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On Complete Convergence of Moving Average Processes for NSD Sequences*

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Abstract—We study the complete convergence of moving-average processes based on an identically distributed doubly infinite sequence of negative superaddative-dependent random variables. As a corollary, the Marcinkiewicz-Zygmund strong law of large numbers is obtained.

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1. INTRODUCTION

We assume that $\{Y_i;\, i = 0, \pm1, \pm2, \ldots\}$ is a doubly infinite sequence of identically distributed random variables with $\mathbb{E}|Y_1| < \infty$. Let $\{a_i;\, i = 0, \pm1, \pm2, \ldots\}$ be an absolutely summable sequence of real numbers and let

$$X_n = \sum_{i=-\infty}^{\infty} a_i Y_{n+i}, \quad n \geq 1, \tag{1}$$

be the moving average process based on the sequence $\{Y_i;\, i = 0, \pm1, \pm2, \ldots\}$. As usual,

$$S_n = \sum_{i=1}^{n} X_i, \quad n \geq 1,$$

denotes the sequence of partial sums.

Under the assumption that $\{Y_i;\, i = 0, \pm1, \pm2, \ldots\}$ is a sequence of independent identically distributed random variables, there are some authors who have studied limit properties for the moving average process $\{X_n;\, n \geq 1\}$. In particular, Ibragimov [13] established the central limit theorem, Burton and Dehling [5] obtained a large deviation principle, and Li et al. [16] gave the complete convergence result for $\{X_n;\, n \geq 1\}$.

Many authors extended the complete convergence of moving average processes in the case of dependent sequences, for example, Zhang [22] for φ-mixing sequences, Liang et al. [15] for NA sequences, Li and Zhang [17] for NA sequences, Chen et al. [6] for φ-mixing sequences, Amini et al. [2] for negative dependent sub-Gaussian sequences, and Yang et al. [21] for AANA Sequences.

The concept of negatively associated (NA) random variables was introduced by Alam and Saxena [1] and carefully studied by Joag-Dev and Proschan [14]. As pointed out and proved in [14], a number of well-known multivariate distribution possess the NA property. Negative association has found important and wide applications in multivariate statistical analysis and reliability. Many investigators discuss applications of NA to Probability, Stochastic Processes, and Statistics.

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Definition 1.1. *Random variables $\{X_i, 1 \le i \le n\}$ are said to be NA if, for every pair of disjoint subsets A_1 and A_2 of $\{1, 2, \dots, n\}$,*

$$Cov(f_1(x_i; i \in A_1), f_2(X_j; j \in A_2)) \le 0,$$

where f_1 and f_2 are increasing in any variable (or decreasing in any variable) measurable functions such that this covariance exists. An infinite family of random variables is NA if every finite subfamily is NA.

The next dependence notion is negatively superadditive dependence which is weaker than NA. The concept of negatively superadditive-dependent (NSD) random variables was introduced by Hu [12] as follows.

Definition 1.2. (Kemperman [15]). *A function $\phi : \mathcal{R}^n \to \mathcal{R}$ is called superadditive if $\phi(\boldsymbol{x} \vee \boldsymbol{y}) + \phi(\boldsymbol{x} \wedge \boldsymbol{y}) \ge \phi(\boldsymbol{x}) + \phi(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{R}^n$, where $\vee$ is for componentwise maximum and $\wedge$ is for componentwise minimum.*

Definition 1.3. (Hu [12]). *A random vector $\boldsymbol{X} = (X_1, X_2, \dots, X_n)$ is said to be NSD if*

$$\mathbb{E}\phi(X_1, X_2, \dots, X_n) \le \mathbb{E}\phi(X_1^*, X_2^*, \dots, X_n^*),$$

where $X_1^*, X_2^*, \dots, X_n^*$ are independent such that X_i^* and X_i have the same distribution for each i, and ϕ is a superadditive function such that the expectations above exist.

A sequence $\{X_n, n \ge 1\}$ of random variables is said to be NSD if, for all $n \ge 1$, the random vector $(X_1, X_2, \dots, X_n)$ is NSD .

Hu [12] gave an example illustrating that NSD does not imply NA and posed an open problem whether NA implies NSD. In addition, he provided some basic properties and three structural theorems of NSD. Christofides and Vaggelatou [7] solved this open problem and indicated that NA implies NSD. The NSD structure is an extension of NA structure and sometimes more useful than that and can be used to get many important probability inequalities. Eghbal et al. [8] derived two maximal inequalities and strong law of large numbers of quadratic forms of NSD random variables, and Eghbal et al. [9] provided some Kolmogorov inequalities for quadratic forms of nonnegative NSD uniformly bounded random variables.

This paper is organized as follows. In Section 2, some preliminary lemmas and inequalities for NSD random variables are provided. In Section 3, complete convergence for moving-average processes of identically distributed doubly infinite sequence and the Marcinkiewicz–Zygmund strong law of large numbers for NSD random variables are presented. These results improve the corresponding results of Amini et al. [2] and Sadeghi and Bozorgnia [19].

Throughout the paper, C denotes a positive constant which may be different in various places. The notation $a_n \ll b_n$ ($a_n \gg b_n$) means that there exists a constant $C > 0$ such that $a_n \le Cb_n$ ($a_n \ge Cb_n$), and $a_n = O(b_n)$ denote that there exists a constant $C > 0$ such that $a_n \le Cb_n$.

2. PRELIMINARY

We provide some preliminary facts needed to prove our main results. The first two lemmas were proved by Hu [12] and Wang et al. [20], respectively.

Lemma 2.1. *If $(X_1, X_2, \dots, X_n)$ is NSD and $g_1, g_2, \dots, g_n$ are nondecreasing functions then $g_1(X_1), g_2(X_2), \dots, g_n(X_n)$ are NSD.*

Lemma 2.2. (Rosenthal − type maximal inequality) *Let $\{X_n;\ n \ge 1\}$ be a sequence of NSD random variables, with $\mathbb{E}|X_i| < \infty$ for each $i \ge 1$, and let $\{X_n^*;\ n \ge 1\}$ be a sequence of independent random variables such that X_i^* and X_i have the same distribution for each $i \ge 1$. Then, for all $n \ge 1$ and $p \ge 1$,*

$$\mathbb{E}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} X_i\right|^p\right) \le \mathbb{E}\left|\sum_{i=1}^{k} X_i^*\right|^p, \tag{2}$$

$$\mathbb{E}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} X_i\right|^p\right) \le 2^{3-p}\sum_{i=1}^{n}\mathbb{E}|X_i|^p \quad \mathit{for}\ 1 < p \le 2, \tag{3}$$

and

$$\mathbb{E}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}X_i\right|^p\right)\le 2\left(\frac{15p}{\ln p}\right)^p\left[\sum_{i=1}^{n}\mathbb{E}|X_i|^p+\left(\sum_{i=1}^{n}\mathbb{E}X_i^2\right)^{p/2}\right]\quad for\ p>2. \tag{4}$$

Definition 2.3. *A real-valued function $l(x)$, positive and measurable on $[A,\infty)$ for some $A>0$, is said to be slowly varying if $\lim_{x\to\infty}(l(\lambda x)/l(x))=1$ for each $\lambda>0$.*

We have the following facts for slowly varying functions.

Lemma 2.4. (Zhidong and Chun [23]). *Let $l(x)>0$ be a slowly varying function as $x\to\infty$. Then we have*

(i) $\lim_{k\to\infty}\sup_{2^k\le x\le 2^{k+1}}\dfrac{l(x)}{l(2^k)}=1$;

(ii) $C2^{kr}l(\eta 2^k)\le\sum_{i=1}^{k}2^{ir}l(\eta 2^i)\le C2^{kr}l(\eta 2^k)$ *for every positive r, η, and integer k;*

(iii) $C2^{kr}l(\eta 2^k)\le\sum_{i=k}^{\infty}2^{ir}l(\eta 2^i)\le C2^{kr}l(\eta 2^k)$ *for every $r<0$, $\eta>0$, and integer k;*

(iv) $\lim_{x\to\infty}x^{\delta}l(x)=\infty$, $\lim_{x\to\infty}x^{-\delta}l(x)=0$ *for every $\delta>0$.*

3. COMPLETE CONVERGENCE

The concept of complete convergence was introduced by Hsu and Robbins [11]. A sequence of random variables $\{X_n;\ n\ge 1\}$ is said to converge completely to a constant θ if

$$\sum_{i=1}^{\infty}\mathbb{P}(|X_n-\theta|>\varepsilon)<\infty\ \text{ for all }\ \varepsilon>0.$$

Hsu and Robbins [11] proved that the sequence of arithmetic means of i.i.d. random variables converges completely to the expected value if the variance of the summands is finite. Erdös [10] proved the converse. The result of Hsu-Robbins-Erdös is a fundamental theorem in probability theory and has been generalized and extended in several directions by many authors.

Now we state the main results of this article.

Theorem 3.1. *Let $l(x)>0$ be a slowly varying function at infinity, $1<p\le 2$, $0<\alpha<1$, and $rp>1$. Suppose that $\{X_n, n\ge 1\}$ is a moving average process based on a sequence $\{Y_i, i=0,\pm1,\pm2,\dots\}$ of identically distributed centered NSD random variables. If $\mathbb{E}|Y_1|^p l(|Y_1|^{1/\alpha})<\infty$ then, for all $\varepsilon>0$,*

$$\sum_{n=1}^{\infty}n^{\alpha p-2}l(n)\mathbb{P}\left\{\max_{1\le k\le n}|S_k|\ge\varepsilon n^{\alpha}\right\}<\infty \tag{5}$$

and

$$\sum_{n=1}^{\infty}n^{\alpha p-2}l(n)\mathbb{P}\left\{\sup_{k\ge n}|S_k/k^{\alpha}|\ge\varepsilon\right\}<\infty. \tag{6}$$

Proof. Recall that

$$\sum_{k=1}^{n}X_k=\sum_{k=1}^{n}\sum_{i=-\infty}^{\infty}a_iY_{i+k}=\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+n}Y_j,\ n\ge 1,\ \text{and}\ \sum_{i=-m}^{m}|a_i|=O(1)\ \text{as}\ m\to\infty.$$

Without loss of generality, we assume that $a_i > 0$ for all $i \geq 1$ (otherwise, we use a_i^+ and a_i^- instead of a_i, respectively, and note that $a_i = a_i^+ - a_i^-$). Denote

$$Y_i^{(n)} = -n^\alpha I(Y_i < -n^\alpha) + Y_i I(|Y_i| \leq n^\alpha) + n^\alpha I(Y_i > n^\alpha), \;\; n \geq 1.$$

So, for $n \geq 1$,

$$Y_i \leq n^\alpha I(|Y_i| > n^\alpha) + Y_i I(|Y_i| > n^\alpha) + Y_i^{(n)}.$$

Then, for any $\varepsilon > 0$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \mathbb{P}\left\{\max_{1\leq k\leq n} |S_k| > n^\alpha \varepsilon\right\} \\
\leq\;& \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \mathbb{P}\left\{\max_{1\leq k\leq n} \left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} [n^\alpha I(|Y_j| > n^\alpha) + Y_j I(|Y_j| > n^\alpha)]\right| > n^\alpha \varepsilon/2\right\} \\
+\;& \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \mathbb{P}\left\{\max_{1\leq k\leq n} \left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_j^{(n)}\right| > n^\alpha \varepsilon/2\right\} \\
=:\;& I_1 + I_2. \qquad (7)
\end{aligned}$$

Hence, in order to prove (5), it suffices to establish that $I_1 < \infty$ and $I_2 < \infty$. Applying Lemma 2.4 (iv) and taking the restriction $\mathbb{E}|Y_1|^p l(|Y_1|^{1/\alpha}) < \infty$ into account, we conclude that

$$\mathbb{E}|Y_1|^{p-\gamma} < \infty \text{ for any } \gamma > 0. \qquad (8)$$

Since $p > 1$ then, by Markov's inequality, we can obtain

$$\begin{aligned}
I_1 \;&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha-2} l(n) \mathbb{E}\left(\max_{1\leq k\leq n} \left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} n^\alpha I(|Y_j| > n^\alpha) + Y_j I(|Y_j| > n^\alpha)\right|\right) \\
&\leq \sum_{n=1}^{\infty} n^{\alpha p-\alpha-2} l(n) \sum_{i=-\infty}^{\infty} a_i \left\{n^{\alpha+1}\mathbb{P}(|Y_1| > n^\alpha) + n\mathbb{E}|Y_1| I(|Y_1| > n^\alpha)\right\} \\
&\ll 2\sum_{n=1}^{\infty} n^{\alpha p-\alpha-1} l(n) \mathbb{E}|Y_1| I(|Y_1| > n^\alpha) \\
&= \sum_{j=0}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} n^{\alpha p-\alpha-1} l(n) \mathbb{E}|Y_1| I(|Y_1| > n^\alpha) \\
&\ll \sum_{j=1}^{\infty} 2^{\alpha(p-1)j} l(2^j) \mathbb{E}|Y_1| I(|Y_1| > 2^{\alpha j}) \;\; \text{(by Lemma 2.4 (i))} \\
&= \sum_{j=1}^{\infty} 2^{\alpha(p-1)j} l(2^j) \sum_{k=j}^{\infty} \mathbb{E}|Y_1| I(2^{\alpha k} < |Y_1| \leq 2^{\alpha(k+1)}) \\
&= \sum_{k=1}^{\infty} \mathbb{E}|Y_1| I(2^{\alpha k} < |Y_1| \leq 2^{\alpha(k+1)}) \sum_{j=1}^{k} 2^{\alpha(p-1)j} l(2^j) \\
&\ll \sum_{k=1}^{\infty} 2^{\alpha(p-1)k} l(2^k) \mathbb{E}|Y_1| I(2^{\alpha k} < |Y_1| \leq 2^{\alpha(k+1)}) \;\; \text{(by Lemma 2.4 (ii))} \\
&\ll \sum_{k=1}^{\infty} 2^{\alpha p k} l(2^k) \mathbb{P}(2^{\alpha k} < |Y_1| \leq 2^{\alpha(k+1)})
\end{aligned}$$

$$\ll \mathbb{E}\{|Y_1|^p l(|Y_1|^{1/\alpha})\} < \infty.$$

To prove that $I_2 < \infty$, we first show

$$n^{-\alpha} \max_{1\le k\le n} \left| \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} \mathbb{E}Y_j^{(n)} \right| \to 0 \ \text{ as } \ n \to \infty. \tag{9}$$

In the case $0 < \alpha < 1$, $p \ge 1$, and $0 < \gamma < \min\{\frac{\alpha p - 1}{\alpha}, p-1\}$, we obtain by (8) the following estimate:

$$\begin{aligned} n^{-\alpha} \max_{1\le k\le n} \left| \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} \mathbb{E}Y_j^{(n)} \right| &\le n^{-\alpha} \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} |\mathbb{E}Y_j^{(n)}| \\ &\le n^{-\alpha} \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} \{\mathbb{E}|Y_1| I(|Y_1| > n^\alpha) + n^\alpha \mathbb{P}(|Y_1| > n^\alpha)\} \\ &\le Cn^{1-\alpha}\mathbb{E}|Y_1| I(|Y_1| > n^\alpha) + n\mathbb{P}(|Y_1| > n^\alpha) \\ &\le Cn^{1-\alpha}\mathbb{E}|Y_1| I(|Y_1| > n^\alpha) \\ &= Cn^{1-\alpha}\mathbb{E}|Y_1|^{p-\gamma} |Y_1|^{1-p+\gamma} I(|Y_1| > n^\alpha) \\ &\le Cn^{1+\alpha\gamma-\alpha p}\mathbb{E}|Y_1|^{p-\gamma} I(|Y_1| > n^\alpha) \to 0 \ \text{ as } n \to \infty. \end{aligned}$$

Hence, from (9) it follows that, for n large enough and every $\varepsilon > 0$,

$$n^{-\alpha} \max_{1\le k\le n} \left| \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} \mathbb{E}Y_j^{(n)} \right| < \varepsilon/2.$$

This implies that

$$I_2 \le \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \mathbb{P}\left\{ \max_{1\le k\le n} \left| \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} (Y_j^{(n)} - \mathbb{E}Y_j^{(n)}) \right| > n^\alpha \varepsilon/4 \right\}.$$

By Lemma 2.1, we can see that, for every fixed $n \ge 1$, the sequence $\{Y_j^{(n)}, j \ge 1\}$ consists of NSD random variables. Thus, by Markov's inequality and Part J of Theorem 1 in [6], it follows from (4) that, for $q > 2$,

$$\begin{aligned} I_2 &\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2} l(n) \mathbb{E}\left(\max_{1\le k\le n} \left| \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} (Y_j^{(n)} - \mathbb{E}Y_j^{(n)}) \right|^q \right) \\ &\le \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2} l(n) \left(\sum_{i=-\infty}^{\infty} a_i \right)^{q-1} \sum_{i=-\infty}^{\infty} a_i \mathbb{E} \max_{1\le k\le n} \left| \sum_{j=i+1}^{i+k} (Y_j^{(n)} - \mathbb{E}Y_j^{(n)}) \right|^q \\ &\le C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2} l(n) \sum_{i=-\infty}^{\infty} a_i \left\{ \sum_{j=i+1}^{i+n} \mathbb{E}\left| Y_j^{(n)} - \mathbb{E}Y_j^{(n)} \right|^q + \left(\sum_{j=i+1}^{i+n} \mathbb{E}\left| Y_j^{(n)} - \mathbb{E}Y_j^{(n)} \right|^2 \right)^{q/2} \right\} \\ &=: I_3 + I_4. \end{aligned}$$

To estimate I_3, by C_r inequality, we have

$$I_3 \ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2} l(n) \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} \mathbb{E}\left\{ \left| Y_j^{(n)} \right|^q + \left| \mathbb{E}Y_j^{(n)} \right|^q \right\}$$

$$\begin{aligned}
&\le 2\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2} l(n) \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} \mathbb{E}\left|Y_j^{(n)}\right|^q \text{ (by Jensen's inequality)}\\
&= \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2} l(n) \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} \mathbb{E}\left|-n^\alpha I(Y_1<-n^\alpha)+Y_1 I(|Y_1|\le n^\alpha)+n^\alpha I(Y_1>n^\alpha)\right|^q\\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-1} l(n)\mathbb{E}\left\{|Y_1| I(|Y_1|\le n^\alpha)+n^\alpha I(|Y_1|>n^\alpha)\right\}^q\\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-1} l(n)\mathbb{E}|Y_1|^q I(|Y_1|\le n^\alpha)+\sum_{n=1}^{\infty} n^{\alpha p-1} l(n)\{\mathbb{P}(|Y_1|>n^\alpha)\}^q \text{ (by } \mathrm{C_r} \text{ inequality)}\\
&=: I_5+I_6.
\end{aligned}$$

To estimate I_5, we have for $p \le 2$:

$$\begin{aligned}
I_5 &= \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-1} l(n)\mathbb{E}|Y_1|^q I(|Y_1|\le n^\alpha)\\
&= \sum_{j=0}^{\infty}\sum_{n=2^j}^{2^{j+1}-1} n^{\alpha p-\alpha q-1} l(n)\mathbb{E}|Y_1|^q I(|Y_1|\le n^\alpha)\\
&\ll \sum_{j=1}^{\infty} 2^{(\alpha p-\alpha q-1)j} 2^j l(2^j)\mathbb{E}|Y_1|^q I(|Y_1|\le 2^{\alpha(j+1)}) \text{ (by Lemma 2.4 (i))}\\
&\ll \sum_{j=1}^{\infty} 2^{\alpha(p-q)j} l(2^j)\sum_{k=1}^{j}\mathbb{E}|Y_1|^q I(2^{\alpha k}<|Y_1|\le 2^{\alpha(k+1)})+\sum_{j=1}^{\infty} 2^{\alpha(p-q)j} l(2^j)\\
&\le \sum_{j=1}^{\infty} 2^{\alpha(p-q)j} l(2^j)\sum_{k=1}^{j}\mathbb{E}|Y_1|^q I(2^{\alpha k}<|Y_1|\le 2^{\alpha(k+1)})+C \text{ (by Lemma 2.4 (iii))}\\
&\ll \sum_{k=1}^{\infty} 2^{\alpha p k} l(2^k)\mathbb{P}(2^{\alpha k}<|Y_1|\le 2^{\alpha(k+1)})+C\\
&\ll \mathbb{E}\{|Y_1|^p l(|Y_1|^{1/\alpha})\}<\infty.
\end{aligned}$$

To estimate I_6, by Lemma 2.4 (ii) and similar argument as those in proving finiteness of I_5, we can establish that $I_6 < \infty$.

Now we prove that $I_4 < \infty$. By C_r inequality, we obtain

$$\begin{aligned}
I_4 &\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2} l(n) \sum_{i=-\infty}^{\infty} a_i \left\{\sum_{j=i+1}^{i+n} \mathbb{E}|Y_j^{(n)}|^2\right\}^{q/2}\\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2} l(n) \sum_{i=-\infty}^{\infty} a_i \left\{n^{2\alpha+1}\mathbb{P}(|Y_1|>n^\alpha)+n\mathbb{E}|Y_1|^2 I(|Y_1|\le n^\alpha)\right\}^{q/2}\\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-2+\frac{q}{2}} l(n)\left\{\mathbb{P}(|Y_1|>n^\alpha)\right\}^{q/2}+\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2+\frac{q}{2}} l(n)\left\{\mathbb{E}|Y_1|^2 I(|Y_1|\le n^\alpha)\right\}^{q/2}\\
&=: I_7+I_8.
\end{aligned}$$

To evaluate I_7, we choose $q > 2$ large enough such that $(\alpha p - 2) + \frac{q}{2}(1 - \alpha p + \alpha\gamma) < -1$. Hence, in the case $0 < \alpha < 1$, $p > 1$, and $0 < \gamma < \min\{\frac{\alpha p-1}{\alpha}, p-1\}$, we obtain

$$\begin{aligned} I_7 &\leq \sum_{n=1}^{\infty} n^{\alpha p-2+\frac{q}{2}-\frac{\alpha q}{2}} l(n) \{\mathbb{E}|Y_1| I(|Y_1| > n^\alpha)\}^{q/2} \\ &= \sum_{n=1}^{\infty} n^{\alpha p-2+\frac{q}{2}-\frac{\alpha q}{2}} l(n) \left\{\mathbb{E}|Y_1|^{p-\gamma}|Y_1|^{1-p+\gamma} I(|Y_1| > n^\alpha)\right\}^{q/2} \\ &\leq \sum_{n=1}^{\infty} n^{(\alpha p-2)+\frac{q}{2}(1-\alpha p+\alpha\gamma)} l(n) \left\{\mathbb{E}|Y_1|^{p-\gamma} I(|Y_1| > n^\alpha)\right\}^{q/2} < \infty. \end{aligned}$$

Finally, to evaluate I_8, with the same choice of q and γ, we have

$$\begin{aligned} I_8 &= \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2+\frac{q}{2}} l(n) \left(\mathbb{E}|Y_1|^{p-\gamma}|Y_1|^{2-p+\gamma} I(|Y_1| \leq n^\alpha)\right)^{q/2} \\ &\leq \sum_{n=1}^{\infty} n^{(\alpha p-2)+\frac{q}{2}(1-\alpha p+\alpha\gamma)} l(n) \left(\mathbb{E}|Y_1|^{p-\gamma} I(|Y_1| \leq n^\alpha)\right)^{q/2} < \infty. \end{aligned}$$

The relation (5) is proven and we now prove (6). By Lemma 2.4 and Lemma 3 in [6], for $r = \alpha p$ and $\varepsilon^* = \varepsilon/4^\alpha$,

$$\begin{aligned} \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \mathbb{P}\left(\sup_{k\geq n} |S_k/k^\alpha| > \varepsilon\right) &= \sum_{j=0}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} n^{\alpha p-2} l(n) \mathbb{P}\left(\sup_{k\geq n} |S_k/k^\alpha| > \varepsilon\right) \\ &\ll \sum_{j=0}^{\infty} 2^{(\alpha p-1)j} l(2^j) \mathbb{P}\left(\sup_{k\geq 2^j} |S_k/k^\alpha| > \varepsilon\right) \\ &\ll \sum_{n=1}^{\infty} n^{(\alpha p-2)} l(n) \mathbb{P}\left(\max_{1\leq k\leq n} |S_k| > \varepsilon^* n^\alpha\right) < \infty. \end{aligned}$$

This completes the proof of the theorem. □

Theorem 3.2. *Let $l(x) > 0$ be a slowly varying and nondecreasing function. Suppose that $\{X_n, n \geq 1\}$ is a moving average process based on a sequence $\{Y_i;\ i = 0, \pm 1, \pm 2, \dots\}$ of identically distributed NSD random variables. If $\mathbb{E}Y_1 = 0$ and $\mathbb{E}|Y_1|^{1/\alpha} l(|Y_1|^{1/\alpha}) < \infty$, with $1/2 < \alpha < 1$, then, for all $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} \frac{l(n)}{n} \mathbb{P}\left\{\max_{1\leq k\leq n} |S_k| \geq \varepsilon n^\alpha\right\} < \infty. \tag{10}$$

Proof. Without loss of generality, we assume that $a_i > 0$ for all $i \geq 1$. Recall that

$$Y_i^{(n)} = -n^\alpha I(Y_i < -n^\alpha) + Y_i I(|Y_i| \leq n^\alpha) + n^\alpha I(Y_i > n^\alpha), \;\; n \geq 1.$$

Then, for any $\varepsilon > 0$,

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{-1} l(n) \mathbb{P}\left\{\max_{1\leq k\leq n} |S_k| > n^\alpha \varepsilon\right\} \\ &\leq \sum_{n=1}^{\infty} n^{-1} l(n) \mathbb{P}\left\{\max_{1\leq k\leq n} |\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} n^\alpha I(|Y_j| > n^\alpha) + Y_j I(|Y_j| > n^\alpha)| > n^\alpha \varepsilon/2\right\} \end{aligned}$$

$$+ \sum_{n=1}^{\infty} n^{-1} l(n) \mathbb{P}\left\{ \max_{1\le k\le n} |\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_j^{(n)}| > n^{\alpha}\varepsilon/2 \right\}$$
$$=: I_1^* + I_2^*. \tag{11}$$

Hence, in order to prove (10), it suffices to show that $I_1^* < \infty$ and $I_2^* < \infty$. Finiteness of I_1^* can be proved similarly to that of I_1.

Prove finiteness of I_2^*. We have

$$\begin{aligned} n^{-\alpha} \max_{1\le k\le n} \left| \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} \mathbb{E}Y_j^{(n)} \right| &\le n^{-\alpha} \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} |\mathbb{E}Y_j^{(n)}| \\ &\le 2n^{1-\alpha}\mathbb{E}|Y_1| I(|Y_1| > n^{\alpha}) \ \ (by\ \mathbb{E}Y_1 = 0) \\ &= 2n^{1-\alpha}\mathbb{E}|Y_1|^{1/\alpha}|Y_1|^{1-1/\alpha} I(|Y_1| > n^{\alpha}) \\ &\le 2\mathbb{E}\{|Y_1|^{1/\alpha} I(|Y_1| > n^{\alpha})\} \to 0 \text{ as } n \to \infty. \end{aligned} \tag{12}$$

By Lemma 1.4, it is easy to show that, for every fixed $n \ge 1$, the sequence $\{Y_j^{(n)};\ j \ge 1\}$ consists of NSD random variables. By analogy with proving the relation $I_2 < \infty$, we have

$$\begin{aligned} I_2^* &\ll \sum_{n=1}^{\infty} n^{-2\alpha-1} l(n) \sum_{i=-\infty}^{\infty} a_i \mathbb{E} \max_{1\le k\le n} \left| \sum_{j=i+1}^{i+k} (Y_j^{(n)} - \mathbb{E}Y_j^{(n)}) \right|^2 \\ &\ll \sum_{n=1}^{\infty} n^{-2\alpha-1} l(n) \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} \mathbb{E}|Y_j^{(n)}|^2 \ \text{(by (3))} \\ &\ll \sum_{n=1}^{\infty} n^{-2\alpha-1} l(n) \left\{ n^{2\alpha+1}\mathbb{P}(|Y_1| > n^{\alpha}) + n\mathbb{E}|Y_1|^2 I(|Y_1| \le n^{\alpha}) \right\} \\ &=: I_3^* + I_4^*. \end{aligned}$$

To estimate I_3^* for $\alpha < 1$, by Lemma 2.4 (i), (ii), we can write

$$\begin{aligned} I_3^* &\le \sum_{n=1}^{\infty} n^{-\alpha} l(n) \mathbb{E}|Y_1| I(|Y_1| > n^{\alpha}) \\ &= \sum_{j=0}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} n^{-\alpha} l(n) \mathbb{E}|Y_1| I(|Y_1| > n^{\alpha}) \\ &\ll \mathbb{E}\{|Y_1|^{1/\alpha} l(|Y_1|^{1/\alpha})\} < \infty. \end{aligned}$$

To estimate I_4^* for $\alpha > 1/2$, by Lemma 2.4 (i), (iii), we obtain

$$\begin{aligned} I_4^* &\le \sum_{n=1}^{\infty} n^{-2\alpha} l(n) \mathbb{E}|Y_1|^2 I(|Y_1| \le n^{\alpha}) \\ &= \sum_{j=0}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} n^{-2\alpha} l(n) \mathbb{E}|Y_1|^2 I(|Y_1| \le n^{\alpha}) \\ &\ll \mathbb{E}\{|Y_1|^{1/\alpha} l(|Y_1|^{1/\alpha})\} < \infty. \end{aligned}$$

These complete the proof. □

As an application of Theorem 3.2, we can deduce the Marcinkiewicz–Zygmund strong law of large numbers as follows.

Corollary 3.3. *Let $\{X_n, n \geq 1\}$ be a moving average process based on a sequence $\{Y_i; i = 0, \pm 1, \pm 2, \ldots\}$ of identically distributed NSD random variables. If $\mathbb{E}Y_1 = 0$ and $\mathbb{E}|Y_1|^{1/\alpha} < \infty$, with $1/2 < \alpha < 1$, then*

$$\frac{S_n}{n^\alpha} \to 0 \ \ a.\, s.$$

Proof. By applying Theorem 3.2, with $l(x) = 1$, we can write

$$\sum_{n=1}^{\infty} n^{-1}\mathbb{P}\left(\max_{1\leq k\leq n} |S_k| > \varepsilon n^\alpha\right) < \infty \ \text{ for all } \ \varepsilon > 0.$$

Hence, for all $\varepsilon > 0$,

$$\begin{aligned} \frac{1}{2}\sum_{r=1}^{\infty}\mathbb{P}\left(\max_{1\leq k\leq 2^{r-1}} |S_k| > \varepsilon 2^{r\alpha}\right) \ &\leq \ \sum_{r=1}^{\infty}\sum_{n=2^{r-1}}^{2^r-1} n^{-1}\mathbb{P}\left(\max_{1\leq k\leq n} |S_k| > \varepsilon n^\alpha\right) \\ &= \sum_{n=1}^{\infty} n^{-1}\mathbb{P}\left(\max_{1\leq k\leq n} |S_k| > \varepsilon n^\alpha\right) < \infty. \end{aligned}$$

So the Borel—Cantelli lemma yields that, as $r \to 0$,

$$2^{-r\alpha}\max_{1\leq k\leq 2^r} |S_k| \to 0 \ \text{ a.s.}$$

For all positive integers n, there exists a nonnegative integer r such that $2^{r-1} \leq n < 2^r$. Thus,

$$n^{-\alpha}|S_n| \leq \max_{2^{r-1}\leq n<2^r} n^{-\alpha}|S_n| \leq 2^{-\alpha(r-1)}\max_{2^{r-1}\leq k<2^r} |S_k| \to 0 \ \text{ a.s.}$$

This implies that $\frac{S_n}{n^\alpha} \to 0$ a.s. □

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