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Some results concerning ideal and classical uniform integrability and mean convergence

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Abstract

In this article, the concept of $\mathcal{J}$-uniform integrability of a sequence of random variables $\{X_k\}$ with respect to $\{a_{nk}\}$ is introduced where $\mathcal{J}$ is a non-trivial ideal of subsets of the set of positive integers and $\{a_{nk}\}$ is an array of real numbers. We show that this concept is weaker than the concept of $\{X_k\}$ being uniformly integrable with respect to $\{a_{nk}\}$ and is more general than the concept of B-statistical uniform integrability with respect to $\{a_{nk}\}$. We give two characterizations of $\mathcal{J}$-uniform integrability with respect to $\{a_{nk}\}$. One of them is a de La Vallée Poussin type characterization. For a sequence of pairwise independent random variables $\{X_k\}$ which is $\mathcal{J}$-uniformly integrable with respect to $\{a_{nk}\}$, a law of large numbers with mean ideal convergence is proved. We also obtain a version without the pairwise independence assumption by strengthening other conditions. Supplements to the classical Mean Convergence Criterion are also established.

Keywords Uniform integrability · Summability methods · Sequence of random variables · Weighted sums · Mean convergence

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1 Introduction

Probability limit theorems are crucial for making advances in mathematical statistics and its applications. The concept of uniform integrability plays an important role in establishing probability limit theorems. It is well known that mean convergence of order $p > 0$ of a sequence of $\mathcal{L}_p$ random variables $\{X_k\}$ to a random variable X implies that $\{X_k\}$ converges in probability to X and that for $p > 0$, convergence in probability of $\{X_k\}$ to X does not guarantee that mean convergence of order p holds, even if $X_k \in \mathcal{L}_p$, $k \geq 1$. However, convergence in probability when combined with an additional uniform integrability condition is equivalent to mean convergence. This equivalence is made precise by the classical Mean Convergence Criterion (see Theorem 8 in Section 6).

The main motivation of summability theory is to make a non-convergent sequence or series converge in a more general sense. Summability theory has many applications (see the discussion in [22] and the references cited in that discussion).

In this article, we introduce the concept of $\mathcal{J}$-uniform integrability of a sequence of random variables $\{X_k\}$ with respect to an array $\{a_{nk}\}$ of real numbers where $\mathcal{J}$ is an ideal of subsets of the set of positive integers. (Technical terms will be defined below.) This concept is more general than the concept of B-statistical uniform integrability with respect to $\{a_{nk}\}$.

Let $x = \{x_k : k \geq 1\}$ be a real sequence and let $B = \{b_{nk} : n \geq 1, k \geq 1\}$ be a summability matrix (an array of real numbers). Assume that the series

$$(Bx)_n = \sum_{k=1}^{\infty} b_{nk} x_k$$

converges for all $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of positive integers. If the sequence $\{(Bx)_n : n \geq 1\}$ is convergent to a real number α, then we say that the sequence x is *B-summable* to the real number α.

A summability matrix B is said to be *regular* if $\lim_{n\to\infty} (Bx)_n = \alpha$ whenever $\lim_{k\to\infty} x_k = \alpha$ (see, [2]).

Let B a non-negative regular summability matrix and let $S \subset \mathbb{N}$. Then the number

$$\delta_B(S) := \lim_{n\to\infty} \sum_{k\in S} b_{nk}$$

is said to be the *B-density* of S whenever the limit exists (see, [3, 4, 16]). Regularity of the summability matrix B yields that $0 \leq \delta_B(S) \leq 1$ whenever $\delta_B(S)$ exists. If we take $B = C$, the Cesàro matrix, then $\delta(S) := \delta_C(S)$ is called the (*natural* or *asymptotic*) *density* of S (see, [11]), where $C = (c_{nk})$ is the summability matrix (Cesàro matrix) defined by

$$c_{nk} = \begin{cases} \frac{1}{n} & , \text{ if } k \leq n \\ 0 & , \text{ otherwise.} \end{cases}$$

A real sequence $x = \{x_k\}$ is said to be B *-statistically convergent* (see, [9, 12]) to a real number α if for any $\varepsilon > 0$,

$$\delta_B(\{k \in \mathbb{N} : |x_k - \alpha| \geq \varepsilon\}) = 0$$

holds. In this case, we write $\text{st}_B - \lim_{k\to\infty} x_k = \alpha$. If we consider the Cesàro matrix then C-statistical convergence is known as *statistical convergence* [10, 23, 25]. In general, B-statistical convergence is regular (i.e., it preserves ordinary limits) and there exist sequences which are B-statistically convergent but not ordinary convergent.

The concept of $\mathcal{J}$-convergence is based on the notion of an ideal $\mathcal{J}$ of subsets of the set $\mathbb{N}$.

Let X be a nonempty set. A family $\mathcal{J}$ of subsets of X is called an *ideal of subsets* of X if

i. $A \cup D \in \mathcal{J}$ whenever $A, D \in \mathcal{J}$,
ii. If $A \in \mathcal{J}$ and $D \subset A$, then $D \in \mathcal{J}$ (see [17]).

An ideal $\mathcal{J}$ of subsets of X is said to be a *non-trivial ideal* if $\mathcal{J} \neq \emptyset$ and $X \notin \mathcal{J}$. If $\{x\} \in \mathcal{J}$ for every $x \in X$, then $\mathcal{J}$ is said to be an *admissible ideal*. Throughout this paper, all ideals are considered as non-trivial ideals of subsets of $\mathbb{N}$.

A real sequence $x = \{x_k\}$ is said to be *$\mathcal{J}$-convergent* to a real number α, if for any $\varepsilon > 0$

$$\{k \in \mathbb{N} : |x_k - \alpha| \geq \varepsilon\} \in \mathcal{J},$$

and in this case we write $\mathcal{J} - \lim_{k\to\infty} x_k = \alpha$ (see [17]).

Let $\mathcal{J}_{\text{fin}}$ be the ideal of all of the finite subsets of $\mathbb{N}$. Then $\mathcal{J}_{\text{fin}}$ is a non-trivial admissible ideal and $\mathcal{J}_{\text{fin}}$-convergence coincides with the ordinary (Cauchy) convergence (see [17]).

Consider the family $\mathcal{J}$ of B-density zero sets for a non-negative regular summability matrix B. Suppose that $A, D \in \mathcal{J}$. Then we have

$$\lim_{n\to\infty} \sum_{k\in A} b_{nk} = \lim_{n\to\infty} \sum_{k\in D} b_{nk} = 0. \tag{1}$$

On the other hand, as the elements of B are non-negative we can write

$$0 \leq \sum_{k\in A\cup D} b_{nk} \leq \sum_{k\in A} b_{nk} + \sum_{k\in D} b_{nk}. \tag{2}$$

Now, by (1) and (2) we obtain

$$\lim_{n\to\infty} \sum_{k\in A\cup D} b_{nk} = 0.$$

So, $A \cup D \in \mathcal{J}$. To see (ii) let $A \in \mathcal{J}$ and $D \subset A$. Then we have

$$\lim_{n\to\infty} \sum_{k\in A} b_{nk} = 0 \tag{3}$$

and

$$0 \leq \sum_{k\in D} b_{nk} \leq \sum_{k\in A} b_{nk}. \tag{4}$$

Now, (3) and (4) yield that

$$\lim_{n\to\infty}\sum_{k\in D} b_{nk} = 0.$$

So, $D \in \mathcal{J}$. Hence, $\mathcal{J}$ is an ideal. It is clear from the Silverman-Toeplitz Theorem (see Theorem 2.3.7 of [2]) that $\mathcal{J}$ is non-trivial and admissible. In this case, $\mathcal{J}$-convergence coincides with B-statistical convergence. Note that the ideal given in (g) of Example 3.1 of [17] does not coincide with B-statistical convergence for any non-negative regular matrix B.

Furthermore, ordinary convergence implies $\mathcal{J}$-convergence and a $\mathcal{J}$-limit is unique whenever $\mathcal{J}$ is an admissible ideal (see [17]).

The concepts of an ideal lower bound and an ideal upper bound of a real sequence are defined in [1]. Let $x = (x_k)$ be a real sequence and let $\mathcal{J}$ be an ideal over $\mathbb{N}$. A real number a is said to be a $\mathcal{J}$-*lower bound* of x if

$$\left\{k \in \mathbb{N} : x_k < a\right\} \in \mathcal{J}$$

and a real number b is said to be a $\mathcal{J}$-*upper bound* of x if

$$\left\{k \in \mathbb{N} : x_k > b\right\} \in \mathcal{J}.$$

As $\emptyset \in \mathcal{J}$, a number is a $\mathcal{J}$-lower (upper) bound of a sequence whenever it is an ordinary lower (upper) bound of it. The converse of this statement is not true in general (see, Theorems 2.1 and 2.2 of [1]).

Now we recall the definitions of the concept of supremum and infimum with respect to an ideal. The supremum of all $\mathcal{J}$-lower bounds of x is called the $\mathcal{J}$-*infimum* of the sequence x and the infimum of all $\mathcal{J}$-upper bounds of x is called the $\mathcal{J}$-*supremum* of the sequence x. The $\mathcal{J}$-infimum of $x = (x_k)$ is denoted by $\mathcal{J}-\inf_{k\in\mathbb{N}} x_k$ and the $\mathcal{J}$-supremum of $x = (x_k)$ is denoted by $\mathcal{J}-\sup_{k\in\mathbb{N}} x_k$ [1]. It is also shown in [1] that

$$\inf_{k\in\mathbb{N}} x_k \leq \mathcal{J}-\inf_{k\in\mathbb{N}} x_k \leq \mathcal{J}-\sup_{k\in\mathbb{N}} x_k \leq \sup_{k\in\mathbb{N}} x_k. \tag{5}$$

Remark 1 Suppose that $\mathcal{J}-\sup_{k\in\mathbb{N}} x_k = K < \infty$. Then for any $\varepsilon > 0$ there exists $\beta < K + \varepsilon$ such that $\left\{k \in \mathbb{N} : x_k > \beta\right\} \in \mathcal{J}$. Thus, we have

$$\left\{k \in \mathbb{N} : x_k > K + \varepsilon\right\} \in \mathcal{J}$$

and

$$\left\{k \in \mathbb{N} : x_k \geq K + \varepsilon\right\} \in \mathcal{J}.$$

Now we recall some previous versions of the notion of uniform integrability.

Definition 1 A sequence of random variables $\{X_k, k \geq 1\}$ is said to be:

1. Uniformly integrable in classical sense if

$$\lim_{c\to\infty}\sup_{k\in\mathbb{N}} \mathbb{E}|X_k| I_{\{|X_k|>c\}} = 0.$$

2. Uniformly integrable with respect to $\{a_{nk}\}$ if

$$\lim_{c\to\infty}\sup_{n\in\mathbb{N}}\sum_{k=1}^{\infty}|a_{nk}|\mathbb{E}|X_k|I_{\{|X_k|>c\}}=0$$

provided that

$$\sup_{n\in\mathbb{N}}\sum_{k=1}^{\infty}|a_{nk}|<\infty.$$

3. B-statistically uniformly integrable with respect to $\{a_{nk}\}$ if

$$\lim_{c\to\infty}\sup_{n\in\mathbb{N}}\ \mathrm{st}_B\sum_{k=1}^{\infty}|a_{nk}|\mathbb{E}|X_k|I_{\{|X_k|>c\}}=0$$

provided that B is regular summability matrix with non-negative elements and

$$\sup_{n\in\mathbb{N}}\ \mathrm{st}_B\sum_{k=1}^{\infty}|a_{nk}|<\infty.$$

The notion of the uniform integrability in classical sense is well known and can be found in any graduate level textbook on probability theory (see, for example, [8] Chapter 4, Section 4.5). The notion of uniform integrability with respect to $\{a_{nk}\}$ was introduced in [20] and [21] and the notion of B-statistical uniform integrability with respect to $\{a_{nk}\}$ was introduced in [22].

The following result which characterizes uniform integrability in the classical sense is known as the classical uniform integrability criterion (see [7] Chapter 4, Section 4.2).

Theorem 1 *A sequence of random variables $\{X_k\}$ is uniformly integrable in the classical sense if and only if*

i. $\sup_{k\in\mathbb{N}}\mathbb{E}|X_k|<\infty$ *and*

ii. for all $\varepsilon>0$, there exists $\delta>0$ such that for every event A with $P(A)<\delta$,

$$\sup_{k\in\mathbb{N}}\mathbb{E}|X_k|I_A<\varepsilon.$$

The condition (ii) is the property that the sequence of set functions $\{Q_k\}$ defined on the σ-algebra of events by $Q_k(A)=\mathbb{E}|X_k|I_A, k\geq 1$ is *uniformly absolutely continuous* with respect to the probability measure P (see Serfling (1980) [24] Chapter 1, Section 1.4).

It is easy to show via examples that (i) and (ii) are independent conditions in the sense that neither implies the other.

Remark 2 It was shown in [14] that (ii) is indeed equivalent to the apparently stronger condition

ii′. For all $\varepsilon>0$, there exists $\delta>0$ such that for every sequence of events $\{A_k\}$ with $P(A_k)<\delta, k\geq 1$,

$$\sup_{k\in\mathbb{N}} \mathbb{E}|X_k|I_{A_k} < \varepsilon.$$

The following classical result of Charles de La Vallée Poussin (see [19] p. 19) provides another characterization of uniform integrability in the classical sense. It is referred to as the classical de La Vallée Poussin criterion for uniform integrability.

Theorem 2 *A sequence of random variables* $\{X_k\}$ *is uniformly integrable in the classical sense if and only if there exists a convex monotone function G defined on* $[0, \infty)$ *with* $G(0) = 0$ *such that*

$$\lim_{x\to\infty} \frac{G(x)}{x} = \infty \text{ and } \sup_{k\in\mathbb{N}} \mathbb{E}G(|X_k|) < \infty.$$

The proof of the necessity half is much more difficult than that of the sufficiency half. On the other hand, the sufficiency half provides a very useful method for establishing uniform integrability in classical sense of a sequence of random variables. For the sufficiency half, the condition that G is a convex monotone function defined on $[0, \infty)$ with $G(0) = 0$ is not needed; it can weakened to the condition that G is a nonnegative Borel measurable function defined on $[0, \infty)$.

An alternative proof of the classical de La Vallée Poussin criterion for uniform integrability was provided by [6].

Remark 3 In the following we will use the following two simple observations. Let $\{d_n, n \geq 1\}$ and $\{e_n, n \geq 1\}$ be two sequences of nonnegative numbers and $\varepsilon > 0$. Then,

1. $\{n \in \mathbb{N} : d_n + e_n \geq \varepsilon\} \subset \{n \in \mathbb{N} : d_n \geq \frac{\varepsilon}{2}\} \cup \{n \in \mathbb{N} : e_n \geq \frac{\varepsilon}{2}\}$.
2. For any $H > 0$: $\{n \in \mathbb{N} : d_n e_n \geq \varepsilon\} \subset \{n \in \mathbb{N} : d_n \geq \frac{\varepsilon}{H}\} \cup \{n \in \mathbb{N} : e_n \geq H\}$.
   We will also need the following two famous inequalities:
3. The c_p-inequality: For any $p \geq 1$ and any two random variables X and Y with finite absolute p^{th} moment,

$$\mathbb{E}|X + Y|^p \leq 2^{p-1}(\mathbb{E}|X|^p + \mathbb{E}|Y|^p).$$

4. The von Bahr-Essen inequality: For any sequence $X_1, X_2, \ldots, X_n$ of mean zero independent random variables with finite absolute p^{th} moment, $1 \leq p \leq 2$, there exists a constant $C_p < \infty$ such that

$$\mathbb{E}\left|\sum_{k=1}^{n} X_k\right|^p \leq C_p \sum_{k=1}^{n} \mathbb{E}|X_k|^p.$$

   In fact, the constant $C_p = 2$ works for all $p \in [1, 2]$. See Theorem 2 of [26].

The plan of the paper is as follows. The concept of $\mathcal{J}$-uniform integrability of a sequence of random variables with respect to an array of real numbers is introduced in Section 2 where $\mathcal{J}$ is a non-trivial ideal of subsets of the set of positive integers and two characterizations (Theorems 3 and 4) of this concept are given in Section 3. Theorem 3 is an analogue of the classical uniform integrability criterion (Theorem 1) and Theorem 4 is an analogue of the classical de La Vallée Poussin criterion for uniform integrability (Theorem 2). In Section 4, the concepts of a sequence of random variables being

$\mathcal{J}$-convergent in the p^{th} mean ($p > 0$) to a random variable and being $\mathcal{J}$-convergent in probability to a random variable are presented and analogues (Theorems 5, 6, and 7) of the classical Mean Convergence Criterion are established. Additional remarks on $\mathcal{J}$-uniform integrability are given in Section 5. In Section 6, supplements (Theorems 10 and 11) to the classical Mean Convergence Criterion are presented.

2 Definition of $\mathcal{J}$-uniform integrability with respect to $\{a_{nk}\}$

Now we are ready to define a new version of uniform integrability which is called $\mathcal{J}$-uniform integrability with respect to $\{a_{nk}\}$. Throughout this paper we assume that $\mathcal{J}$ is an ideal over $\mathbb{N}$ and $\{a_{nk}\}$ is an array of real numbers such that $\mathcal{J}-\sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}| < \infty$. Without loss of generality we assume that

$$\mathcal{J}-\sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}| = 1. \tag{6}$$

Definition 2 A sequence of random variables $\{X_k\}$ is said to be *$\mathcal{J}$-uniformly integrable with respect to* $\{a_{nk}\}$ if

$$\lim_{c\to\infty} \mathcal{J}-\sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}|X_k| I_{\{|X_k|>c\}} = 0.$$

Considering (5), we see that if a sequence of random variables $\{X_k\}$ is uniformly integrable with respect to $\{a_{nk}\}$, then it is $\mathcal{J}$-uniformly integrable with respect to $\{a_{nk}\}$. Note that the notion of uniform integrability with respect to $\{a_{nk}\}$ is more general than the notion of uniform integrability in the classical sense, (see [20] and [21]). Therefore, uniform integrability implies $\mathcal{J}$-uniform integrability with respect to $\{a_{nk}\}$. Furthermore, if we consider the ideal of B-density zero sets, then $\mathcal{J}$-uniformly integrability with respect to $\{a_{nk}\}$ is reduces to B-statistical uniform integrability with respect to $\{a_{nk}\}$.

In the following example, we show that a sequence of random variables can be $\mathcal{J}$-uniformly integrable with respect to an array, but not uniformly integrable with respect to this array.

Example 1 Consider the infinite set of prime numbers $\{2, 3, 5, \dots\} = \{p_1, p_2, \dots\} \subset \mathbb{N}$. Next, consider the following subsets of $\mathbb{N}$:

$$\widetilde{\Delta}_i = \{jp_i, j \in \mathbb{N}\}, i \geq 1.$$

Of course, the sets $\widetilde{\Delta}_i$ are not disjoint. To make them disjoint, consider the following procedure producing a modified sequence of sets $\{\Delta_i, i \in \mathbb{N}\}$:

$$\Delta_1 = \widetilde{\Delta_1} \text{ and } \Delta_i = \widetilde{\Delta}_i \setminus (\cup_{j=1}^{i-1} \Delta_j), i \geq 2.$$

In order to cover the set of natural numbers $\mathbb{N}$, we need to include the number 1 in one of these sets. We put it into the set Δ_2, for example.

First of all, we note that sets Δ_i are infinite because the sequence $A_i = \{p_i, p_i^2, \dots, p_i^n, \dots\} \subset \Delta_i$. By the Fundamental Theorem of Arithmetic, sets $\{\Delta_i, i \in \mathbb{N}\}$ form a partition of the set $\mathbb{N}$.

Let $\mathcal{F}$ be the class of all subsets $A \subset \mathbb{N}$ that have a nonempty intersection with only a finite number of Δ_i's.

Note that the class $\mathcal{F}$ is nonempty because the aforementioned sequence A_i intersects only with one Δ_i, $i \geq 1$ and hence belongs to $\mathcal{F}$. Also, $\mathcal{F}$ contains all finite subsets of $\mathbb{N}$.

We now verify that $\mathcal{F}$ is an ideal:

1. For any $A, D \in \mathcal{F}$, the union $A \cup D$ intersects only with Δ_i's that A or D intersect. Because both A and D intersect with only finite number of Δ_i's, we have that $A \cup D$ intersects with only finite number of Δ_i's as well; that is, $A \cup D \in \mathcal{F}$.
2. If $A \in \mathcal{F}$ and $D \subset A$, then $D \in \mathcal{F}$ because D intersects with no larger number of Δ_i's than A, which is finite.

In the following, the set of even numbers Δ_1 plays an important role.

Note that the ideal $\mathcal{F}$ is non-trivial because $\mathbb{N} \notin \mathcal{F}$, and the ideal $\mathcal{F}$ is admissible because for any $n \in \mathbb{N}$, $\{n\} \in \mathcal{F}$.

Define an array $\{a_{nk}, n \geq 1, k \geq 1\}$ of real numbers by

$$a_{nk} = \begin{cases} 1, & \text{if } k \leq n, n \in \Delta_1 \\ \frac{1}{n}, & \text{if } k \leq n, n \notin \Delta_1 \\ 0, & \text{if } k > n. \end{cases}$$

Then

$$\sum_{k=1}^{\infty} a_{nk} = \begin{cases} n, & \text{if } n \in \Delta_1 \\ 1, & \text{if } n \notin \Delta_1. \end{cases} \tag{7}$$

Obviously, the usual supremum is infinite:

$$\sup_{n\geq 1} \sum_{k=1}^{\infty} a_{nk} = \infty.$$

At the same time, the $\mathcal{F}-\sup_{n\geq 1} \sum_{k=1}^{\infty} a_{nk}$ is finite, namely

$$\mathcal{F}-\sup_{n\geq 1} \sum_{k=1}^{\infty} a_{nk} = 1.$$

To see this, denote $b_n = \sum_{k=1}^{\infty} a_{nk}, n \geq 1$. The number 1 is an $\mathcal{F}$- upper bound of the sequence $\{b_n, n \geq 1\}$ because by (7)

$$\{n \in \mathbb{N} : b_n > 1\} = \Delta_1 \in \mathcal{F}.$$

Next, any number $b < 1$ cannot be an $\mathcal{F}$- upper bound of the sequence $\{b_n, n \geq 1\}$ because in this case by (7),

$$\{n \in \mathbb{N} : b_n > b\} = \mathbb{N} \notin \mathcal{F}.$$

Let $\{X_k, k \geq 1\}$ be a sequence of identically distributed random variables with finite mean. For any $c \geq 0$, denote

$$h(c) = \mathbb{E}|X_1| I_{\{|X_1|>c\}}.$$

Then

$$\sum_{k=1}^{\infty} a_{nk} \mathbb{E}|X_k| I_{\{|X_k|>c\}} = \begin{cases} nh(c), & \text{if } n \in \Delta_1 \\ h(c), & \text{if } n \notin \Delta_1. \end{cases}$$

If we assume that $h(c) > 0$ for all $c > 0$ (for example, the random variables are normal, or exponential, etc.), then the sequence $\{X_k, k \geq 1\}$ is not uniformly integrable with respect to the array $\{a_{nk}\}$ because

$$\sup_{n\geq 1} \sum_{k=1}^{\infty} a_{nk} \mathbb{E}|X_k| I_{\{|X_k|>c\}} = \sup_{n\geq 1} nh(c) = \infty$$

and hence

$$\lim_{c\to\infty} \sup_{n\geq 1} \sum_{k=1}^{\infty} a_{nk} \mathbb{E}|X_k| I_{\{|X_k|>c\}} = \infty.$$

At the same time,

$$\mathcal{F} - \sup_{n\geq 1} \sum_{k=1}^{\infty} a_{nk} \mathbb{E}|X_k| I_{\{|X_k|>c\}} = \sup_{n\geq 1} h(c) = h(c)$$

by the same argument as above. Hence

$$\lim_{c\to\infty} \mathcal{F} - \sup_{n\geq 1} \sum_{k=1}^{\infty} a_{nk} \mathbb{E}|X_k| I_{\{|X_k|>c\}} = \lim_{c\to\infty} h(c) = 0.$$

Therefore, the sequence $\{X_k, k \geq 1\}$ is $\mathcal{F}$-uniformly integrable with respect to the array $\{a_{nk}\}$.

3 Two characterizations of $\mathcal{J}$-uniformly integrability with respect to an array of constants

The following theorem is a characterization of $\mathcal{J}$-uniformly integrability with respect to $\{a_{nk}\}$ and is an analogue of the classical uniform integrability criterion Theorem 1.

Theorem 3 *A sequence of random variables $\{X_k\}$ is $\mathcal{J}$-uniformly integrable with respect to $\{a_{nk}\}$ if and only if the following two conditions hold*:

i. $\mathcal{J} - \sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}|X_k| < \infty,$

ii. *For every $\varepsilon > 0$, there exist $\mu(\varepsilon) > 0$ such that*

$$\mathcal{J}-\sup_{n\in\mathbb{N}}\sum_{k=1}^{\infty}|a_{nk}|\mathbb{E}|X_k|I_{H_k}\leq\varepsilon$$

for any sequence $\{H_k\}$ *of events with*

$$\mathcal{J}-\sup_{n\in\mathbb{N}}\sum_{k=1}^{\infty}|a_{nk}|P(H_k)\leq\mu(\varepsilon). \tag{8}$$

Proof Assume that $\{X_k\}$ is a $\mathcal{J}$-uniformly integrable sequence of random variables with respect to $\{a_{nk}\}$. To prove **(i)** we fix $\varepsilon>0$. Then, there exists $\alpha>0$ such that

$$\left\{n\in\mathbb{N}:\sum_{k=1}^{\infty}|a_{nk}|\mathbb{E}|X_k|I_{\{|X_k|>\alpha\}}\geq\varepsilon/2\right\}\in\mathcal{J}. \tag{9}$$

From (6) and Remark 1 we have for any $\varepsilon>0$ that

$$\left\{n\in\mathbb{N}:\sum_{k=1}^{\infty}|a_{nk}|\geq1+\varepsilon/(2\alpha)\right\}\in\mathcal{J}. \tag{10}$$

Now,

$$\begin{aligned}&\left\{n\in\mathbb{N}:\sum_{k=1}^{\infty}|a_{nk}|<1+\varepsilon/(2\alpha)\right\}\\ &\subset\left\{n\in\mathbb{N}:\sum_{k=1}^{\infty}|a_{nk}|\mathbb{E}|X_k|I_{\{|X_k|\leq\alpha\}}\leq\alpha\sum_{k=1}^{\infty}|a_{nk}|<\alpha+\varepsilon/2\right\}.\end{aligned}$$

Taking complements of these sets, we obtain

$$\left\{n\in\mathbb{N}:\sum_{k=1}^{\infty}|a_{nk}|\mathbb{E}|X_k|I_{\{|X_k|\leq\alpha\}}\geq\alpha+\varepsilon/2\right\}\subset\left\{n\in\mathbb{N}:\sum_{k=1}^{\infty}|a_{nk}|\geq1+\varepsilon/(2\alpha)\right\}.$$

As $\mathcal{J}$ is an ideal by (10) we have $\left\{n\in\mathbb{N}:\sum_{k=1}^{\infty}|a_{nk}|\mathbb{E}|X_k|I_{\{|X_k|\leq\alpha\}}\geq\alpha+\varepsilon/2\right\}\in\mathcal{J}$. On the other hand, we have

$$\begin{aligned}&\left\{n\in\mathbb{N}:\sum_{k=1}^{\infty}|a_{nk}|\mathbb{E}|X_k|\geq\alpha+\varepsilon\right\}\\ &=\left\{n\in\mathbb{N}:\left(\sum_{k=1}^{\infty}|a_{nk}|\mathbb{E}|X_k|I_{\{|X_k|\leq\alpha\}}-\alpha\right)+\left(\sum_{k=1}^{\infty}|a_{nk}|\mathbb{E}|X_k|I_{\{|X_k|>\alpha\}}\right)\geq\varepsilon\right\}\\ &\subset\left\{n\in\mathbb{N}:\sum_{k=1}^{\infty}|a_{nk}|\mathbb{E}|X_k|I_{\{|X_k|\leq\alpha\}}\geq\alpha+\varepsilon/2\right\}\\ &\cup\left\{n\in\mathbb{N}:\sum_{k=1}^{\infty}|a_{nk}|\mathbb{E}|X_k|I_{\{|X_k|>\alpha\}}\geq\varepsilon/2\right\}\\ &\text{(by(1) of Remark3).}\end{aligned}$$

Now, since $\mathcal{J}$ is an ideal we obtain

$$\left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}|X_k| \geq \alpha + \varepsilon \right\} \in \mathcal{J}.$$

Therefore, any real number larger than $\alpha + \varepsilon$ is a $\mathcal{J}$-upper bound of the sequence $\left\{ \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}|X_k|, n \geq 1 \right\}$ which proves **(i).**

To prove **(ii)**, let us choose $\mu(\varepsilon) = \dfrac{\varepsilon}{2\alpha}$. For any sequence $\{H_k\}$ of events such that (8) is satisfied we obtain

$$\left\{ n \in \mathbb{N} : \alpha \sum_{k=1}^{\infty} |a_{nk}| P(H_k) > \varepsilon/2 \right\} \in \mathcal{J}. \tag{11}$$

Furthermore, we get by (1) of Remark 3 that

$$\begin{aligned}
&\left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}|X_k| I_{H_k} > \varepsilon \right\} \\
&\subset \left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}|X_k| I_{H_k \cap \{|X_k| \leq \alpha\}} > \varepsilon/2 \right\} \\
&\quad \cup \left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}|X_k| I_{H_k \cap \{|X_k| > \alpha\}} > \varepsilon/2 \right\} \\
&\subset \left\{ n \in \mathbb{N} : \alpha \sum_{k=1}^{\infty} |a_{nk}| P(H_k) > \varepsilon/2 \right\} \\
&\quad \cup \left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}|X_k| I_{\{|X_k| > \alpha\}} > \varepsilon/2 \right\}.
\end{aligned}$$

Considering that $\mathcal{J}$ is an ideal we have by (9) and (11) that

$$\left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}|X_k| I_{H_k} > \varepsilon \right\} \in \mathcal{J}.$$

Hence, **(ii)** holds.

Conversely, let **(i)** and **(ii)** hold. If $\mathcal{J} - \sup_{n \in \mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}|X_k| = K < \infty$, then by Remark 1 we obtain

$$\left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}|X_k| > K + \varepsilon \right\} \in \mathcal{J}. \tag{12}$$

Now, using Markov's inequality we get for any $c > 0$ that

$$\left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} |a_{nk}| P(|X_k| > c) > \frac{K + \varepsilon}{c} \right\} \subset \left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}|X_k| > K + \varepsilon \right\}.$$

As $\mathcal{J}$ is an ideal, by (12) we have

$$\left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} |a_{nk}| P(|X_k| > c) > \frac{K+\varepsilon}{c} \right\} \in \mathcal{J}. \tag{13}$$

Now, let $c > \frac{K+\varepsilon}{\mu}$. Then, we get

$$\left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} |a_{nk}| P(|X_k| > c) > \mu \right\} \subset \left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} |a_{nk}| P(|X_k| > c) > \frac{K+\varepsilon}{c} \right\}. \tag{14}$$

Hence we have by (13) and (14) that

$$\left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} |a_{nk}| P(|X_k| > c) > \mu \right\} \in \mathcal{J}. \tag{15}$$

Using the inequality

$$\mathbb{E}|X_k| I_{\{|X_k|>c\}} \geq c P(|X_k| > c)$$

and applying **(ii)** for the sequence of events $\left\{ |X_k| > c \right\}$ we have

$$\left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}|X_k| I_{\{|X_k|>c\}} > \varepsilon \right\} \subset \left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} |a_{nk}| P(|X_k| > c) > \mu \right\}.$$

By (15) we have $\left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}|X_k| I_{\{|X_k|>c\}} > \varepsilon \right\} \in \mathcal{J}$. Hence, we obtain

$$\mathcal{J} - \sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}|X_k| I_{\{|X_k|>c\}} \leq \varepsilon$$

which completes the proof. □

Remark 4 If we take $\mathcal{J}$ as the ideal of B-density zero sets for a non-negative regular summability matrix B, then by Theorem 3 we immediately get Theorem 1 of [22].

Being motivated by the classical de La Vallée Poussin type characterizations of uniform integrability Theorem 2, we give the following de La Vallée Poussin type characterization of $\mathcal{J}$-uniform integrability with respect to $\{a_{nk}\}$.

Theorem 4 *A sequence of random variables* $\{X_k\}$ *is* $\mathcal{J}$*-uniformly integrable with respect to* $\{a_{nk}\}$ *if and only if there exists a Borel measurable function* $\varphi : (0, \infty) \to (0, \infty)$ *such that* $\lim_{t\to\infty} \dfrac{\varphi(t)}{t} = \infty$ *and*

$$\mathcal{J} - \sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}\varphi(|X_k|) < \infty.$$

Proof Suppose that $\{X_k\}$ is $\mathcal{J}$-uniformly integrable with respect to $\{a_{nk}\}$. Then we can choose a sequence of positive integers $\{i_m\}$ such that for any $m \in \mathbb{N}$

$$\mathcal{J}-\sup_{n\in\mathbb{N}}\sum_{k=1}^{\infty}|a_{nk}|\mathbb{E}|X_k|I_{\{|X_k|>i_m\}}=K_m<\frac{1}{2^m}.$$

Thus, for any fixed $m\in\mathbb{N}$, by Remark 1 with $K=K_m$ and $\varepsilon=\frac{1}{2^m}-K_m>0$ we obtain that

$$\left\{n\in\mathbb{N}:\sum_{k=1}^{\infty}|a_{nk}|\mathbb{E}|X_k|I_{\{|X_k|>i_m\}}>\frac{1}{2^m}\right\}\in\mathcal{J}. \tag{16}$$

Furthermore, using the fact that if $\sum_{m=1}^{\infty}c_m>\sum_{m=1}^{\infty}c'_m$, then there exists $m_0\in\mathbb{N}$ such that $c_{m_0}>c'_{m_0}$, where c_m and c'_m are real numbers. This yields that there exists $m_0\in\mathbb{N}$ such that

$$\begin{aligned}&\left\{n\in\mathbb{N}:\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}|a_{nk}|\mathbb{E}|X_k|I_{\{|X_k|>i_m\}}>\sum_{m=1}^{\infty}\frac{1}{2^m}\right\}\\&\subset\left\{n\in\mathbb{N}:\sum_{k=1}^{\infty}|a_{nk}|\mathbb{E}|X_k|I_{\{|X_k|>i_{m_0}\}}>\frac{1}{2^{m_0}}\right\}.\end{aligned} \tag{17}$$

Considering $\sum_{m=1}^{\infty}\frac{1}{2^m}=1$ and $\mathcal{J}$ is an ideal we obtain by (16) and (17) that

$$\left\{n\in\mathbb{N}:\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}|a_{nk}|\mathbb{E}|X_k|I_{\{|X_k|>i_m\}}>1\right\}\in\mathcal{J}. \tag{18}$$

On the other hand, there exists a Borel measurable function (see, [20]) $\varphi:(0,\infty)\to(0,\infty)$ such that $\lim_{t\to\infty}\frac{\varphi(t)}{t}=\infty$ and for any $k\in\mathbb{N}$

$$\mathbb{E}\varphi\big(|X_k|\big)\le\sum_{m=1}^{\infty}\sum_{j=i_m}^{\infty}P\big(|X_k|>j\big).$$

Therefore, we have that (see [20])

$$\begin{aligned}&\left\{n\in\mathbb{N}:\sum_{k=1}^{\infty}|a_{nk}|\mathbb{E}\varphi\big(|X_k|\big)>1\right\}\subset\left\{n\in\mathbb{N}:\sum_{k=1}^{\infty}\left(|a_{nk}|\sum_{m=1}^{\infty}\sum_{j=i_m}^{\infty}P\big(|X_k|>j\big)\right)>1\right\}\\&\subset\left\{n\in\mathbb{N}:\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}|a_{nk}|\mathbb{E}|X_k|I_{\{|X_k|>i_m\}}>1\right\}.\end{aligned} \tag{19}$$

Now, by considering (18) and (19) we get

$$\left\{n\in\mathbb{N}:\sum_{k=1}^{\infty}|a_{nk}|\mathbb{E}\varphi\big(|X_k|\big)>1\right\}\in\mathcal{J}$$

which yields that $\mathcal{J}-\sup_{n\in\mathbb{N}}\sum_{k=1}^{\infty}|a_{nk}|\mathbb{E}\varphi(|X_k|)\le 1$.

Conversely, assume that there exists such a function φ that satisfies assumptions and let $\varepsilon>0$. Then considering Remark 1 we can write

$$\left\{n \in \mathbb{N} : \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}\varphi\big(|X_k|\big) > K + \varepsilon\right\} \in \mathcal{J}, \tag{20}$$

where $K := \mathcal{J} - \sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}\varphi(|X_k|)$. Moreover, there exists $\alpha > 0$ such that $\frac{\varphi(t)}{t} > \frac{K+\varepsilon+1}{\varepsilon}$ whenever $t > \alpha$. Thus, we obtain

$$\begin{aligned}
&\left\{n \in \mathbb{N} : \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}|X_k| I_{\{|X_k|>\alpha\}} > \varepsilon\right\} \\
&\subset \left\{n \in \mathbb{N} : \tfrac{\varepsilon}{K+\varepsilon+1} \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}\varphi\big(|X_k|\big) I_{\{|X_k|>\alpha\}} > \varepsilon\right\} \\
&= \left\{n \in \mathbb{N} : \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}\varphi\big(|X_k|\big) I_{\{|X_k|>\alpha\}} > K + \varepsilon + 1\right\} \\
&\subset \left\{n \in \mathbb{N} : \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}\varphi\big(|X_k|\big) I_{\{|X_k|>\alpha\}} > K + \varepsilon\right\}.
\end{aligned}$$

Hence, since $\mathcal{J}$ is an ideal, (20) yields that

$$\left\{n \in \mathbb{N} : \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}|X_k| I_{\{|X_k|>\alpha\}} > \varepsilon\right\} \in \mathcal{J}.$$

Hence, $\{X_k\}$ is $\mathcal{J}$-uniformly integrable with respect to $\{a_{nk}\}$. □

4 Laws of large numbers with mean ideal convergence

Let $\{X_k\}$ be a sequence of random variables with $\mathbb{E}|X_k|^p < \infty$, $k \geq 1$ where $p > 0$. If

$$\mathcal{J} - \lim_{k\to\infty} \mathbb{E}\big|X_k - X\big|^p = 0,$$

then $\{X_k\}$ is said to be *$\mathcal{J}$-convergent in the p^{th} mean* to the random variable X.

A sequence $\{X_k\}$ is said to be *$\mathcal{J}$-convergent to X in probability* if for any $\varepsilon, \nu > 0$

$$\big\{k \in \mathbb{N} : P(\big(\big|X_k - X\big|\big) \geq \nu\big) \geq \varepsilon\big\} \in \mathcal{J}$$

and we write $X_k \overset{\mathcal{J},P}{\to} X$. Various versions of such convergence can be found in [13, 15]. Considering the extended Markov's inequality, we get that $\mathcal{J}$-convergence in p^{th} mean implies $\mathcal{J}$-convergence in probability for $p > 0$.

Remark 5 Let $0 < p < q$. Assume that a sequence of random variables $\{X_k\}$ is $\mathcal{J}$-convergent to a random variable X in q^{th} mean. Then we have for any $\varepsilon > 0$ that

$$\big\{k \in \mathbb{N} : \mathbb{E}\big|X_k - X\big|^q \geq \varepsilon^{q/p}\big\} \in \mathcal{J}.$$

On the other hand, as $\big(\mathbb{E}\big|X_k - X\big|^p\big)^{1/p} \leq \big(\mathbb{E}\big|X_k - X\big|^q\big)^{1/q}$, we obtain for any $\varepsilon > 0$

$$\big\{k \in \mathbb{N} : \mathbb{E}\big|X_k - X\big|^p \geq \varepsilon\big\} \subset \big\{k \in \mathbb{N} : \mathbb{E}\big|X_k - X\big|^q \geq \varepsilon^{q/p}\big\}. \tag{21}$$

Since $\mathcal{J}$ is an ideal, from (21) we obtain

$$\left\{k \in \mathbb{N} : \mathbb{E}\left|X_k - X\right|^p \geq \varepsilon\right\} \in \mathcal{J},$$

which yields that $\{X_k\}$ is $\mathcal{J}$-convergent to a random variable X in p^{th} mean.

We need the following lemma for proving a main result. Recall that we say that two random variables X and Y are *uncorrelated*, if their covariance is zero, that is $\text{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)] = 0$. A sequence of random variables is said to be uncorrelated, if any two terms of it are uncorrelated.

Remark 6 The random series in Lemma 1, and Theorems 5–7 are assumed to be a.s. convergent for each $n \geq 1$. Of course, a.s. convergence is automatic for any $n \geq 1$ in which $a_{nk} = 0$ for all large k.

Lemma 1 *Let $\{X_k\}$ be an uncorrelated sequence of uniformly bounded random variables and let $\{a_{nk}\}$ be an array such that (6) holds and*

$$\mathcal{J}-\lim_{n\to\infty} \sup_{k\in\mathbb{N}} \left|a_{nk}\right| = 0. \tag{22}$$

Then

$$\mathcal{J}-\lim_{n\to\infty} \mathbb{E}\left(\sum_{k=1}^{\infty} a_{nk}(X_k - \mathbb{E}X_k)\right)^2 = 0. \tag{23}$$

Proof By (6) and Remark 1 by taking $\varepsilon = 1$ we obtain

$$\left\{n \in \mathbb{N} : \sum_{k=1}^{\infty} \left|a_{nk}\right| \geq 2\right\} \in \mathcal{J}. \tag{24}$$

Let $H > 0$ denote a uniform bound of $\{X_k\}$. Then by (22) and Remark 1 we have that for any $\varepsilon > 0$

$$\left\{n \in \mathbb{N} : \sup_{k\in\mathbb{N}} \left|a_{nk}\right| \geq \varepsilon/2H^2\right\} \in \mathcal{J}. \tag{25}$$

Considering the assumption that correlations are 0, we obtain

$$\left\{n \in \mathbb{N} : \mathbb{E}\left(\sum_{k=1}^{\infty} a_{nk}(X_k - \mathbb{E}X_k)\right)^2 \geq \varepsilon\right\}$$
$$= \left\{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk}^2 \mathbb{E}(X_k - \mathbb{E}X_k)^2 \geq \varepsilon\right\}$$
(by (2) of Remark 3)
$$\subset \left\{n \in \mathbb{N} : H^2 \sup_{k\in\mathbb{N}} |a_{nk}| \sum_{k=1}^{\infty} |a_{nk}| \geq \varepsilon\right\}$$
$$\subset \left\{n \in \mathbb{N} : \sup_{k\in\mathbb{N}} |a_{nk}| \geq \varepsilon/2H^2\right\} \cup \left\{n \in \mathbb{N} : \sum_{k=1}^{\infty} |a_{nk}| \geq 2\right\}.$$

From this, since $\mathcal{J}$ is an ideal by (24) and (25), we have

$$\left\{n \in \mathbb{N} : \mathbb{E}\left(\sum_{k=1}^{\infty} a_{nk}(X_k - \mathbb{E}X_k)\right)^2 \geq \varepsilon\right\} \in \mathcal{J}.$$

which yields (23). □

With the preliminaries accounted for, we can now state and prove the main results concerning $\mathcal{J}$-convergence in the p^{th} mean.

Theorem 5 *Let $\{X_k\}$ be a sequence independent random variables, $1 < p \leq 2$, and let $\{a_{nk}\}$ be an array such that (6) and (22) hold. If $\{|X_k|^p\}$ is $\mathcal{J}$-uniformly integrable with respect to $\{|a_{nk}|^p\}$, then*

$$\mathcal{J}-\lim_{n\to\infty} \mathbb{E}\left|\sum_{k=1}^{\infty} a_{nk}(X_k - \mathbb{E}X_k)\right|^p = 0$$

and, a fortiori,

$$\sum_{k=1}^{\infty} a_{nk}(X_k - \mathbb{E}X_k) \overset{\mathcal{J},P}{\to} 0.$$

Proof From $\mathcal{J}$-uniform integrability of $\{|X_k|^p\}$ with respect to $\{|a_{nk}|^p\}$, for any $\varepsilon > 0$ there exists $\alpha > 0$ (we write α instead of $\alpha^{1/p}$) such that

$$\mathcal{J}-\sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}|^p \mathbb{E}|X_k|^p I_{\{|X_k|>\alpha\}} < \varepsilon/(C_p 2^{2p}),$$

where C_p is the constant from the von Bahr-Essen inequality, (4) of Remark 3. By Remark 1 this implies

$$\left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} \left|a_{nk}\right|^p \mathbb{E}|X_k|^p I_{\{|X_k|>\alpha\}} \geq \varepsilon/(C_p 2^{2p}) \right\} \in \mathcal{J}. \tag{26}$$

Now, let us define

$$W_k = X_k I_{\{|X_k|\leq\alpha\}} \text{ and } T_k = X_k I_{\{|X_k|>\alpha\}}$$

for any $k \in \mathbb{N}$.

It is clear that the random variables $\{W_k - \mathbb{E}W_k\}$ are independent and hence uncorrelated with uniform bound 2α. Considering Lemma 1 and Remark 5 with $q = 2$, we obtain

$$\mathcal{J} - \lim_{n\to\infty} \mathbb{E}\left|\sum_{k=1}^{\infty} a_{nk}(W_k - \mathbb{E}W_k)\right|^p = 0.$$

Thus we get that

$$\left\{ n \in \mathbb{N} : \mathbb{E}\left|\sum_{k=1}^{\infty} a_{nk}(W_k - \mathbb{E}W_k)\right|^p \geq \varepsilon/2^p \right\} \in \mathcal{J}. \tag{27}$$

Furthermore, the random variables $\{T_k - \mathbb{E}T_k\}$ have mean zero and are independent. By the von Bahr-Essen inequality, (4) of Remark 3, we get

$$\begin{aligned} &\left\{ n \in \mathbb{N} : \mathbb{E}\left|\sum_{k=1}^{\infty} a_{nk}(T_k - \mathbb{E}T_k)\right|^p \geq \varepsilon/2^p \right\} \\ &\subset \left\{ n \in \mathbb{N} : C_p \sum_{k=1}^{\infty} |a_{nk}|^p \mathbb{E}|T_k - \mathbb{E}T_k|^p \geq \varepsilon/2^p \right\} \\ &\subset \left\{ n \in \mathbb{N} : 2^p C_p \sum_{k=1}^{\infty} |a_{nk}|^p \mathbb{E}|T_k|^p \geq \varepsilon/2^p \right\} \\ &\text{(by the } c_p\text{-inequality, (3) of Remark 3)} \\ &= \left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} |a_{nk}|^p \mathbb{E}|X_k|^p I_{\{|X_k|>\alpha\}} \geq \varepsilon/(C_p 2^{2p}) \right\}. \end{aligned}$$

From this and (26), since $\mathcal{J}$ is an ideal, we have

$$\left\{ n \in \mathbb{N} : \mathbb{E}\left|\sum_{k=1}^{\infty} a_{nk}(T_k - \mathbb{E}T_k)\right|^p \geq \varepsilon/2^p \right\} \in \mathcal{J}. \tag{28}$$

Finally, since

$$\left\{ n \in \mathbb{N} : \mathbb{E}\left| \sum_{k=1}^{\infty} a_{nk}(X_k - \mathbb{E}X_k) \right|^p \geq \varepsilon \right\}$$
$$= \left\{ n \in \mathbb{N} : \mathbb{E}\left| \left(\sum_{k=1}^{\infty} a_{nk}(W_k - \mathbb{E}W_k) \right) + \left(\sum_{k=1}^{\infty} a_{nk}(T_k - \mathbb{E}T_k) \right) \right|^p \geq \varepsilon \right\}$$
$$\subset \left\{ n \in \mathbb{N} : 2^{p-1} \left(\mathbb{E}\left| \sum_{k=1}^{\infty} a_{nk}(W_k - \mathbb{E}W_k) \right|^p + \mathbb{E}\left| \sum_{k=1}^{\infty} a_{nk}(T_k - \mathbb{E}T_k) \right|^p \right) \geq \varepsilon \right\}$$
$$\text{(by the } c_p\text{-inequality, (3) of Remark 3)}$$
$$\subset \left\{ n \in \mathbb{N} : \mathbb{E}\left| \sum_{k=1}^{\infty} a_{nk}(W_k - \mathbb{E}W_k) \right|^p \geq \varepsilon/2^p \right\}$$
$$\cup \left\{ n \in \mathbb{N} : \mathbb{E}\left| \sum_{k=1}^{\infty} a_{nk}(T_k - \mathbb{E}T_k) \right|^p \geq \varepsilon/2^p \right\}$$
$$\text{(by (1) of Remark 3).}$$

By (27) and (28) we obtain

$$\left\{ n \in \mathbb{N} : \mathbb{E}\left| \sum_{k=1}^{\infty} a_{nk}(X_k - \mathbb{E}X_k) \right|^p \geq \varepsilon \right\} \in \mathcal{J}.$$

Hence, the proof is completed. □

For the case $p = 1$, the assumption of the independence in Theorem 5 can be relaxed to the assumption of the pairwise independence.

Theorem 6 *Let $\{X_k\}$ be a sequence of pairwise independent random variables and let $\{a_{nk}\}$ be an array such that (6) and (22) hold. If $\{X_k\}$ is $\mathcal{J}$-uniformly integrable with respect to $\{a_{nk}\}$, then*

$$\mathcal{J}-\lim_{n\to\infty} \mathbb{E}\left| \sum_{k=1}^{\infty} a_{nk}(X_k - \mathbb{E}X_k) \right| = 0$$

and, a fortiori,

$$\sum_{k=1}^{\infty} a_{nk}(X_k - \mathbb{E}X_k) \overset{\mathcal{J},P}{\to} 0.$$

The proof of Theorem 6 repeats the proof of Theorem 5 (pairwise random variables are also uncorrelated), with the only difference being that in (28) we do not need to apply the von Bahr-Essen inequality, (4) of Remark 3.

For the case $0 < p < 1$, the assumption of any kind of independence can be dropped by strengthening the other conditions. We consider this fact in the following theorem.

Theorem 7 *Let $0 < p < 1$ and let $\{a_{nk}\}$ be an array satisfying (22) and*

$$\mathcal{J}-\sup_{n\in\mathbb{N}}\sum_{k=1}^{\infty}\left|a_{nk}\right|^{p}<\infty.$$

If $\left\{|X_k|^p\right\}$ *is a* $\mathcal{J}$*-uniformly integrable sequence of random variables with respect to* $\left\{\left|a_{nk}\right|^p\right\}$, *then*

$$\mathcal{J}-\lim_{n\to\infty}\mathbb{E}\left|\sum_{k=1}^{\infty}a_{nk}X_k\right|^{p}=0$$

and, *a fortiori*,

$$\sum_{k=1}^{\infty}a_{nk}X_k\overset{\mathcal{J},P}{\to}0.$$

Proof For any $\varepsilon>0$ there exists $\alpha>0$ such that

$$\mathcal{J}-\sup_{n\in\mathbb{N}}\sum_{k=1}^{\infty}\left|a_{nk}\right|^{p}\mathbb{E}|X_k|^{p}I_{\{|X_k|>\alpha\}}<\varepsilon/2$$

which yields that

$$\left\{n\in\mathbb{N}:\sum_{k=1}^{\infty}\left|a_{nk}\right|^{p}\mathbb{E}|X_k|^{p}I_{\{|X_k|>\alpha\}}>\varepsilon/2\right\}\in\mathcal{J}. \tag{29}$$

By (6) and Remark 1 by taking $\varepsilon=1$ we have that

$$\left\{n\in\mathbb{N}:\sum_{k=1}^{\infty}\left|a_{nk}\right|^{p}\geq 2\right\}\in\mathcal{J}. \tag{30}$$

On the other hand, $\mathcal{J}-\lim_{n\to\infty}\sup_{k\in\mathbb{N}}\left|a_{nk}\right|=0$ yields that

$$\left\{n\in\mathbb{N}:\sup_{k\in\mathbb{N}}\left|a_{nk}\right|\geq(\varepsilon/2\alpha)^{1/1-p}\right\}\in\mathcal{J}. \tag{31}$$

Now, defining sequences of random variables $\left\{W_k\right\}$ and $\left\{T_k\right\}$ as in Theorem 5 we get

$$\begin{aligned}
&\left\{n\in\mathbb{N}:\mathbb{E}\left|\sum_{k=1}^{\infty}a_{nk}W_k\right|\geq\varepsilon\right\}\\
&\subset\left\{n\in\mathbb{N}:\sum_{k=1}^{\infty}\left|a_{nk}\right|\mathbb{E}\left|W_k\right|\geq\varepsilon\right\}\\
&\subset\left\{n\in\mathbb{N}:\alpha\sup_{k\in\mathbb{N}}\left|a_{nk}\right|^{1-p}\sum_{k=1}^{\infty}\left|a_{nk}\right|^{p}\geq\varepsilon\right\}\\
&\subset\left\{n\in\mathbb{N}:\sup_{k\in\mathbb{N}}\left|a_{nk}\right|^{1-p}\geq\varepsilon/(2\alpha)\right\}\cup\left\{n\in\mathbb{N}:\sum_{k=1}^{\infty}\left|a_{nk}\right|^{p}\geq 2\right\}\\
&\text{(by (2) of Remark 3).}
\end{aligned}$$

Therefore, (30) and (31) imply

$$\left\{ n \in \mathbb{N} : \mathbb{E}\left| \sum_{k=1}^{\infty} a_{nk} W_k \right| \geq \varepsilon \right\} \in \mathcal{J}.$$

Hence, $\mathcal{J}-\lim_{n\to\infty} \mathbb{E}\left| \sum_{k=1}^{\infty} a_{nk} W_k \right| = 0$. By Remark 5 we get that

$$\mathcal{J}-\lim_{n\to\infty} \mathbb{E}\left| \sum_{k=1}^{\infty} a_{nk} W_k \right|^p = 0. \tag{32}$$

Thus, we obtain

$$\left\{ n \in \mathbb{N} : \mathbb{E}\left| \sum_{k=1}^{\infty} a_{nk} W_k \right|^p \geq \varepsilon/2 \right\} \in \mathcal{J}.$$

Moreover, since $p < 1$, we obtain

$$\begin{aligned}\left\{ n \in \mathbb{N} : \mathbb{E}\left| \sum_{k=1}^{\infty} a_{nk} T_k \right|^p \geq \varepsilon/2 \right\} &\subset \left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} |a_{nk}|^p \mathbb{E}|T_k|^p \geq \varepsilon/2 \right\} \\ &= \left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} |a_{nk}|^p \mathbb{E}|X_k|^p I_{\{|X_k|>\alpha\}} \geq \varepsilon/2 \right\}.\end{aligned}$$

Hence, by (29) we get

$$\left\{ n \in \mathbb{N} : \mathbb{E}\left| \sum_{k=1}^{\infty} a_{nk} T_k \right|^p \geq \varepsilon/2 \right\} \in \mathcal{J}. \tag{33}$$

Next, we have

$$\begin{aligned}&\left\{ n \in \mathbb{N} : \mathbb{E}\left| \sum_{k=1}^{\infty} a_{nk} X_k \right|^p \geq \varepsilon \right\} \\ &\subset \left\{ n \in \mathbb{N} : \mathbb{E}\left| \sum_{k=1}^{\infty} a_{nk} W_k \right|^p + \mathbb{E}\left| \sum_{k=1}^{\infty} a_{nk} T_k \right|^p \geq \varepsilon \right\} \\ &\subset \left\{ n \in \mathbb{N} : \mathbb{E}\left| \sum_{k=1}^{\infty} a_{nk} W_k \right|^p \geq \varepsilon/2 \right\} \cup \left\{ n \in \mathbb{N} : \mathbb{E}\left| \sum_{k=1}^{\infty} a_{nk} T_k \right|^p \geq \varepsilon/2 \right\} \\ &\text{(by (1) of Remark 3).}\end{aligned}$$

Hence, by (32) and (33), we get

$$\left\{ n \in \mathbb{N} : \mathbb{E}\left| \sum_{k=1}^{\infty} a_{nk} X_k \right|^p \geq \varepsilon \right\} \in \mathcal{J}$$

which completes the proof. □

5 Additional remarks on $\mathcal{J}$-uniform integrability

Let $y = (y_k)$ be a real sequence. If $\mathcal{J}_{fin} - \sup_{k\in\mathbb{N}} y_k = s < \infty$, then from Remark 1 we have

$$\left\{k \in \mathbb{N} : y_k > s + \varepsilon\right\} \in \mathcal{J}_{fin},$$

which means there exist finitely many $k \in \mathbb{N}$ such that $y_k > s + \varepsilon$. Hence, $\sup_{k\in\mathbb{N}} y_k < \infty$.

If a sequence of random variables $\{X_k\}$ is $\mathcal{J}_{fin}$-uniformly integrable with respect to $\left\{a_{nk}\right\}$, then from Theorem 4 there exists a Borel measurable function $\varphi : (0,\infty) \to (0,\infty)$ such that $\lim_{t\to\infty} \dfrac{\varphi(t)}{t} = \infty$ and

$$\mathcal{J}_{fin} - \sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}\varphi|X_k| < \infty.$$

From the observation presented above, we have

$$\sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}\varphi|X_k| < \infty.$$

Now, from the de La Vallée Poussin type characterizations of uniform integrability with respect to $\{a_{nk}\}$ (Theorem 3 of [20]), $\{X_k\}$ is uniformly integrable with respect to $\{a_{nk}\}$. On the other hand, from (5) it is clear that if $\{X_k\}$ is uniformly integrable with respect to $\{a_{nk}\}$, then it is $\mathcal{J}_{fin}$-uniformly integrable with respect to $\{a_{nk}\}$. Hence, $\mathcal{J}_{fin}$-uniform integrability with respect to $\{a_{nk}\}$ is equivalent to uniform integrability with respect to $\{a_{nk}\}$.

If we take $\mathcal{J}$ as the ideal of B-density zero subsets of $\mathbb{N}$ for a non-negative regular summability matrix B, then by Theorem 6 we obtain Theorem 3 of [22] and if we take $\mathcal{J}$ as $\mathcal{J}_{fin}$, then by Theorem 6 we obtain Theorem 4 of [20]. Furthermore, if we take $\mathcal{J}$ as the $\mathcal{J}_{fin}$ and if we take $\{a_{nk}\}$ to be the Cesàro array, then by Theorem 6 we obtain Theorem 1 of [5].

If we take $\mathcal{J}$ to be the ideal of B-density zero subsets of $\mathbb{N}$ for a non-negative regular summability matrix B, then by Theorem 7 we obtain Theorem 4 of [22] and if we take $\mathcal{J}$ to be the $\mathcal{J}_{fin}$, by Theorem 7 we obtain Theorem 5 of [20].

6 Supplements to the classical mean convergence criterion

We close by presenting two new results, Theorems 10 and 11, regarding uniform integrability in the classical sense.

The following theorem which is known as the classical Mean Convergence Criterion establishes the relationship between convergence in $\mathcal{L}_p$ for a sequence of random variables, convergence in probability, and uniform integrability in the classical sense (see, for example, [7] Chapter 4, Section 4.2).

Theorem 8 *Let $\{X_k\}$ be a sequence of random variables, let X be a random variable, and let $p > 0$.*

i. If $\{X_k\}$ is a sequence of $\mathcal{L}_p$ random variables and if $X_k \xrightarrow{\mathcal{L}_p} X$, then $X \in \mathcal{L}_p, X_k \xrightarrow{P} X$, and $\{|X_k|^p\}$ is uniformly integrable in the classical sense.

ii. If $\{|X_k|^p\}$ *is uniformly integrable in the classical sense and if* $X_k \xrightarrow{P} X$, *then* $X \in \mathcal{L}_p$ *and* $X_k \xrightarrow{\mathcal{L}_p} X$.

Let us now introduce the following notation. For a sequence of random variables $\{X_k\}$ and $p > 0$, let the sequence of set functions $\{Q_{p,k}\}$ be defined on the σ-algebra of events by

$$Q_{p,k}(A) = \mathbb{E}|X_k|^p I_A, k \geq 1.$$

We note that if $\{|X_k|^p\}$ is uniformly integrable in the classical sense, then by Theorem 1 the sequence of set functions $\{Q_{p,k}\}$ is uniformly absolutely continuous with respect to the probability measure P. Consequently, Theorem 8 (ii) is an immediate consequence of the following result which may be found in [18] Chapter 3, Section 9.

Theorem 9 *Let* $\{X_k\}$ *be a sequence of* $\mathcal{L}_p$ *random variables where* $p > 0$ *and let* X *be a random variable. If the sequence of set functions* $\{Q_{p,k}\}$ *is uniformly absolutely continuous with respect to the probability measure* P *and if* $X_k \xrightarrow{P} X$, *then*

$$X \in \mathcal{L}_p \text{ and } X_k \xrightarrow{\mathcal{L}_p} X.$$

We now present an improved version of Theorem 9.

Theorem 10 *Let* $\{X_k\}$ *be a sequence of random variables, let* X *be a random variable, and let* $p > 0$. *If the sequence of set functions* $\{Q_{p,k}\}$ *is uniformly absolutely continuous with respect to the probability measure* P *and if* $X_k \xrightarrow{P} X$, *then*

$$X_k \in \mathcal{L}_p, k \geq 1, X \in \mathcal{L}_p, X_k \xrightarrow{\mathcal{L}_p} X,$$

and $\{|X_k|^p\}$ *is uniformly integrable in the classical sense.*

Proof We first show that $X_k \in \mathcal{L}_p, k \geq 1$. By the uniformly absolutely continuous hypothesis and Remark 2, there exists $\delta > 0$ such that for every sequence of events $\{A_k\}$ with $P(A_k) < \delta, k \geq 1$,

$$\sup_{k \in \mathbb{N}} Q_{p,k}(A_k) < 1.$$

Thus for $k \geq 1$, choosing $c_k > 0$ so that $P(|X_k| > c_k) < \delta$, we have

$$\mathbb{E}|X_k|^p = Q_{p,k}([|X_k| \leq c_k]) + Q_{p,k}([|X_k| > c_k]) \leq c_k^p + 1 < \infty$$

and so $X_k \in \mathcal{L}_p, k \geq 1$.

Thus by Theorem 9,

$$X \in \mathcal{L}_p \text{ and } X_k \xrightarrow{\mathcal{L}_p} X.$$

Then by Theorem 8 (i), $\{|X_k|^p\}$ is uniformly integrable in the classical sense. □

While the equivalences between (ii), (iii), (iv), and (v) in the next theorem are well known and indeed are contained in Theorems 8 and 9, it is the added equivalence between (i) and (ii) which establishes the novelty of the theorem.

Theorem 11 *Let $\{X_k\}$ be a sequence of $\mathcal{L}_p$ random variables for some $p > 0$ and suppose that $X_k \xrightarrow{P} X$ for some random variable X. Then the following five statements are equivalent:*

i. $\lim_{k\to\infty} \mathbb{E}|X_k|^p = \mathbb{E}|X|^p < \infty$.
ii. $|X_k|^p \xrightarrow{\mathcal{L}_1} |X|$.
iii. $\{|X_k|^p\}$ *is uniformly integrable in the classical sense.*
iv. *The sequence of set functions $\{Q_{p,k}\}$ is uniformly absolutely continuous with respect to the probability measure P.*
v. $X_k \xrightarrow{\mathcal{L}_p} X$.

Proof It follows from Theorem 1 that (iii) implies (iv). Moreover, it follows from Theorem 9 that (iv) implies (v) and it follows from Theorem 8 (i) that (v) implies (iii). Thus (iii), (iv), and (v) are equivalent.

Next, let $Y_k = |X_k|^p, k \geq 1$ and $Y = |X|^p$. Since the function $|\cdot|^p$ is continuous, $Y_k \xrightarrow{P} Y$. To complete the proof, we must show that the following three statements are equivalent:

i$'$. $\lim_{k\to\infty} \mathbb{E}Y_k = \mathbb{E}Y < \infty$.
ii$'$. $Y_k \xrightarrow{\mathcal{L}_1} Y$.
iii$'$. $\{Y_k\}$ is uniformly integrable in the classical sense.

Now (ii$'$) and (iii$'$) are equivalent by Theorem 8 (with $p = 1$). Also,

$$|\mathbb{E}Y_k - \mathbb{E}Y| = |\mathbb{E}(Y_k - Y)| \leq \mathbb{E}|Y_k - Y|$$

and so (ii$'$) implies (i$'$). It remains to show that (i$'$) implies (ii$'$). It follows from the elementary identity

$$|a - b| = a + b - 2\min\{a, b\}$$

that

$$\begin{aligned} \mathbb{E}|Y_k - Y| &= \mathbb{E}(Y_k + Y - 2\min\{Y_k, Y\}) \\ &= \mathbb{E}Y_k + \mathbb{E}Y - 2\mathbb{E}(\min\{Y_k, Y\}), k \geq 1 \end{aligned} \tag{34}$$

Note that for all $k \geq 1$,

$$0 \leq \min\{Y_k, Y\} \leq Y \in \mathcal{L}_1 (\text{by (i}'))$$

and $Y_k \xrightarrow{P} Y$ ensures that

$$\min\{Y_k, Y\} = \frac{Y_k + Y}{2} - \frac{|Y_k - Y|}{2} \xrightarrow{P} \frac{Y + Y}{2} - 0 = Y.$$

Then by the Lebesgue dominated convergence theorem,

$$\lim_{k\to\infty} \mathbb{E}(\min\{Y_k, Y\}) = \mathbb{E}Y. \tag{35}$$

Then by (34), (i$'$), and (35),

$$\lim_{k\to\infty} \mathbb{E}|Y_k - Y| = \mathbb{E}Y + \mathbb{E}Y - 2\mathbb{E}Y = 0$$

and so (ii′) holds. □

In the following example, it is shown that the implication (i) $\Longrightarrow$ (ii) in Theorem 11 can fail if the condition $X_k \xrightarrow{P} X$ is dispensed with.

Example 2 Let $p = 1, C_2 > C_1 > 0$,

$$X_k = C_2 I_A + C_1 I_{A^c}, k \geq 1 \text{ and } X = C_1 I_A + C_2 I_{A^c}$$

where A is an event with $P(A) = \frac{1}{2}$. Now for $0 < \varepsilon < C_2 - C_1$,

$$P(|X_k - X| \geq \varepsilon) = P(|(C_2 - C_1)I_A + (C_1 - C_2)I_{A^c}| \geq \varepsilon) = 1 \nrightarrow 0$$

so $X_k \overset{P}{\nrightarrow} X$. The condition (i) holds since

$$\mathbb{E}|X_k| = \frac{C_2 + C_1}{2} = \mathbb{E}|X|, k \geq 1$$

but the condition (ii) fails since

$$\mathbb{E}\big||X_k| - |X|\big| = \mathbb{E}|(C_2 - C_1)I_A + (C_1 - C_2)I_{A^c}| = C_2 - C_1 \nrightarrow 0.$$

The following example of a sequence of $\mathcal{L}_1$ random variables $X_k, k \geq 1$ is such that X_k converges in probability to a random variable X and the conditions (i)–(v) of Theorem 11 fail.

Example 3 Let $p = 1$ and let $X_k, k \geq 1$ be a sequence of random variables with

$$P(X_k = 0) = 1 - (1/k), P(X_k = k) = 1/k, k \geq 1.$$

Then X_k converges in probability to 0 and $\mathbb{E}X_k = 1$ does not converge to 0. Thus condition (i) of Theorem 11 fails and so by Theorem 11, the conditions (ii)–(v) also fail.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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