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Special cases in order statistics for the alternative parametrization of the Generalized Power Function Distribution

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ABSTRACT

This paper investigates the order statistics based on the moment-generating function for the Generalized Power Function Distribution for the two different forms of this distribution. In this paper, we continue our investigation on the distribution of order statistics assuming that the original sample $\underline{Y} = (Y_1, Y_2, \dots, Y_n)$ of size n is taken from a population that follows the Generalized Power Function Distribution. Two different forms of the Generalized Power Function Distribution with its special cases are presented and some order statistics related to these different forms are discussed. The main technique is the consideration of the moment-generating function.

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1. Introduction

The Generalized Power Function Distribution (GPDF) is one of the useful lifetime distribution models which offers a good fit to various sets of failure data. This paper investigates the moment generating function and moments for the special cases of the order statistics for the GPDF by the classical method of moment generating functions [1].

The GPDF [2] is defined as

$$f(y) = \frac{p}{\sigma b^p} \left(\frac{y - \theta}{\sigma} + a \right)^{p-1}, \quad (1)$$

$$\theta - a\sigma \leq y \leq \theta + (b - a)\sigma, 0 \text{ otherwise,}$$

where $p \geq 1$, and a and b are defined as:

$$b = (p + 1) \sqrt{\frac{p + 2}{p}}, \quad a = \sqrt{p(p + 2)}. \quad (2)$$

It is well known that the GPDF is a special case of Beta (p, q) when $q = 1$ distribution [2].

The following is the alternative parametrization of the GPDF [3]:

$$f(y) = \frac{p}{(b - a)^p} (y - a)^{p-1}, \quad p > 1, a \leq y \leq b. \quad (3)$$

In the following, we will always assume that parameter $p > 1$, and we will not specify this anymore.

Figure 1 shows the probability density function for the GPDF for the parametrization (3) for different

values for the shape p , scale b and location a parameters.

2. Parametrization (3)

Let $\underline{Y} = (Y_1, Y_2, \dots, Y_n)$ be a random sample of size n from a population which follows the GPDF with the probability density function (3). Al Mutairi [4] derived the probability density function of the r -th order statistic $Y_{(r)}$, $1 \leq r \leq n$, $Y_{(1)} \leq Y_{(2)} \leq \dots \leq Y_{(r)} \leq \dots \leq Y_{(n)}$, obtained from this random sample in the following form:

$$g_{Y_{(r)}}(y) = rp \binom{n}{r} \sum_{k=0}^{n-r} \binom{n-r}{k} \frac{(-1)^k}{(b-a)^{(n-k)p}} (y-a)^{(n-k)p-1},$$

$$a \leq y \leq b, 0 \text{ otherwise.}$$

In addition, Al Mutairi [4] derived the moment generating function for the order statistic $Y_{(r)}$, $1 \leq r \leq n$ as follows:

$$M_{Y_{(r)}}(u) = \sum_{k=0}^{n-r} C_k e^{au} \left\{ \sum_{v=0}^{\infty} \frac{[(b-a)u]^v}{v!} \frac{(n-k)p}{(n-k)p+v} \right\}, \quad (4)$$

where

$$C_k = \frac{(-1)^k}{n-k} r \binom{n}{r} \binom{n-r}{k}.$$

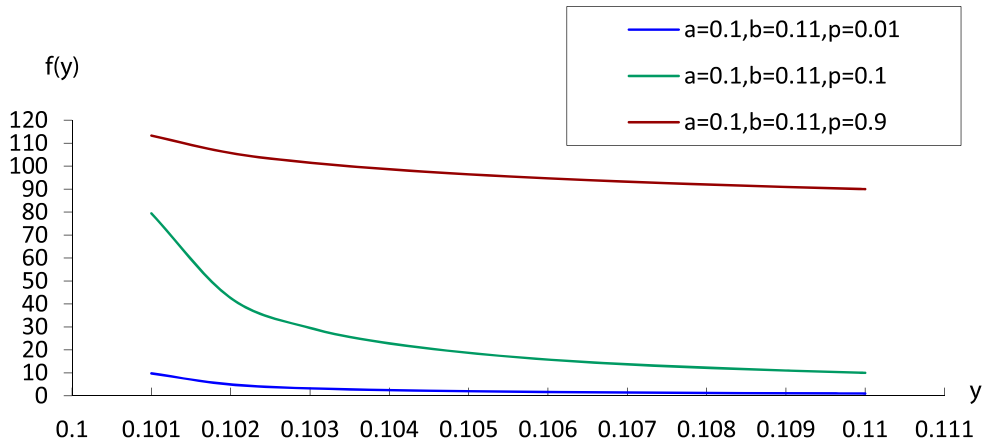


Figure 1. Probability density functions for the GPDF of the form (3).

Hence the m -th moment of the order statistic $Y_{(r)}$ is given as:

$$E(Y_{(r)}^m) = \frac{\sum_{v=0}^{n-r} C_k(n-k)p^{(n-k)p-1} \binom{(n-k)p-1}{v}}{(b-a)^{p(r+k)}} \times \frac{(-a)^v}{(n-k)p-v+m} [b^{(n-k)p-v+m} - a^{(n-k)p-v+m}]. \quad (5)$$

In particular, the first moment of the order statistics $Y_{(r)}$ is obtained by setting $m = 1$ in Equation (5) and this leads to

$$E(Y_{(r)}) = \frac{\sum_{k=0}^{n-r} C_k(n-k)p^{(n-k)p-1} \binom{(n-k)p-1}{v}}{(b-a)^{(n-k)p}} \times \frac{(-a)^v [b^{(n-k)p-v+1} - a^{(n-k)p-v+1}]}{(n-k)p-v+1}.$$

The second non-central moment $\mu_{Y_{(r)}}^{(2)}$ is obtained by setting $m = 2$ in Equation (5). The same as in Al Mutairi [4], we obtain

$$E(Y_{(r)}^2) = \frac{\sum_{k=0}^{n-r} C_k(n-k)p^{(n-k)p-1} \binom{(n-k)p-1}{v}}{(b-a)^{(n-k)p}} \times \frac{(-a)^v [b^{(n-k)p-v+2} - a^{(n-k)p-v+2}]}{(n-k)p-v+2}$$

2.1. Special case

Let $a = 0$ and $b = 1$ in Equation (3) to get one of the most important special cases which is known as a standard Power Function distribution that has many applications. For example, see ref. [3] where data representing the failure time in minutes of an electrical insulation device are considered and have been modelled as a standard Power Function distribution.

To derive some properties for this special case, we consider the moment generating function of the order statistic $Y_{(r)}$ (Equation (4)) as follows:

$$M_{Y_{(r)}}(u) = \sum_{k=0}^{n-r} C_k \left[\sum_{v=0}^{\infty} \frac{(u)^v}{v!} \frac{(n-k)p}{(n-k)p+v} \right] = \sum_{k=0}^{n-r} C_k \left[1 + \frac{(n-k)pu}{[(n-k)p+1]!} + \frac{(n-k)pu^2}{[(n-k)p+2]!} + \dots \right]. \quad (6)$$

Then, by differentiating Equation (6) and setting $u = 0$, the first moment (mean) $\mu_{Y_{(r)}}$ of the order statistic $Y_{(r)}$ for the GPDF with $a = 0$ and $b = 1$ is obtained as

$$\frac{d}{du} M_{Y_{(r)}}(u) = \sum_{k=0}^{n-r} C_k \left[\frac{(n-k)p}{(n-k)p+1} + \frac{2(n-k)pu}{((n-k)p+2)!} + \dots \right] \mu_{Y_{(r)}} = \frac{d}{du} M_{Y_{(r)}}(u)|_{u=0} = \sum_{k=0}^{n-r} C_k \left[\frac{(n-k)p}{(n-k)p+1} \right].$$

In the same way, the second non-central moment $\mu_{Y_{(r)}}^{(2)}$ for the order statistic for the Standard Power Function Distribution is computed as

$$\mu_{Y_{(r)}}^{(2)} = \sum_{k=0}^{n-r} C_k \left[\frac{(n-k)p}{(n-k)p+2} \right].$$

The variance $\sigma_{Y_{(r)}}^2$ of the order statistic $Y_{(r)}$ for the GPDF with $a = 0$ and $b = 1$ can be obtained as follows:

$$\begin{aligned} \sigma_{Y_{(r)}}^2 &= \mu_{Y_{(r)}}^{(2)} - (\mu_{Y_{(r)}})^2 \\ &= \sum_{k=0}^{n-r} C_k \left\{ \frac{(n-k)p}{(n-k)p+2} \right\} - \left\{ \sum_{k=0}^{n-r} C_k \left[\frac{(n-k)p}{(n-k)p+1} \right] \right\}^2 \\ &= \sum_{k=0}^{n-r} C_k \left\{ \frac{(n-k)p[(n-k)p+1]^2}{[(n-k)p+2][(n-k)p+1]^2} - \frac{\sum_{k=0}^{n-r} C_k(n-k)^2 p^2 [(n-k)p+2]}{[(n-k)p+2][(n-k)p+1]^2} \right\}. \end{aligned}$$

3. Alternative parametrization (1)

A special case of the alternative parametrization of the GPDF can be obtained by setting $\theta = 0$ and $\sigma = 1$ in Equation (1). Then we obtain

$$f(y) = \frac{p}{b^p} (y + a)^{p-1}, \quad -a \leq y \leq b - a, \quad (7)$$

where p is the shape parameter, b is the scale parameter and a is the location parameter [3]. Moreover, a special case of Equation (7) can be obtained by setting $a = 0$ and $b = 1$ which leads to the same formula of Standard GPDF that is discussed in Section 2.1.

The probability density function of the r -th order statistic $Y_{(r)}$ is given [4] as follows:

$$g_{Y_{(r)}}(y) = rp \binom{n}{r} \sum_{t=0}^{n-r} \binom{n-r}{t} \frac{(-1)^t}{b^{(n-t)p}} (y + a)^{(n-t)p-1}, \quad -a \leq y \leq b - a. \quad (8)$$

See refs. [5] and [6].

With preliminaries accounted for, we can now formulate and prove the following theorem.

Theorem 1: If $Y_{(r)}$ is the r -th order statistic for a sample from the population with GPDF of the form (7), then:

(i) The moment generating function $M_{Y_{(r)}}(u)$ of the order statistic $Y_{(r)}$ is given as:

$$M_{Y_{(r)}}(u) = \sum_{t=0}^{n-r} C_t e^{-au} \left\{ \sum_{v=0}^{\infty} \frac{[bu]^v}{v!} \frac{(n-t)p}{(n-t)p + v} \right\}. \quad (9)$$

(ii) The first moment (mean) $\mu_{Y_{(r)}}$, second non-central moment $\mu_{Y_{(r)}}^{(2)}$ and the variance $\sigma_{Y_{(r)}}^2$ of the order statistic $Y_{(r)}$ are given in [7] as

$$\mu_{Y_{(r)}} = \sum_{t=0}^{n-r} C_t \left(\frac{(n-t)bp}{(n-t)p + 1} - a \right) \quad (10)$$

$$\mu_{Y_{(r)}}^{(2)} = \sum_{t=0}^{n-r} C_t \left(a^2 - \frac{2a(n-t)bp}{(n-t)p + 1} + \frac{(n-t)b^2p}{(n-t)p + 2} \right) \quad (11)$$

$$\sigma_{Y_{(r)}}^2 = \sum_{k=0}^{n-r} C_k \left[\frac{(n-t)^3 p^3 \left[(b-a)^2 - \sum_{t=0}^{n-r} C_t (b-a)^2 \right] + (n-t)^2 p^2 \left[(b^2 - 4ab + 3a^2) + (b-a)^2 + 2 \sum_{t=0}^{n-r} C_t a (b-a) - 2 \sum_{t=0}^{n-r} C_t (b-a)^2 \right]}{[(n-t)p + 2][(n-t)p + 1]^2} \right] \\ = \frac{(n-t)p[2a + (b^2 - 4ab + 3a^2) - \sum_{t=0}^{n-r} C_t a^2 + 4 \sum_{t=0}^{n-r} C_t a (b-a)] + 2a^2 - 2 \sum_{t=0}^{n-r} C_t a^2}{[(n-t)p + 2][(n-t)p + 1]^2} \quad (12)$$

(iii) The m -th moment $E(Y_{(r)}^m)$ of the order statistic $Y_{(r)}$ is

$$E(Y_{(r)}^m) = \frac{\sum_{v=0}^{n-r} C_t (n-t)p \binom{(n-t)p-1}{v}}{(b-a)^{p(r+k)}} \times \frac{a^v}{(n-t)p - v + m} \times [(b-a)^{(n-t)p-v+m} - (-a)^{(n-t)p-v+m}]. \quad (13)$$

Proof: (i) The moment generating function of the order statistic $Y_{(r)}$ can be obtained by the considering the probability density function

$$g_{Y_{(r)}}(y_r) = \frac{n!}{(r-1)!(n-r)!} \left[\frac{(y_r - a)^p}{(b-a)^p} \right]^{r-1} \left[1 - \frac{(y_r - a)^p}{(b-a)^p} \right]^{n-r} \times \frac{p}{(b-a)^p} (y_r - a)^{p-1}, \quad -a \leq y_r \leq b - a. \quad (14)$$

This leads to

$$M_{Y_{(r)}}(u) = E(e^{uy_r}) = \int_{-a}^{b-a} e^{uy_r} \sum_{t=0}^{n-r} C_t f(y_r) dy_r,$$

where $f(y_r)$ is given by

$$f(y_r) = \frac{(n-k)p(y_r - a)^{(n-k)p-1}}{(b-a)^{(n-k)p}}. \quad (15)$$

Then using the following substitution:

$$y_r = bx - a \text{ hence } dy_r = bdx.$$

The limits of integration become as follows:

$$y_r = -a \Rightarrow x = 0, y_r = b - a \Rightarrow x = 1.$$

Thus, the previous equation becomes

$$\begin{aligned}
 M_{Y_{(r)}}(u) &= \sum_{t=0}^{n-r} C_t \int_0^1 e^{[bux-au]} \frac{(n-t)p[bx]^{(n-t)p-1}}{b^{(n-t)p-1}} dx \\
 &= \sum_{t=0}^{n-r} C_t (n-t) p e^{-au} \int_0^1 e^{bux} x^{(n-t)p-1} dx \\
 &= \sum_{t=0}^{n-r} C_t (n-t) p e^{-au} \int_0^1 x^{(n-t)p-1} \\
 &\quad \times \left\{ 1 + \frac{bux}{1!} + \frac{b^2 u^2 x^2}{2!} + \dots \right\} dx \\
 &= \sum_{t=0}^{n-r} C_t (n-t) p e^{-au} \left\{ \frac{1}{(n-t)p} + \frac{bu}{((n-t)p+1)1!} \right. \\
 &\quad \left. + \frac{b^2 u^2}{((n-t)p+2)2!} + \dots \right\} \\
 &= \sum_{t=0}^{n-r} C_t e^{-au} \left\{ 1 + \frac{(n-t)bp u}{((n-t)p+1)1!} \right. \\
 &\quad \left. + \frac{(n-t)b^2 p u^2}{((n-t)p+2)2!} + \dots \right\} \\
 &= \sum_{t=0}^{n-r} C_t e^{-au} \left\{ \sum_{v=0}^{\infty} \frac{[bu]^v}{v!} \frac{(n-t)p}{(n-t)p+v} \right\}. \quad (16)
 \end{aligned}$$

This formula proves Equation (9) in Theorem 1, which provides the moment generating function $M_{Y_{(r)}}(u)$ of the order statistic $Y_{(r)}$.

(ii) The first moment (mean) $\mu_{Y_{(r)}}$ for the order statistic $Y_{(r)}$ is calculated by differentiating Equation (9) with respect to u and setting $u = 0$:

$$\begin{aligned}
 \sigma_{Y_{(r)}}^2 &= \mu_{Y_{(r)}}^{(2)} - (\mu_{Y_{(r)}})^2 \\
 &= \sum_{k=0}^{n-r} C_t \frac{(n-t)^3 p^3 [(b-a)^2 - \sum_{t=0}^{n-r} C_t (b-a)^2] + (n-t)^2 p^2 [(b^2 - 4ab + 3a^2) + (b-a)^2 + 2 \sum_{t=0}^{n-r} C_t a(b-a) - 2 \sum_{t=0}^{n-r} C_t (b-a)^2]}{[(n-t)p+2][(n-t)p+1]^2} \\
 &= \frac{+(n-t)p[2a + (b^2 - 4ab + 3a^2) - \sum_{t=0}^{n-r} C_t a^2 + 4 \sum_{t=0}^{n-r} C_t a(b-a)] + 2a^2 - 2 \sum_{t=0}^{n-r} C_t a^2}{[(n-t)p+2][(n-t)p+1]^2},
 \end{aligned}$$

$$\begin{aligned}
 \frac{dM_{Y_{(r)}}(u)}{du} &= \\
 &\sum_{t=0}^{n-r} C_t \left\{ -ae^{-au} \left[1 + \frac{(n-t)bp u}{((n-t)p+1)1!} + \frac{(n-t)b^2 p u^2}{((n-t)p+2)2!} + \dots \right] \right. \\
 &\left. + e^{-au} \left[\frac{(n-t)bp}{((n-t)p+1)1!} + \frac{2(n-t)b^2 p u}{((n-t)p+2)2!} + \dots \right] \right\}.
 \end{aligned}$$

Then, by setting $u = 0$, we obtain

$$\begin{aligned}
 \mu_{Y_{(r)}} &= \frac{d}{du} M_{Y_{(r)}}(u)|_{u=0} \\
 &= \sum_{t=0}^{n-r} C_t \left[\frac{(n-t)bp}{(n-t)p+1} - a \right],
 \end{aligned}$$

which represents the first moment (mean) $\mu_{Y_{(r)}}$ for the order statistic $Y_{(r)}$.

The second non-central moment $\mu_{Y_{(r)}}^{(2)}$ of the order statistics is calculated as:

$$\begin{aligned}
 \frac{d^2 M_{Y_{(r)}}(u)}{du^2} &= \sum_{t=0}^{n-r} C_t \left\{ a^2 e^{-au} \left[1 + \frac{(n-t)bp u}{((n-t)p+1)} \right. \right. \\
 &\quad \left. \left. + \frac{(n-t)b^2 u^2 p}{2((n-t)p+2)} + \dots \right] \right. \\
 &\quad \left. - a e^{-au} \left[\frac{(n-t)bp}{((n-t)p+1)} + \frac{(n-t)b^2 u p}{((n-t)p+2)} + \dots \right] \right. \\
 &\quad \left. + e^{-au} \left[\frac{(n-t)b^2 p}{((n-t)p+2)} \right] \right\}.
 \end{aligned}$$

Then, by setting $u = 0$, we obtain

$$\begin{aligned}
 \mu_{Y_{(r)}}^{(2)} &= \frac{d^2}{du^2} M_{Y_{(r)}}(u)|_{u=0} = \sum_{t=0}^{n-r} C_t \left\{ a^2 - a \left[\frac{(n-t)bp}{(n-t)p+1} \right] \right. \\
 &\quad \left. - a \left[\frac{(n-t)bp}{(n-t)p+1} \right] + \frac{(n-t)b^2 p}{(n-t)p+2} \right\} \\
 &= \sum_{t=0}^{n-r} C_t \left\{ a^2 - \frac{2a(n-t)bp}{(n-t)p+1} + \frac{(n-t)b^2 p}{(n-t)p+2} \right\},
 \end{aligned}$$

which represents the second non-central moment $\mu_{Y_{(r)}}^{(2)}$ of the order statistic $Y_{(r)}$.

In addition, the variance $\sigma_{Y_{(r)}}^2$ of the order statistic $Y_{(r)}$ can be calculated as follows: which represents the variance $\sigma_{Y_{(r)}}^2$ for the order statistic $Y_{(r)}$.

(iii) Finally, the m -th moment $E(Y_{(r)}^m)$ for the order statistic is

$$\begin{aligned}
 E(Y_{(r)}^m) &= \int_{-a}^{b-a} y_r^m \sum_{t=0}^{n-r} C_t f_{Z_t}(y_r) dy_r \\
 &= \int_{-a}^{b-a} y_r^m \sum_{t=0}^{n-r} C_t \frac{(n-t)p(y_r+a)^{(n-t)p-1}}{b^{(n-t)p}} dy_r.
 \end{aligned}$$

Using the binomial expansion, we obtain the following:

$$\begin{aligned}
 E(Y_{(r)}^m) &= \frac{\sum_{t=0}^{n-r} C_t(n-t)p}{b^{(n-t)p}} \int_{-a}^{b-a} y_r^m \sum_{\nu=0}^{(n-t)p-1} \binom{(n-t)p-1}{\nu} \\
 &\quad \times a^\nu y_r^{(n-t)p-\nu-1} dy_r \\
 &= \frac{\sum_{t=0}^{n-r} C_t(n-t)p}{b^{(n-t)p}} \sum_{\nu=0}^{(n-t)p-1} \binom{(n-t)p-1}{\nu} \\
 &\quad \times a^\nu \int_{-a}^{b-a} y_r^{(n-t)p-\nu-1+m} dy_r \\
 &= \frac{\sum_{t=0}^{n-r} C_t(n-t)p}{b^{(n-t)p}} \sum_{\nu=0}^{(n-t)p-1} \binom{(n-t)p-1}{\nu} \\
 &\quad \times \frac{a^\nu}{(n-t)p-\nu+m} \\
 &\quad \times [(b-a)^{(n-t)p-\nu+m} - (-a)^{(n-t)p-\nu+m}].
 \end{aligned}$$

This result proves Equation (13) in Theorem 1, which represents the m -th moment $E(Y_{(r)}^m)$ of the order statistic $Y_{(r)}$.

Now we consider first two moments for (7). We can obtain the first moment (mean) $\mu_{Y_{(r)}}$ of the order statistic $Y_{(r)}$ by setting $m = 1$ in Equation (13) as follows:

$$\begin{aligned}
 E(Y_{(r)}) &= \frac{\sum_{t=0}^{n-r} C_t(n-t)p}{b^{(n-t)p}} \sum_{\nu=0}^{(n-t)p-1} \binom{(n-t)p-1}{\nu} \\
 &\quad \times \frac{a^\nu}{(n-t)p-\nu+1} \\
 &\quad \times [(b-a)^{(n-t)p-\nu+1} - (-a)^{(n-t)p-\nu+1}].
 \end{aligned}$$

Similarly, by setting $m = 2$ in Equation (13), we get:

$$\begin{aligned}
 E(Y_{(r)}^2) &= \frac{\sum_{t=0}^{n-r} C_t(n-t)p}{b^{(n-t)p}} \sum_{\nu=0}^{(n-t)p-1} \binom{(n-t)p-1}{\nu} \\
 &\quad \times \frac{a^\nu}{(n-t)p-\nu+2} \\
 &\quad \times [(b-a)^{(n-t)p-\nu+2} - (-a)^{(n-t)p-\nu+2}]. \quad (17)
 \end{aligned}$$

This gives the second non-central moment $\mu_{Y_{(r)}}^{(2)}$ for GPDF.

4. Conclusion

In this paper, two different forms of the GPDF with its special cases are presented and some order statistics related to these different forms are discussed. The main technique is the consideration of the moment generating function.

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References

- [1] Hogg RV, Craig AT. Introduction to mathematical statistics. 4th ed. New York (NY): Macmillan Publishing Co., Inc., Collier Macmillan Publishers; 1978.
- [2] Johnson NL, Kotz S, Balakrishnan N. Practical statistical process control – A tool for quality manufacturing. J R Stat Soc Ser A Stat Soc. 1996;159(2):343.
- [3] Sultan KS, Moshref ME, Childs A. Record values from generalized power function distribution and associated inference. J Appl Stat Sci. 2002;11(2):145–157.
- [4] Al Mutairi AO. Distribution of the order statistics of the generalized power function. Adv Appl Stat. 2016;49(2):87–104.
- [5] Allam SM, Datta TK. Response spectrum analysis of suspension bridges for random ground motion. J Bridge Eng. 2002;7(6):325–337.
- [6] Saluja GS, Postolache M, Kurdi A. Convergence of three-step iterations for nearly asymptotically nonexpansive mappings in CAT (k) spaces. J Inequal Appl. 2015;(1):1–18. DOI:10.1186/s13660-015-0670-z
- [7] Maiti R, Biswas A. Coherent forecasting for over-dispersed time series of count data. Braz J Probab Statistics. 2015;29(4):747–766.