \frac{S_n}{a_n} = \infty \quad \text{a.s.} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n} P\left(\frac{|S_n|}{a_n} \geq \varepsilon\right) = \infty \quad \text{for all } \varepsilon > 0.$$

Proof According to the work of Rogozin [8] and Heyde [9], it follows from (1.16) that there does not exist a constant $0 < \lambda < \infty$ such that

$$\limsup_{n\to\infty} \frac{S_n}{a_n} = \lambda \quad \text{a.s.}$$

Thus, since $\{X_n;\ n \geq 1\}$ are symmetric, either

$$\lim_{n\to\infty} \frac{S_n}{a_n} = 0 \quad \text{a.s.} \quad \text{or} \quad \limsup_{n\to\infty} \frac{S_n}{a_n} = \infty \quad \text{a.s.},$$

and the conclusion follows from Theorem 2.

Corollary 2 *Let $\{X,\ X_n;\ n \geq 1\}$ be a sequence of i.i.d.* **B***-valued random variables. Let $h(\cdot) : [0, \infty) \to (0, \infty)$ be continuous, nondecreasing, and slowly varying at infinity. Set $a_n = \sqrt{nh(n)},\ n \geq 1$. Then we have*

(i) *The relations* (1.8) *and* (1.9) *are equivalent*;

(ii) *If $S_n/a_n \to_P 0$, then for all constants $0 \leq \lambda < \infty$, the relations* (1.12) *and* (1.13) *are equivalent.*

Under the conditions of Corollary 1, if any of (1.8) or (1.9) holds, then

$$E(X) = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} P(\|X\| \geq c\sqrt{nh(n)}) < \infty \quad \text{for some } 0 < c < \infty$$

or, equivalently,

$$E(X) = 0 \ \text{ and } \ E\left(\Psi^{-1}(\|X\|)\right) < \infty, \tag{1.17}$$

where $\Psi^{-1}(t)$ is the inverse function of $\Psi(t) = \sqrt{th(t)}$. We leave it to the reader to verify that in type 2 Banach spaces, (1.17) implies (1.15). Hence Theorems 1 and 2 yield the following corollary:

Corollary 3 *Let $\{X,\ X_n;\ n \geq 1\}$ be a sequence of i.i.d. random variables taking values in a Banach space $\mathbf{B}$ of type 2. Let $h(\cdot) : [0, \infty) \to (0, \infty)$ be continuous, nondecreasing, and slowly varying at infinity and suppose that (1.17) holds. Set $a_n = \sqrt{nh(n)}$, $n \geq 1$. Then, for all constants $0 \leq \lambda < \infty$, the relations (1.12) and (1.13) are equivalent.*

Let $\{X,\ X_n;\ n \geq 1\}$ be a sequence of i.i.d. real-valued random variables. Let $p \geq 1$. Einmahl and Li ([10, Corollary 1]) proved that, for all constants $0 \leq \lambda < \infty$,

$$\limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2n(\log\log n)^p}} = \lambda \ \text{ a.s.} \tag{1.18}$$

if and only if

$$\begin{cases} E(X) = 0, \quad E\left(\frac{X^2}{(\log\log(3+|X|))^p}\right) < \infty, \\ \quad \text{and } \limsup_{x\to\infty}(\log\log x)^{1-p}E\left(X^2 I(|X| \leq x)\right) = \lambda^2. \end{cases} \tag{1.19}$$

Combining our Corollary 3 and Corollary 1 of Einmahl and Li [10], one can see that, for all constants $0 \leq \lambda < \infty$, (1.18) and (1.19) are each equivalent to

$$\sum_{n=3}^{\infty} \frac{1}{n} P\left(\frac{|S_n|}{\sqrt{2n(\log\log n)^p}} \geq \varepsilon\right) \begin{cases} < \infty, & \text{if } \varepsilon > \lambda, \\ = \infty, & \text{if } \varepsilon < \lambda. \end{cases} \tag{1.20}$$

Clearly, for the particularly important case of $p = 1$, which is related to the Hartman–Wintner–Strassen law of the iterated logarithm, for all constants $0 \leq \lambda < \infty$, the following three statements are equivalent:

$$\sum_{n=3}^{\infty} \frac{1}{n} P\left(\frac{|S_n|}{\sqrt{2n\log\log n}} \geq \varepsilon\right) \begin{cases} < \infty, & \text{if } \varepsilon > \lambda, \\ = \infty, & \text{if } \varepsilon < \lambda. \end{cases} \tag{1.21}$$

$$\limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2n\log\log n}} = \lambda \ \text{ a.s.,} \tag{1.22}$$

$$E(X) = 0 \ \text{ and } \ E\left(X^2\right) = \lambda^2. \tag{1.23}$$

Hartman and Wintner [11] proved that (1.23) implies (1.22) and the converse is due to Strassen [12]. The implication "(1.23) $\Longrightarrow$ (1.21)" should be due to Davis ([13], Theorem 4) which was remedied by Li, Wang, and Rao ([14], Corollary 2.3). For the implication "(1.21) $\Longrightarrow$ (1.23)", see Gut ([15], Theorem 6.2).

Substantially simpler proofs of Strassen's [12] converse were discovered by Feller [16], Heyde [17], and Steiger and Zaremba [18]. Martikainen [19], Rosalsky [20], and Pruitt [21] simultaneously and independently obtained a "one-sided" converse to the Hartman–Wintner [11] law of the iterated logarithm. Specifically, they proved that, if

$$0 \leq \lim_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = \lambda < \infty \ \text{ a.s.,}$$

then (1.23) holds.

Let $v > 0$. Einmahl and Li ([10, Corollary 2]) proved that, for all constants $0 \leq \lambda < \infty$,

$$\limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2n(\log n)^v}} = \lambda \ \text{ a.s.,} \tag{1.24}$$

if and only if

$$\begin{cases} E(X)=0, \quad E\left(\dfrac{X^2}{(\log(e+|X|))^v}\right)<\infty, \\ \text{and} \quad \limsup_{x\to\infty} \dfrac{\log\log x}{(\log x)^v} E\left(X^2 I(|X|\le x)\right)=2^v\lambda^2. \end{cases} \tag{1.25}$$

Combining our Corollary 3 and Corollary 2 of Einmahl and Li [10], one can see that for all constants $0 \le \lambda < \infty$, (1.24) and (1.25) are each equivalent to

$$\sum_{n=2}^{\infty} \frac{1}{n} P\left(\frac{|S_n|}{\sqrt{2n(\log n)^v}} \ge \varepsilon\right) \begin{cases} <\infty, & \text{if } \varepsilon>\lambda, \\ =\infty, & \text{if } \varepsilon<\lambda. \end{cases}$$

A version of the equivalence between (1.21) and (1.22) in a Banach space setting was obtained by Li [22] who proved that there exists a constant $0 \le \lambda < \infty$ such that

$$\sum_{n=3}^{\infty} \frac{1}{n} P\left(\frac{\|S_n\|}{\sqrt{2n\log\log n}} \ge \varepsilon\right) < \infty \quad \text{for all} \quad \varepsilon > \lambda$$

if and only if

$$\limsup_{n\to\infty} \frac{\|S_n\|}{\sqrt{2n\log\log n}} < \infty \quad \text{a.s.} \tag{1.26}$$

Ledoux and Talagrand ([23, Theorem 1.1]) showed that (1.26) holds if and only if

$$\begin{cases} E(X)=0, \quad E\left(\dfrac{\|X\|^2}{\log\log(3+\|X\|)}\right)<\infty, \\ E\left(\phi^2(X)\right)<\infty \quad \text{for all} \quad \phi\in\mathbf{B}^*, \\ \text{and} \quad \left\{\dfrac{S_n}{\sqrt{2n\log\log n}};\ n\ge 3\right\} \quad \text{is bounded in probability.} \end{cases}$$

If

$$\frac{S_n}{\sqrt{2n\log\log n}} \to_P 0,$$

then, from our Corollary 3 and Theorem 5.1 of Ledoux and Talagrand [24], for all constants $0 \le \lambda < \infty$, the following three statements are equivalent:

$$\sum_{n=3}^{\infty} \frac{1}{n} P\left(\frac{\|S_n\|}{\sqrt{2n\log\log n}} \ge \varepsilon\right) \begin{cases} <\infty, & \text{if } \varepsilon>\lambda, \\ =\infty, & \text{if } \varepsilon<\lambda; \end{cases}$$

$$\lim_{n\to\infty} \frac{\|S_n\|}{\sqrt{2n\log\log n}} = \lambda \quad \text{a.s.};$$

$$\begin{cases} E(X)=0, \quad E\left(\dfrac{\|X\|^2}{\log\log(3+\|X\|)}\right)<\infty, \\ \text{and} \quad \sup\left\{E\left(\phi^2(X)\right);\ \phi\in\mathbf{B}^*, \|\phi\|\le 1\right\}=\lambda^2. \end{cases}$$

2 Proof of Theorem 1

The following lemmas will be used to prove Theorem 1:

Lemma 1 *Let $\{U_n;\ n \ge 1\}$ be a sequence of $\mathbf{B}$-valued random variables, let $\{U'_n;\ n \ge 1\}$ be an independent copy of $\{U_n;\ n \ge 1\}$. If there exists a constant $b > 0$ such that $\liminf_{n\to\infty} P(\|U_n\| \le b) > 0$, then we have*

(i) $\limsup_{n\to\infty} \|U_n\| < \infty$ *a.s. if and only if* $\limsup_{n\to\infty} \|U_n - U'_n\| < \infty$ *a.s.*;

(ii) *There exists a constant $0 \le b_1 < \infty$ such that*

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\|U_n\| \ge \varepsilon\right) < \infty \quad \textit{for all} \quad \varepsilon > b_1$$

if and only if there exists a constant $0 \le b_2 < \infty$ *such that*

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\|U_n - U_n'\| \ge \varepsilon\right) < \infty \quad \text{for all} \quad \varepsilon > b_2.$$

Proof Part (i) is just a special case of Theorem 3 of Li [25]. To prove Part (ii), let $q > 0$ be such that $P(\|U_n\| \le b) \ge q$ for all large n. Then since

$$\{\|U_n\| \le \varepsilon, \|U_n'\| \ge 2\varepsilon\} \subset \{\|U_n - U_n'\| \ge \varepsilon\}, \quad \varepsilon > 0, \ n \ge 1,$$

we have, for all large n and $\varepsilon \ge b$, that

$$P\left(\|U_n| \ge 2\varepsilon\right) \le (1/q) P\left(\|U_n - U_n'\| \ge \varepsilon\right) \le (2/q) P\left(\|U_n\| \ge \varepsilon/2\right).$$

Part (ii) follows immediately from this.

The following lemma is one of Lévy's inequalities in a Banach space setting; see, e.g., Araujo and Giné ([26, p. 102]) or Ledoux and Talagrand ([27, p. 47]).

Lemma 2 *Let* $\{V_i;\ 1 \le i \le n\}$ *be a finite sequence of independent symmetric* $\mathbf{B}$*-valued random variables, and set* $T_j = V_1 + \cdots + V_j,\ j = 1, \ldots, n$. *Then*

$$P\left(\max_{1 \le j \le n} \|T_j\| \ge t\right) \le 2P\left(\|T_n\| \ge t\right), \ t > 0.$$

The following lemma is due to Li, Zhang, and Rosalsky [7].

Lemma 3 *Let* $\{k_n;\ n \ge 1\}$ *be a sequence of integers such that* $2^{n-1} \le k_n < 2^n,\ n \ge 1$. *Then, for every integer* $n \ge 1$ *and each integer* $0 \le m < k_{n+1}$, *there exist* n *numbers* $w_i \in \{0, 1, 2, 3\},\ i = 1, 2, \ldots, n$ *depending only on* m *such that*

$$m = w_1 k_1 + w_2 k_2 + \cdots + w_n k_n.$$

Proof of Theorem 1 Set $I(n) = \{i;\ 2^{n-1} \le i < 2^n\},\ n \ge 1$. Let $\{X', X_n';\ n \ge 1\}$ be an independent copy of $\{X, X_n;\ n \ge 1\}$ and let $T_n = V_1 + \cdots + V_n,\ n \ge 1$ where $V_n = X_n - X_n',\ n \ge 1$.

We first prove that (1.8) implies (1.9). Note that the Kolmogorov zero-one law (see, e.g., Chow and Teicher ([28, Theorem 3.3]), (1.4), and (1.8) imply that there exists a constant $0 \le b_0 < \infty$ such that

$$\limsup_{n \to \infty} \frac{\left\|\sum_{i \in I(n)} V_i\right\|}{a_{2^n}} = b_0 \quad \text{a.s.}$$

Hence, by the Borel–Cantelli lemma and identical distributions, we have that

$$\sum_{n=1}^{\infty} P\left(\frac{\|T_{2^n}\|}{a_{2^n}} \ge \varepsilon\right) < \infty \quad \text{for all} \quad \varepsilon > b_0. \tag{2.1}$$

By (1.4), there exists a constant $1 < \tau < \infty$ such that

$$a_{2n} \le \tau a_n, \quad n \ge 1. \tag{2.2}$$

Now by (2.2) and Lemma 2, for $\varepsilon > 0$,

$$\begin{aligned} P\left(\frac{\|T_k\|}{a_k} \ge \varepsilon\right) &\le P\left(\max_{1 \le j \le 2^n} \frac{\|T_j\|}{a_{2^{n-1}}} \ge \varepsilon\right) \\ &\le 2P\left(\frac{\|T_{2^n}\|}{a_{2^{n-1}}} \ge \varepsilon\right) \\ &\le 2P\left(\frac{\|T_{2^n}\|}{a_{2^n}} \ge \varepsilon/\tau\right), \quad k \in I(n). \end{aligned}$$

Let $\lambda = \tau b_0$. Then $0 \le \lambda < \infty$ and, on account of (2.1),

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n} P\left(\frac{\|T_n\|}{a_n} \ge \varepsilon\right) &= \sum_{n=1}^{\infty} \sum_{k \in I_n} \frac{1}{k} P\left(\frac{\|T_k\|}{a_k} \ge \varepsilon\right) \\ &\le 2 \sum_{n=1}^{\infty} P\left(\frac{\|T_{2^n}\|}{a_{2^n}} \ge \varepsilon/\tau\right) \\ &< \infty \quad \text{for all } \varepsilon > \lambda.\end{aligned}$$

Thus, by Lemma 1 (ii), (1.9) follows.

We now show that (1.9) implies (1.8). Note that, for all $\varepsilon > \lambda$,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n} P\left(\frac{\|S_n\|}{a_n} \ge \varepsilon\right) &\ge \sum_{n=1}^{\infty} \sum_{k \in I_n} \frac{1}{k} \min_{j \in I_n} P\left(\frac{\|S_j\|}{a_j} \ge \varepsilon\right) \\ &\ge \frac{1}{2} \sum_{n=1}^{\infty} \min_{j \in I_n} P\left(\frac{\|S_j\|}{a_j} \ge \varepsilon\right).\end{aligned}$$

Hence, for fixed $\varepsilon > \lambda$, (1.9) implies that there exists a sequence $\{k_n;\ n \ge 1\}$ of integers depending only on $\varepsilon > \lambda$ and the distribution of X such that $2^{n-1} \le k_n < 2^n,\ n \ge 1$ and

$$\sum_{n=1}^{\infty} P\left(\frac{\|S_{k_n}\|}{a_{k_n}} \ge \varepsilon\right) < \infty. \tag{2.3}$$

It is easy to see that (1.4) implies

$$c \triangleq \sup_{n \ge 1} \sum_{k=0}^{n+1} \frac{a_{2^k}}{a_{2^n}} < \infty. \tag{2.4}$$

For fixed integers $n \ge 1$ and m with $2^{n-1} \le m < 2^n (\le k_{n+1})$, by Lemma 3, there exist n numbers $w_i \in \{0, 1, 2, 3\},\ i = 1, 2, \ldots, n$, depending only on m such that $m = \sum_{i=1}^{n} w_i k_i$. Write

$$l_1 = w_n k_n,\ l_2 = w_n k_n + w_{n-1} k_{n-1}, \ldots,\ l_n = w_n k_n + w_{n-1} k_{n-1} + \ldots + w_1 k_1 = m.$$

Then $l_1 \le l_2 \le \cdots \le l_n$ and $l_i - l_{i-1} = w_{n-i+1} k_{n-i+1},\ i = 1, 2, \ldots, n$, where $l_0 = 0$. Then

$$S_m = \sum_{i=1}^{n} Y_i,$$

where

$$Y_i = \sum_{l_{i-1} < j \le l_i} X_j,\ i = 1, 2, \ldots, n.$$

Note that

$$\begin{aligned}\frac{\|S_m\|}{a_m} &\le \sum_{i=1}^{n-n_0} \frac{\|Y_i\|}{a_m} + \frac{\|\sum_{i=n-n_0+1}^{n} Y_i\|}{a_m} \\ &\le \sum_{i=1}^{n-n_0} \left(\frac{a_{2^{n-i+1}}}{a_{2^{n-1}}}\right) \frac{\|Y_i\|}{a_{k_{n-i+1}}} + \frac{\|\sum_{i=n-n_0+1}^{n} Y_i\|}{a_{2^{n-1}}},\end{aligned} \tag{2.5}$$

where $1 \le n_0 < n$. Since

$$\sum_{i=n-n_0+1}^{n} w_{n-i+1} k_{n-i+1} \le 6 \times 2^{n_0},$$

and Y_i and $S_{w_{n-i+1} k_{n-i+1}}$ have the same distribution, $i = 1, 2, \ldots, n$, in view of (2.5) and (2.4), we get

$$P\left(\frac{\|S_m\|}{a_m} \ge 6c\varepsilon\right) \le P\left(\sum_{i=1}^{n-n_0} \left(\frac{a_{2^{n-i+1}}}{a_{2^{n-1}}}\right) \frac{\|Y_i\|}{a_{k_{n-i+1}}} \ge 3c\varepsilon\right) + P\left(\frac{\|\sum_{i=n-n_0+1}^{n} Y_i\|}{a_{2^{n-1}}} \ge 3c\varepsilon\right)$$

$$\le \sum_{i=1}^{n-n_0} P\left(\frac{\|Y_i\|}{a_{k_{n-i+1}}} \ge 3\varepsilon\right) + \max_{1\le j\le 6\times 2^{n_0}} P\left(\frac{\|S_j\|}{a_{2^{n-1}}} \ge 3c\varepsilon\right)$$

$$\le 3\sum_{i=n_0+1}^{n} P\left(\frac{\|S_{k_i}\|}{a_{k_i}} \ge \varepsilon\right) + \max_{1\le j\le 6\times 2^{n_0}} P\left(\frac{\|S_j\|}{a_{2^{n-1}}} \ge 3c\varepsilon\right).$$

Hence

$$\max_{2^{n-1}\le m<2^n} P\left(\frac{\|S_m\|}{a_m} \ge 6c\varepsilon\right) \le 3\sum_{i=n_0+1}^{n} P\left(\frac{\|S_{k_i}\|}{a_{k_i}} \ge \varepsilon\right) + \max_{1\le j\le 6\times 2^{n_0}} P\left(\frac{\|S_j\|}{a_{2^{n-1}}} \ge 3c\varepsilon\right).$$

Thus, recalling (2.3), we conclude that

$$\begin{aligned}\limsup_{n\to\infty} \max_{2^{n-1}\le m<2^n} P\left(\frac{\|S_m\|}{a_m} \ge 6c\varepsilon\right) &\le 3\sum_{i=n_0+1}^{\infty} P\left(\frac{\|S_{k_i}\|}{a_{k_i}} \ge \varepsilon\right)\\ &\longrightarrow 0 \ \text{ as } \ n_0\to\infty \ \text{ for all } \ \varepsilon>\lambda.\end{aligned} \tag{2.6}$$

Let $S'_n = \sum_{i=1}^n X'_i,\ n\ge 1$. Then $\{S'_n;\ n\ge 1\}$ is an independent copy of $\{S_n;\ n\ge 1\}$. Clearly, (1.9) implies that

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\frac{\|S_n - S'_n\|}{a_n} \ge \varepsilon\right) < \infty \ \text{ for all } \ \varepsilon > 2\lambda. \tag{2.7}$$

By applying Lemma 2, for all $\varepsilon > 2\tau^2\lambda$ and for all $k\in I(n+1),\ n\ge 1$, we have

$$\begin{aligned}P\left(\max_{j\in I(n)} \frac{\|S_j - S'_j\|}{a_j} \ge \varepsilon\right) &\le P\left(\max_{j\in I(n)} \|S_j - S'_j\| \ge \varepsilon a_{2^{n-1}}\right)\\ &\le P\left(\max_{1\le j\le k} \|S_j - S'_j\| \ge \varepsilon a_{2^{n-1}}\right)\\ &\le 2P\left(\|S_k - S'_k\| \ge \varepsilon a_{2^{n-1}}\right)\\ &\le 2P\left(\|S_k - S'_k\| \ge (\varepsilon/\tau^2) a_k\right) \quad \text{(by (2.2))},\end{aligned}$$

and this, together with (2.7), ensures that

$$\sum_{n=1}^{\infty} P\left(\max_{j\in I(n)} \frac{\|S_j - S'_j\|}{a_j} \ge \varepsilon\right) < \infty \ \text{ for all } \ \varepsilon > 2\tau^2\lambda.$$

Hence, by the Borel–Cantelli lemma,

$$\limsup_{n\to\infty} \frac{\|S_n - S'_n\|}{a_n} < \infty \ \text{ a.s.} \tag{2.8}$$

Thus, in view of Lemma 1 (i), (1.8) follows from (2.8) and (2.6). The proof of Theorem 1 is therefore complete.

3 Proofs of Theorems 3 and 4

For the proofs of Theorems 3 and 4 we need the following three lemmas. The first lemma, i.e., Lemma 4, is due to Petrov [29] (see Petrov ([30, Theorem 2.3]).

Lemma 4 *Let $\{V_i;\ 1\le i\le n\}$ be a finite sequence of independent real-valued random variables, and set $T_j = V_1 + \cdots + V_j,\ j=1,\ldots,n$. If*

$$\min_{1\le j\le n-1} P\left(T_n - T_j \ge -b\right) \ge q,$$

for some constants $b\ge 0$ and $q>0$, then

$$P\left(\max_{1\le j\le n} T_j \ge t\right) \le (1/q) P\left(T_n \ge t-b\right) \quad \text{for all real } t. \tag{3.1}$$

The following lemma is a version of Lemma 4 in a Banach space setting.

Lemma 5 *Let $\{V_i;\ 1 \le i \le n\}$ be a finite sequence of independent* $\mathbf{B}$*-valued random variables, and set $T_j = V_1 + \cdots + V_j,\ j = 1, \ldots, n$. If*

$$\min_{1 \le j \le n-1} P\left(\|T_n - T_j\| \le b\right) \ge q, \tag{3.2}$$

for some constants $b \ge 0$ and $q > 0$, then

$$P\left(\max_{1 \le j \le n} \|T_j\| \ge t\right) \le (1/q) P\left(\|T_n\| \ge t - b\right) \text{ for all } t \ge 0. \tag{3.3}$$

Proof Our proof of (3.3) is a modification of Petrov's [29] proof of (3.1). Let $\kappa_q(Y)$ denote a quantile of order $q,\ 0 < q < 1$, for a real-valued random variable Y. We first show that

$$P\left(\max_{1 \le j \le n} \left(\|T_j\| - \kappa_q(\|T_n - T_j\|)\right) \ge t\right) \le (1/q) P\left(\|T_n\| \ge t\right) \quad \text{for all } t \ge 0. \tag{3.4}$$

We write

$$\begin{aligned}
M_j &= \max_{1 \le k \le j} \left(\|T_k\| - \kappa_q(\|T_n - T_k\|)\right), \quad j = 1, 2, \ldots, n, \\
D_1 &= \left\{\|T_1\| - \kappa_q(\|T_n - T_1\|) \ge t\right\}, \\
D_j &= \left\{M_{j-1} < t, \|T_j\| - \kappa_q(\|T_n - T_j\|) \ge t\right\}, \quad j = 2, \ldots, n, \\
E_j &= \left\{\|T_n - T_j\| - \kappa_q(\|T_n - T_j\|) \le 0\right\}, \quad j = 1, 2, \ldots, n.
\end{aligned}$$

Then we have

$$P\left(M_n \ge t\right) = \sum_{j=1}^{n} P\left(D_j\right) \tag{3.5}$$

since

$$\{M_n \ge t\} = \bigcup_{j=1}^{n} D_j \quad \text{and} \quad P\left(D_k \cap D_j\right) = 0 \quad \text{for} \quad k \ne j.$$

Furthermore

$$P\left(E_j\right) \ge q, \quad j = 1, \ldots, n. \tag{3.6}$$

Note that

$$\bigcup_{j=1}^{n} \left(D_j \cap E_j\right) \subset \left\{\|T_n\| \ge t\right\}$$

and

$$P\left(\|T_n\| \ge t\right) \ge P\left(\bigcup_{j=1}^{n} \left(D_j \cap E_j\right)\right) = \sum_{j=1}^{n} P\left(D_j \cap E_j\right) = \sum_{j=1}^{n} P\left(D_j\right) P\left(E_j\right),$$

since the events D_j and E_j are independent. Taking into account (3.5) and (3.6) we conclude that

$$P\left(\|T_n\| \ge t\right) \ge q \sum_{j=1}^{n} P\left(D_j\right) = q P\left(M_n \ge t\right),$$

thus proving (3.4). By (3.2), there exists a set of quantiles $\kappa_q\left(\|T_n - T_j\|\right),\ 1 \le j \le n-1$, such that

$$\kappa_q\left(\|T_n - T_j\|\right) \le b, \quad j = 1, \ldots, n-1$$

and hence, for every $t \ge 0$,

$$\left\{\max_{1 \le j \le n} \|T_j\| \ge t\right\} \subset \left\{\max_{1 \le j \le n} \left(\|T_j\| - \kappa_q(\|T_n - T_j\|)\right) \ge t - b\right\}.$$

Thus, from (3.4), (3.3) follows. The lemma is proved.

The following lemma, which is an extension of the divergence half of the Borel–Cantelli lemma, is due to Baum, Katz, and Stratton [31]. The formulation presented here is that of Petrov ([30, Lemma 7.5]).

Lemma 6 *Let $\{B_n;\ n \geq 1\}$ be a sequence of events such that $P(B_n) \geq \alpha$ for all large n, where α is a positive constant. If the following pairs of events are independent for every n : A_n and B_n, A_n and $B_n \cap \overline{A_{n-1} \cap B_{n-1}}$, A_n and $B_n \cap \overline{A_{n-1} \cap B_{n-1}} \cap \overline{A_{n-2} \cap B_{n-2}}, \ldots$, (here $\bar{A}$ is the complement of A) and if $\sum_{n=1}^{\infty} P(A_n) = \infty$, then $P\left(A_n \cap B_n \text{ i.o.}\right) \geq \alpha$.*

Proof of Theorem 3 Obviously, we need to show only that, for an arbitrary constant $0 \leq \lambda < \infty$,

$$\limsup_{n\to\infty} \frac{\|S_n\|}{a_n} \leq \lambda \ \text{ a.s.}, \tag{3.7}$$

if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\frac{\|S_n\|}{a_n} \geq \varepsilon\right) < \infty \ \text{ for all } \ \varepsilon > \lambda. \tag{3.8}$$

We first prove that (3.8) implies (3.7). To see this, let $\varepsilon > \lambda \geq 0$ and $\eta > 0$ be arbitrary. By (1.10), there exists a constant $\beta_0 > 1$ such that, for every $\beta \in (1, \beta_0)$,

$$a_{[\beta n]} \leq (1+\eta) a_n \text{ for all sufficiently large } n. \tag{3.9}$$

Hence, for all large n,

$$P\left(\max_{\beta^{n-1} < m \leq \beta^n} \frac{\|S_m\|}{a_m} \geq (1+3\eta)^2 \varepsilon\right) \leq P\left(\max_{\beta^{n-1} < m \leq \beta^n} \frac{\|S_m\|}{a_{[\beta^n]}} \geq (1+3\eta)\varepsilon\right).$$

Note that (1.11) ensures that there exists a constant $q_1 > 0$ depending on $\eta\varepsilon$ such that, for all sufficiently large n,

$$\min_{0 \leq k \leq n} P\left(\frac{\|S_n - S_k\|}{a_n} \leq \eta\varepsilon\right) = \min_{1 \leq k \leq n} P\left(\frac{\|S_k\|}{a_n} \leq \eta\varepsilon\right) \geq q_1;$$

here and below $S_0 = 0$. It then follows from Lemma 5 that, for all sufficiently large n,

$$P\left(\max_{\beta^{n-1} < m \leq \beta^n} \frac{\|S_m\|}{a_m} \geq (1+3\eta)^2 \varepsilon\right) \leq (1/q_1) P\left(\frac{\|S_{[\beta^n]}\|}{a_{[\beta^n]}} \geq (1+2\eta)\varepsilon\right). \tag{3.10}$$

Again recalling (1.11), for all large n and $m \in [[\beta^n], [\beta^{n+1}]-1]$, another application of Lemma 5 yields

$$\begin{aligned} & P\left(\frac{\|S_{[\beta^n]}\|}{a_{[\beta^n]}} \geq (1+2\eta)\varepsilon\right) \\ & \leq P\left(\max_{[\beta^n] \leq j \leq m} \frac{\|S_j\|}{a_{[\beta^n]}} \geq (1+2\eta)\varepsilon\right) \\ & \leq (1/q_1) P\left(\frac{\|S_m\|}{a_{[\beta^n]}} \geq (1+\eta)\varepsilon\right) \\ & \leq (1/q_1) P\left(\frac{(1+\eta)\|S_m\|}{a_{[\beta^{n+1}]}} \geq (1+\eta)\varepsilon\right) \ \text{(by (3.9))} \\ & \leq (1/q_1) P\left(\frac{\|S_m\|}{a_m} \geq \varepsilon\right). \end{aligned} \tag{3.11}$$

Since $[\beta^{n+1}] - [\beta^n] \sim \frac{\beta-1}{\beta}\left([\beta^{n+1}] - 1\right)$, it follows from (3.10) and (3.11) that, for all large n,

$$\begin{aligned} & P\left(\max_{\beta^{n-1} < m \leq \beta^n} \frac{\|S_m\|}{a_m} \geq (1+3\eta)^2 \varepsilon\right) \\ & \leq \frac{1}{q_1^2} \frac{\sum_{m=[\beta^n]}^{[\beta^{n+1}]-1} P\left(\frac{\|S_m\|}{a_m} \geq \varepsilon\right)}{[\beta^{n+1}] - [\beta^n]} \\ & \leq \frac{2\beta}{q_1^2(\beta-1)} \frac{\sum_{m=[\beta^n]}^{[\beta^{n+1}]-1} P\left(\frac{\|S_m\|}{a_m} \geq \varepsilon\right)}{[\beta^{n+1}] - 1} \\ & \leq \frac{2\beta}{q_1^2(\beta-1)} \sum_{m=[\beta^n]}^{[\beta^{n+1}]-1} \frac{1}{m} P\left(\frac{\|S_m\|}{a_m} \geq \varepsilon\right). \end{aligned}$$

Then, by (3.8) and the Borel–Cantelli lemma,

$$P\left(\max_{\beta^{n-1}<m\leq\beta^{n}}\frac{\|S_m\|}{a_m}\geq(1+3\eta)^2\varepsilon \text{ i.o.}\right)=0$$

whence

$$\limsup_{n\to\infty}\frac{\|S_n\|}{a_n}\leq(1+3\eta)^2\varepsilon \quad \text{a.s.}$$

Letting $\eta\downarrow 0$ and $\varepsilon\downarrow\lambda$, (3.7) follows.

We now prove that (3.7) implies (3.8). The authors take great pleasure in acknowledging that the proof of this implication was inspired by Petrov ([30, Theorem 7.5]). Let $0\leq\lambda<\infty$ and suppose that (3.7) holds. Then

$$P\left(\frac{\|S_n\|}{a_n}\geq\varepsilon \text{ i.o.}\right)=0 \quad \text{for all } \varepsilon>\lambda. \tag{3.12}$$

Let $\varepsilon>\lambda$ be arbitrary and let $\varepsilon_1\in(\lambda,\varepsilon)$. For an arbitrary nondecreasing sequence of positive integers $k_n\to\infty$ and arbitrary $\eta>0$, consider the events

$$A_n=\left\{\frac{\|S_{k_n}-S_{k_{n-1}}\|}{a_{k_n}}\geq\varepsilon_1+\eta\right\},\quad B_n=\left\{\frac{\|S_{k_{n-1}}\|}{a_{k_n}}\leq\eta\right\},\quad n\geq 1,$$

where $k_0=0$. Obviously, by (3.12),

$$P\left(A_n\cap B_n \;\text{ i.o.}\right)\leq P\left(\frac{\|S_n\|}{a_n}\geq\varepsilon_1 \text{ i.o.}\right)=0.$$

Therefore, by (1.11) and Lemma 6, we conclude that

$$\sum_{n=1}^{\infty}P\left(\frac{\|S_{k_n}-S_{k_{n-1}}\|}{a_{k_n}}\geq\varepsilon_1+\eta\right)=\sum_{n=1}^{\infty}P\left(A_n\right)<\infty.$$

Thus, for every $\beta>1$ and every integer $r\geq 1$, putting $k_n=[\beta^{rn+i}],\ n\geq 1$ for $i=0,1,\ldots,r-1$, we have, for all $i=0,1,\ldots,r-1$, that

$$\sum_{n=1}^{\infty}P\left(\frac{\|S_{[\beta^{rn+i}]}-S_{[\beta^{rn+i-r}]}\|}{a_{[\beta^{rn+i}]}}\geq\varepsilon_1+\eta\right)<\infty$$

and hence

$$\begin{aligned}
&\sum_{n=1}^{\infty}P\left(\frac{\|S_{[\beta^n]-[\beta^{n-r}]}\|}{a_{[\beta^n]}}\geq\varepsilon_1+\eta\right)\\
&=\sum_{n=1}^{\infty}P\left(\frac{\|S_{[\beta^n]}-S_{[\beta^{n-r}]}\|}{a_{[\beta^n]}}\geq\varepsilon_1+\eta\right)\\
&=\sum_{i=0}^{r-1}\sum_{n=1}^{\infty}P\left(\frac{\|S_{[\beta^{rn+i}]}-S_{[\beta^{rn+i-r}]}\|}{a_{[\beta^{rn+i}]}}\geq\varepsilon_1+\eta\right)\\
&<\infty.
\end{aligned} \tag{3.13}$$

We first choose $\eta>0$ such that $\varepsilon_1+2\eta<\varepsilon/(1+2\eta)$. By (1.10), secondly we choose $\beta_0>1$ such that (3.9) holds for every $\beta\in(1,\beta_0)$. We then choose a $\beta\in(1,\beta_0)$ and a positive integer r such that $\beta/\left(1-\beta^{-r}\right)<\beta_0$. Let $j_n=[\beta^n]-[\beta^{n-r}],\ n\geq r$. Note that

$$\lim_{n\to\infty}\frac{[\beta^n]}{[\beta^{n-1}]-[\beta^{n-1-r}]}=\frac{\beta}{1-\beta^{-r}}.$$

It then follows from (3.9) that, for all large n and every $m\in(j_{n-1},j_n]$,

$$P\left(\frac{\|S_m\|}{a_m}\geq\varepsilon\right)\leq P\left(\frac{\|S_m\|}{a_{[\beta^n]}}\geq\frac{\varepsilon}{1+2\eta}\right)\leq P\left(\frac{\|S_m\|}{a_{[\beta^n]}}\geq\varepsilon_1+2\eta\right).$$

Now it is easy to see that (1.11) and Lemma 5 ensure that there exists a constant $q_2>0$ depending on η such that, for all large n and every $m\in(j_{n-1},j_n]$,

$$P\left(\frac{\|S_m\|}{a_{[\beta^n]}}\geq\varepsilon_1+2\eta\right)\leq(1/q_2)P\left(\frac{\|S_{j_n}\|}{a_{[\beta^n]}}\geq\varepsilon_1+\eta\right).$$

Thus, for all large n and every $m \in (j_{n-1}, j_n]$,

$$P\left(\frac{\|S_m\|}{a_m} \geq \varepsilon\right) \leq (1/q_2) P\left(\frac{\|S_{j_n}\|}{a_{[\beta^n]}} \geq \varepsilon_1 + \eta\right). \tag{3.14}$$

Since $\lim_{n\to\infty} j_n/j_{n-1} = \beta$, it follows from (3.14) and (3.13) that, for some sufficiently large n_0,

$$\begin{aligned}\sum_{m=j_{n_0-1}+1}^{\infty} \frac{1}{m} P\left(\frac{\|S_m\|}{a_m} \geq \varepsilon\right) &\leq (1/q_2) \sum_{n=n_0}^{\infty} \sum_{m=j_{n-1}+1}^{j_n} \frac{1}{m} P\left(\frac{\|S_{j_n}\|}{a_{[\beta^n]}} \geq \varepsilon_1 + \eta\right) \\ &\leq (\beta/q_2) \sum_{n=n_0}^{\infty} P\left(\frac{\|S_{j_n}\|}{a_{[\beta^n]}} \geq \varepsilon_1 + \eta\right) \\ &< \infty.\end{aligned}$$

Since $\varepsilon > \lambda$ is arbitrary, (3.8) follows. This completes the proof of Theorem 3.

Proof of Theorem 4 Using Lemma 4 and the same argument as in the proof of Theorem 3 with some obvious modifications, for example, replacing

$$A_n = \left\{\frac{\|S_{k_n} - S_{k_{n-1}}\|}{a_{k_n}} \geq \varepsilon_1 + \eta\right\}, \quad B_n = \left\{\frac{\|S_{k_{n-1}}\|}{a_{k_n}} \leq \eta\right\}, \quad n \geq 1$$

by

$$A_n = \left\{\frac{S_{k_n} - S_{k_{n-1}}}{a_{k_n}} \geq \varepsilon_1 + \eta\right\}, \quad B_n = \left\{\frac{S_{k_{n-1}}}{a_{k_n}} \geq -\eta\right\}, \quad n \geq 1,$$

the conclusion of Theorem 4 follows.

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