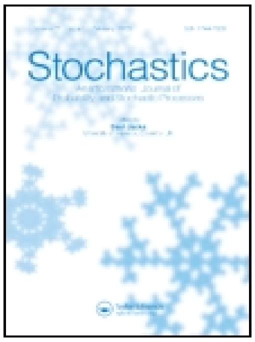

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Complete moment convergence for arrays of rowwise NSD random variables

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ABSTRACT

In this paper, the complete convergence and complete moment convergence for arrays of rowwise negatively superadditive dependent (NSD, in short) random variables are investigated. Some sufficient conditions to prove the complete convergence and the complete moment convergence are presented. The results obtained in the paper generalize and improve some corresponding ones for independent random variables and negatively associated random variables.

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1. Introduction

Firstly, let us recall some definitions of the negative dependence. The first one is the concept of negatively associated (NA) random variables, which was introduced by Alam and Saxena [1] and carefully studied by Joag-Dev and Proschan [10].

Definition 1.1: A finite collection of random variables $X_1, X_2, \ldots, X_n$ is said to be NA if for every pair of disjoint subsets A_1, A_2 of $\{1, 2, \ldots, n\}$,

$$Cov\{f(X_i : i \in A_1), g(X_j : j \in A_2)\} \leq 0,$$

whenever f and g are coordinatewise nondecreasing such that this covariance exists. An infinite sequence $\{X_n, n \geq 1\}$ of random variables is NA if every finite subcollection is NA.

An array $\{X_{ni}, i \geq 1, n \geq 1\}$ of random variables is said to be rowwise NA if for all $n \geq 1$, $\{X_{ni}, i \geq 1\}$ is NA.

The next one is the concept of negatively superadditive dependence, which is based on the superadditive function introduced by Kemperman [11] as follows.

Definition 1.2: (cf. [11]) A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is called superadditive if $\phi(\mathbf{x} \vee \mathbf{y}) + \phi(\mathbf{x} \wedge \mathbf{y}) \geq \phi(\mathbf{x}) + \phi(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}$, where $\vee$ is for componentwise maximum and $\wedge$ is for componentwise minimum.

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Definition 1.3: (cf. [9]) A random vector $\mathbf{X} = (X_1, X_2, \dots, X_n)$ is said to be negatively superadditive dependent (NSD, in short) if

$$E\phi(X_1, X_2, \dots, X_n) \leq E\phi(X_1^*, X_2^*, \dots, X_n^*), \tag{1.1}$$

where $X_1^*, X_2^*, \dots, X_n^*$ are independent such that X_i^* and X_i have the same distribution for each i and ϕ is a superadditive function such that the expectations in (1.1) exist.

A sequence $\{X_n, n \geq 1\}$ of random variables is said to be NSD if for every $n \geq 1$, $(X_1, X_2, \dots, X_n)$ is NSD.

An array $\{X_{ni}, i \geq 1, n \geq 1\}$ of random variables is said to be rowwise NSD if for all $n \geq 1$, $\{X_{ni}, i \geq 1\}$ is NSD.

The concept of NSD random variables, which was introduced by Hu [9], was based on the class of superadditive functions. Hu [9] gave an example illustrating that NSD does not imply negative association (NA, in short, see [10]), and posed an open problem whether NA implies NSD. In addition, Hu provided some basic properties and three structural theorems for NSD random variables. Christofides and Vaggelatou [5] solved this open problem and indicated that NA implies NSD. Negatively superadditive dependent structure is an extension of negatively associated structure and sometimes more useful than it and can be used to get many important probability inequalities and moment inequalities. Eghbal et al. [6] derived two maximal inequalities and strong law of large numbers of quadratic forms of NSD random variables under the assumption that $\{X_i, i \geq 1\}$ is a sequence of nonnegative NSD random variables with $EX_i^r < \infty$ for all $i \geq 1$ and some $r > 1$. [13] established the strong limit theorems for NSD random variables. Wang et al. [15] obtained the complete convergence for arrays of rowwise NSD random variables and gave its applications to nonparametric regression models. Shen et al. [14] gave some applications of the Rosenthal-type inequality for negatively superadditive dependent random variables, and so forth. For more details about the probability limiting behavior, one can refer to Wu [21–23], Wu and Jiang [24], Wang et al. [16,17], and so forth. The main purpose of the paper is to further study the complete convergence and complete moment convergence for arrays of rowwise NSD random variables.

Throughout the paper, let $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ be an array of random variables defined on a probability space $(\Omega, \mathscr{F}, P)$, $\{k_n, n \geq 1\}$ be a sequence of positive integers such that $\lim_{n\to\infty} k_n = \infty$, and $\{c_n, n \geq 1\}$ be a sequence of positive constants. Let C be a positive constant not depending on n, which may be different in various places. $I(A)$ denotes the indicator function of set A. Denote $X_+ = XI(X > 0)$.

The following concept of slowly varying function will be used in this work.

Definition 1.4: A real-valued function $l(x)$, positive and measurable on $(0, \infty)$, is said to be slowly varying if

$$\lim_{x\to\infty} \frac{l(\lambda x)}{l(x)} = 1 \quad \text{for each } \lambda > 0.$$

The concept of complete convergence was introduced by Hsu and Robbins [8] as follows: a sequence of random variables $\{U_n, n \geq 1\}$ is said to *converge completely* to a constant C if $\sum_{n=1}^{\infty} P(|U_n - C| > \varepsilon) < \infty$ for all $\varepsilon > 0$.

In view of the Borel–Cantelli lemma, this implies that $U_n \to C$ almost surely (a.s.). The converse is true if the $\{U_n, n \geq 1\}$ are independent. Hsu and Robbins [8] proved that the sequence of arithmetic means of independent and identically distributed (i.i.d.) random variables converges completely to the expected value if the variance of the summands is finite. Erdös [7] proved the converse. The result of Hsu–Robbins–Erdös is a fundamental theorem in probability theory and has been generalized and extended in several directions by many authors. Recently, Chen et al. [3] established the following complete convergence result for arrays of rowwise NA random variables.

Theorem 1.1: *Let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of rowwise NA random variables and $\{c_n, n \geq 1\}$ be a sequence of positive constants. Suppose that the following conditions are satisfied*:

(i) *for every $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} c_n \sum_{i=1}^{k_n} P(|X_{ni}| > \varepsilon) < \infty, \tag{1.2}$$

(ii) *for some $\delta > 0$, there exists $J \geq 1$ such that*

$$\sum_{n=1}^{\infty} c_n \left(\sum_{i=1}^{k_n} Var(X_{ni} I(|X_{ni}| \leq \delta)) \right)^J < \infty. \tag{1.3}$$

Then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n P\left(\max_{1 \leq m \leq k_n} \left| \sum_{i=1}^{m} (X_{ni} - EX_{ni} I(|X_{ni}| \leq \delta)) \right| > \varepsilon \right) < \infty. \tag{1.4}$$

Chow [4] generalized the concept of complete convergence and introduced the concept of complete moment convergence, which is more general than complete convergence. Let $\{Z_n, n \geq 1\}$ be a sequence of random variables and $a_n > 0$, $b_n > 0$, $q > 0$. If

$$\sum_{n=1}^{\infty} a_n E\{b_n^{-1}|Z_n| - \varepsilon\}_+^q < \infty \quad \text{for all } \varepsilon > 0,$$

then the above result was called the complete moment convergence.

Chow [4] obtained the following complete moment convergence result for i.i.d. random variables.

Theorem 1.2: *Suppose that $\{X_n, n \geq 1\}$ is a sequence of i.i.d. random variables with $EX_1 = 0, \alpha > 1/2, p \geq 1$ and $\alpha p > 1$. If $E[|X_1|^p + |X_1| \log(1 + |X_1|)] < \infty$, then for all $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{\alpha p - 2 - \alpha} E\left\{ \max_{1 \leq j \leq n} \left| \sum_{k=1}^{j} X_k \right| - \varepsilon n^{\alpha} \right\}_+ < \infty. \tag{1.5}$$

Wang and Zhao [19] extended the result of Theorem 1.2 for i.i.d. random variables to the case of NA random variables, and obtained the following result.

Theorem 1.3: *Let $\{X_k, k \geq 1\}$ be a sequence of NA random variables with $EX_k = 0$. Suppose that there exists a constant $C > 0$ such that $\sup_{k\geq 1} P(|X_k| > x) \leq CP(|X| > x)$ for all $x > 0$. Let $l(x) > 0$ be a slowly varying function as $x \to \infty$. If for $\alpha > 1/2$, $\alpha p > 1$ and $1 \leq q < p$, $E|X|^p l(|X|^{1/\alpha}) < \infty$, then*

$$\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q} l(n) E\left\{\max_{1\leq j\leq n}\left|\sum_{k=1}^{j} X_k\right| - \varepsilon n^{\alpha}\right\}_+^q < \infty \quad \textit{for all } \varepsilon > 0. \tag{1.6}$$

Wu [25] improved the result of Theorem 1.3 under weaker conditions. Inspired by Chen et al. [3], Wang and Zhao [19] and Wu [25], we will extend and improve the results of Theorem 1.3 for NA random variables to the case of NSD random variables.

2. Preliminaries

To prove the main results of the paper, we need the following important lemmas. The first one is a basic property for slowly varying function, which was established by Bai and Su [2].

Lemma 2.1: *If $l(x) > 0$ is a slowly varying function as $x \to \infty$, then*

(i) $\lim_{x\to\infty} \frac{l(x+u)}{l(x)}$ *for each $u > 0$;*
(ii) $\lim_{k\to\infty} \sup_{2^k \leq x < 2^{k+1}} \frac{l(x)}{l(2^k)} = 1$;
(iii) $\lim_{x\to\infty} x^{\delta} l(x) = \infty$, $\lim_{x\to\infty} x^{-\delta} l(x) = 0$ *for each $\delta > 0$;*
(iv) $c_1 2^{kr} l(\varepsilon 2^k) \leq \sum_{j=1}^{k} 2^{jr} l(\varepsilon 2^j) \leq c_2 2^{kr} l(\varepsilon 2^k)$ *for every $r > 0, \varepsilon > 0$, positive integer k, and some constants $c_1 > 0, c_2 > 0$;*
(v) $c_3 2^{kr} l(\varepsilon 2^k) \leq \sum_{j=k}^{\infty} 2^{jr} l(\varepsilon 2^j) \leq c_4 2^{kr} l(\varepsilon 2^k)$ *for every $r < 0, \varepsilon > 0$, positive integer k, and some constants $c_3 > 0, c_4 > 0$.*

The following one was presented by Hu [9].

Lemma 2.2: *Let $(X_1, X_2, \ldots, X_n)$ be NSD.*

(i) $(-X_1, -X_2, \ldots, -X_n)$ *is also NSD.*
(ii) *If $g_1, g_2, \ldots, g_n$ are all nondecreasing functions, then $(g_1(X_1), g_2(X_2), \ldots, g_n(X_n))$ is NSD.*

Remark 2.1: Let $(X_1, X_2, \ldots, X_n)$ be NSD. Together with (i) and (ii) in Lemma 2.2, we can see that if $g_1, g_2, \ldots, g_n$ are all nondecreasing (or all nonincreasing) functions, then $(g_1(X_1), g_2(X_2), \ldots, g_n(X_n))$ are NSD.

The next one is the Kolmogorov exponential type inequality for NSD random variables, which was established by Wang et al. [15].

Lemma 2.3: *Let $\{X_n, n \geq 1\}$ be a sequence of NSD random variables with zero mean and finite second moments. Denote $S_n = \sum_{i=1}^{n} X_i$ and $B_n = \sum_{i=1}^{n} EX_i^2$ for each $n \geq 1$. Then for all $x > 0$, $y > 0$ and $n \geq 1$*

$$P\left(\max_{1\leq k\leq n} |S_k| \geq x\right) \leq 2P\left(\max_{1\leq k\leq n} |X_k| > y\right) + 8\left(\frac{2B_n}{3xy}\right)^{x/12y}.$$

With Lemma 2.3 accounted for, we can get the following complete convergence for arrays of rowwise NSD random variables, which is a generalization of Theorem 1.1. The proof is similar to that of Theorem 1.1 or Lemma 3.1 of Shen [12]. So we omit the details.

Lemma 2.4: *Let $\{X_{ni}, 1 \le i \le k_n, n \ge 1\}$ be an array of rowwise NSD random variables and $\{c_n, n \ge 1\}$ be a sequence of positive constants. Suppose that the following conditions are satisfied*:

(i) *for every $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} c_n \sum_{i=1}^{k_n} P(|X_{ni}| > \varepsilon) < \infty, \tag{2.1}$$

(ii) *for some $\delta > 0$, there exists $J \ge 1$ such that*

$$\sum_{n=1}^{\infty} c_n \left(\sum_{i=1}^{k_n} Var(X_{ni} I(|X_{ni}| \le \delta)) \right)^J < \infty. \tag{2.2}$$

Then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n P\left(\max_{1 \le m \le k_n} \left| \sum_{i=1}^{m} (X_{ni} - EX_{ni} I(|X_{ni}| \le \delta)) \right| > \varepsilon \right) < \infty. \tag{2.3}$$

The following concept of stochastic domination will be used in this work.

Definition 2.1: A sequence $\{X_n, n \ge 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$P(|X_n| > x) \le CP(|X| > x)$$

for all $x \ge 0$ and $n \ge 1$.

By the definition of stochastic domination and integration by parts, we can get the following property for stochastic domination. For the details of the proof, one can refer to Wu [20] or Wang et al. [18].

Lemma 2.5: *Let $\{X_n, n \ge 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X. For any $\alpha > 0$ and $b > 0$, the following two statements hold*:

$$E|X_n|^\alpha I\left(|X_n| \le b\right) \le C_1 \left[E|X|^\alpha I\left(|X| \le b\right) + b^\alpha P\left(|X| > b\right) \right], \tag{2.4}$$

$$E|X_n|^\alpha I\left(|X_n| > b\right) \le C_2 E|X|^\alpha I\left(|X| > b\right), \tag{2.5}$$

where C_1 and C_2 are positive constants. Consequently, $E|X_n|^\alpha \le CE|X|^\alpha$, where C is a positive constant.

3. Main results and their proofs

Our main results are as follows.

Theorem 3.1: *Let $q \geq 1$, $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ be an array of rowwise NSD random variables, and $\{c_n, n \geq 1\}$ be a sequence of positive constants. Suppose that the following conditions are satisfied:*

(i) *for every $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} E|X_{nk}|^q I(|X_{nk}| > \varepsilon) < \infty, \tag{3.1}$$

(ii) *for some $\delta > 0$, there exists $\eta > q$ such that*

$$\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} EX_{nk}^2 I(|X_{nk}| \leq \delta) \right)^{\eta} < \infty, \tag{3.2}$$

and

$$\sum_{k=1}^{k_n} E|X_{nk}|^q I\left(|X_{nk}| > \frac{\delta}{128\eta}\right) \to 0, \quad \text{as } n \to \infty. \tag{3.3}$$

Then for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n E \left\{ \max_{1 \leq m \leq k_n} \left| \sum_{k=1}^{m} (X_{nk} - EX_{nk} I(|X_{nk}| \leq \delta)) \right| - \varepsilon \right\}_+^q < \infty. \tag{3.4}$$

Proof: For fixed $n \geq 1$, denote $S_m = \sum_{k=1}^{m} (X_{nk} - EX_{nk} I(|X_{nk}| \leq \delta))$ for $m = 1, 2, \ldots, k_n$. For any fixed $\varepsilon > 0$, without loss of generality, we may assume that $0 < \varepsilon < \delta$. It is easily seen that

$$\begin{aligned}
&\sum_{n=1}^{\infty} c_n E \left\{ \max_{1 \leq m \leq k_n} |S_m| - \varepsilon \right\}_+^q = \sum_{n=1}^{\infty} c_n \int_0^{\infty} P\left(\max_{1 \leq m \leq k_n} |S_m| - \varepsilon > t^{1/q} \right) \mathrm{d}t \\
&\quad = \sum_{n=1}^{\infty} c_n \left[\int_0^{\delta^q} P\left(\max_{1 \leq m \leq k_n} |S_m| > \varepsilon + t^{1/q} \right) \mathrm{d}t + \int_{\delta^q}^{\infty} P\left(\max_{1 \leq m \leq k_n} |S_m| > \varepsilon + t^{1/q} \right) \mathrm{d}t \right] \\
&\quad \leq \delta^q \sum_{n=1}^{\infty} c_n P\left(\max_{1 \leq m \leq k_n} |S_m| > \varepsilon \right) + \sum_{n=1}^{\infty} c_n \int_{\delta^q}^{\infty} P\left(\max_{1 \leq m \leq k_n} |S_m| > t^{1/q} \right) \mathrm{d}t \\
&\quad \doteq H_1 + H_2.
\end{aligned} \tag{3.5}$$

In order to prove (3.4), we only need to show that $H_1 < \infty$ and $H_2 < \infty$. Noting that (3.1) implies (2.1) and (3.2) implies (2.2), we have $H_1 < \infty$ by Lemma 2.4. In the following, we will show that $H_2 < \infty$.

For $t \geq \delta^q$, denote for $1 \leq k \leq k_n$ and $n \geq 1$ that

$$\begin{aligned}
Y_{nk} &= -t^{1/q} I(X_{nk} < -t^{1/q}) + X_{nk} I(|X_{nk}| \leq t^{1/q}) + t^{1/q} I(X_{nk} > t^{1/q}), \\
Z_{nk} &= -t^{1/q} I(X_{nk} < -t^{1/q}) + t^{1/q} I(X_{nk} > t^{1/q}).
\end{aligned}$$

It is easily seen that

$$\begin{aligned}
P\left(\max_{1\leq m\leq k_n} |S_m| > t^{1/q}\right) &= P\left(\max_{1\leq m\leq k_n} |S_m| > t^{1/q},\ |X_{nk}| > t^{1/q} \quad \text{for some } 1 \leq k \leq k_n\right) \\
&\quad + P\left(\max_{1\leq m\leq k_n} |S_m| > t^{1/q},\ |X_{nk}| \leq t^{1/q} \quad \text{for all } 1 \leq k \leq k_n\right) \\
&\leq \sum_{k=1}^{k_n} P(|X_{nk}| > t^{1/q}) + P\Bigg(\max_{1\leq m\leq k_n} \Bigg|\sum_{k=1}^{m} (X_{nk} I(|X_{nk}| \leq t^{1/q}) \\
&\qquad - EX_{nk} I(|X_{nk}| \leq \delta))\Bigg| > t^{1/q}\Bigg),
\end{aligned}$$

which implies that

$$\begin{aligned}
H_2 &\leq \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \int_{\delta^q}^{\infty} P\left(|X_{nk}| > t^{1/q}\right) \mathrm{d}t \\
&\quad + \sum_{n=1}^{\infty} c_n \int_{\delta^q}^{\infty} P\left(\max_{1\leq m\leq k_n} \left|\sum_{k=1}^{m} (X_{nk} I(|X_{nk}| \leq t^{1/q}) - EX_{nk} I(|X_{nk}| \leq \delta))\right| > t^{1/q}\right) \mathrm{d}t \\
&\doteq H_3 + H_4.
\end{aligned}$$

It follows by (3.1) that

$$H_3 \leq \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} E|X_{nk}|^q I(|X_{nk}| > \delta) < \infty.$$

To prove $H_2 < \infty$, it suffices to show $H_4 < \infty$. For $t \geq \delta^q$, we have

$$\begin{aligned}
&P\left(\max_{1\leq m\leq k_n} \left|\sum_{k=1}^{m} (X_{nk} I(|X_{nk}| \leq t^{1/q}) - EX_{nk} I(|X_{nk}| \leq \delta))\right| > t^{1/q}\right) \\
&\quad = P\left(\max_{1\leq m\leq k_n} \left|\sum_{k=1}^{m} (Y_{nk} - EY_{nk} - Z_{nk} + EZ_{nk} + EX_{nk} I(\delta < |X_{nk}| \leq t^{1/q}))\right| > t^{1/q}\right) \\
&\quad \leq P\Bigg(\max_{1\leq m\leq k_n} \left|\sum_{k=1}^{m} (Y_{nk} - EY_{nk} - Z_{nk} + EZ_{nk})\right| \\
&\qquad\qquad + \max_{1\leq m\leq k_n} \left|\sum_{k=1}^{m} EX_{nk} I(\delta < |X_{nk}| \leq t^{1/q})\right| > t^{1/q}\Bigg).
\end{aligned} \tag{3.6}$$

It follows by (3.3) that

$$\begin{aligned}
&\max_{t\geq\delta^q} t^{-1/q} \max_{1\leq m\leq k_n}\left|\sum_{k=1}^{m} EX_{nk}I(\delta<|X_{nk}|\leq t^{1/q})\right| \\
&\leq \max_{t\geq\delta^q} t^{-1/q}\sum_{k=1}^{k_n} E|X_{nk}|I(\delta<|X_{nk}|\leq t^{1/q}) \\
&\leq \max_{t\geq\delta^q}\sum_{k=1}^{k_n} E\frac{|X_{nk}|}{\delta}I(\delta<|X_{nk}|\leq t^{1/q}) \\
&\leq \delta^{-q}\sum_{k=1}^{k_n} E|X_{nk}|^q I(|X_{nk}|>\delta) \ \to\ 0, \quad \text{as} \quad n\to\infty,
\end{aligned}$$

which yields that for all n large enough,

$$\max_{1\leq m\leq k_n}\left|\sum_{k=1}^{m} EX_{nk}I(\delta<|X_{nk}|\leq t^{1/q})\right| < \frac{t^{1/q}}{2}, \quad \text{for all } t\geq\delta q. \tag{3.7}$$

Combining (3.6) and (3.7), we can get that for all n large enough and $t\geq\delta q$,

$$\begin{aligned}
&P\left(\max_{1\leq m\leq k_n}\left|\sum_{k=1}^{m}(X_{nk}I(|X_{nk}|\leq t^{1/q})-EX_{nk}I(|X_{nk}|\leq\delta))\right|>t^{1/q}\right) \\
&\leq P\left(\max_{1\leq m\leq k_n}\left|\sum_{k=1}^{m}(Y_{nk}-EY_{nk})-\sum_{k=1}^{m}(Z_{nk}-EZ_{nk})\right|>t^{1/q}/2\right). \\
&\leq P\left(\max_{1\leq m\leq k_n}\left|\sum_{k=1}^{m}(Y_{nk}-EY_{nk})\right|>\frac{t^{1/q}}{4}\right)+P\left(\max_{1\leq m\leq k_n}\left|\sum_{k=1}^{m}(Z_{nk}-EZ_{nk})\right|>\frac{t^{1/q}}{4}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
H_4 &\leq C\sum_{n=1}^{\infty}c_n\int_{\delta^q}^{\infty}P\left(\max_{1\leq m\leq k_n}\left|\sum_{k=1}^{m}(Z_{nk}-EZ_{nk})\right|>\frac{t^{1/q}}{4}\right)\mathrm{d}t \\
&\quad + C\sum_{n=1}^{\infty}c_n\int_{\delta^q}^{\infty}P\left(\max_{1\leq m\leq k_n}\left|\sum_{k=1}^{m}(Y_{nk}-EY_{nk})\right|>\frac{t^{1/q}}{4}\right)\mathrm{d}t \\
&\doteq CH_5+CH_6.
\end{aligned} \tag{3.8}$$

It follows by Markov's inequality and (3.1) that

$$\begin{aligned}
H_5 &\leq C\sum_{n=1}^{\infty}c_n\sum_{k=1}^{k_n}\int_{\delta^q}^{\infty}t^{-1/q}E|Z_{nk}|\mathrm{d}t \leq C\sum_{n=1}^{\infty}c_n\sum_{k=1}^{k_n}\int_{\delta^q}^{\infty}P(|X_{nk}|>t^{1/q})\mathrm{d}t \\
&\leq C\sum_{n=1}^{\infty}c_n\sum_{k=1}^{k_n}E|X_{nk}|^q I(|X_{nk}|>\delta)<\infty.
\end{aligned}$$

Next, we will prove $H_6 < \infty$. Denote $B_n = \sum_{k=1}^{k_n} E(Y_{nk} - EY_{nk})^2$. It is easily seen that for fixed $n \geq 1$, $\{Y_{nk} - EY_{nk}, 1 \leq k \leq k_n\}$ are still NSD random variables by Lemma 2.2. Applying Lemma 2.3 with $x = t^{1/q}/4$ and $y = t^{1/q}/(48\eta)$, we have

$$\begin{aligned} H_6 &\leq C \sum_{n=1}^{\infty} c_n \int_{\delta^q}^{\infty} P\left(\max_{1\leq k\leq k_n} |Y_{nk} - EY_{nk}| > t^{1/q}/(48\eta)\right) \mathrm{d}t \\ &\quad + C \sum_{n=1}^{\infty} c_n \int_{\delta^q}^{\infty} \left(\frac{B_n}{t^{2/q}}\right)^{\eta} \mathrm{d}t \\ &\doteq CH_7 + CH_8. \end{aligned}$$

By (3.3), we can see that for all n large enough,

$$\sum_{k=1}^{k_n} P\left(|X_{nk}| > \frac{\delta}{128\eta}\right) \leq \sum_{k=1}^{k_n} E|X_{nk}|^q I\left(|X_{nk}| > \frac{\delta}{128\eta}\right) < \frac{1}{256\eta},$$

which implies that for all n large enough,

$$\begin{aligned} &\max_{t\geq\delta^q} \max_{1\leq k\leq k_n} t^{-1/q}|EY_{nk}| \leq \max_{t\geq\delta^q} \max_{1\leq k\leq k_n} t^{-1/q}E|Y_{nk}| \\ &\quad\leq \max_{t\geq\delta^q} \max_{1\leq k\leq k_n} [t^{-1/q}E|X_{nk}|I(|X_{nk}| \leq \delta/128\eta) \\ &\qquad + t^{-1/q}E|X_{nk}|I(\delta/128\eta < |X_{nk}| \leq t^{1/q}) + P(|X_{nk}| > t^{1/q})] \\ &\quad\leq \max_{t\geq\delta^q} \max_{1\leq k\leq k_n} \left[t^{-1/q}\delta/128\eta + P(|X_{nk}| > \delta/128\eta) + P(|X_{nk}| > t^{1/q})\right] \\ &\quad\leq \frac{1}{128\eta} + \sum_{k=1}^{k_n} P\left(|X_{nk}| > \frac{\delta}{128\eta}\right) + \sum_{k=1}^{k_n} P(|X_{nk}| > \delta) \\ &\quad\leq \frac{1}{128\eta} + 2\sum_{k=1}^{k_n} P\left(|X_{nk}| > \frac{\delta}{128\eta}\right) < \frac{1}{64\eta}. \end{aligned}$$

Hence, we can get that for all n large enough,

$$\max_{1\leq k\leq k_n} E|Y_{nk}| < \frac{t^{1/q}}{64\eta}, \quad t \geq \delta^q. \tag{3.9}$$

Noting that $|Y_{nk}| \leq |X_{nk}|$, we have by (3.9) and (3.1) that

$$\begin{aligned} H_7 &\leq C \sum_{n=1}^{\infty} c_n \int_{\delta^q}^{\infty} P\left(\max_{1\leq k\leq k_n} |X_{nk}| > \frac{t^{1/q}}{192\eta}\right) \mathrm{d}t \\ &\leq C \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \int_{\delta^q}^{\infty} P\left(|X_{nk}| > \frac{t^{1/q}}{192\eta}\right) \mathrm{d}t \\ &\leq C \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} E|X_{nk}|^q I\left(|X_{nk}| > \frac{\delta}{192\eta}\right) < \infty. \end{aligned}$$

Since $\eta > q \geq 1$, it follows by the C_r-inequality that

$$
\begin{aligned}
H_8 &\leq C\sum_{n=1}^{\infty} c_n \int_{\delta^q}^{\infty} (t^{-2/q}B_n)^{\eta} \mathrm{d}t \leq C\sum_{n=1}^{\infty} c_n \int_{\delta^q}^{\infty} \left(t^{-2/q}\sum_{k=1}^{k_n} EY_{nk}^2\right)^{\eta} \mathrm{d}t \\
&= C\sum_{n=1}^{\infty} c_n \int_{\delta^q}^{\infty} \left[t^{-2/q}\sum_{k=1}^{k_n} EX_{nk}^2 I(|X_{nk}| \leq t^{1/q}) + \sum_{k=1}^{k_n} P\left(|X_{nk}| > t^{1/q}\right)\right]^{\eta} \mathrm{d}t \\
&\leq C\sum_{n=1}^{\infty} c_n \int_{\delta^q}^{\infty} \left[t^{-2/q}\sum_{k=1}^{k_n} EX_{nk}^2 I(|X_{nk}| \leq \delta)\right]^{\eta} \mathrm{d}t \\
&\quad + C\sum_{n=1}^{\infty} c_n \int_{\delta^q}^{\infty} \left[t^{-2/q}\sum_{k=1}^{k_n} EX_{nk}^2 I\left(\delta < |X_{nk}| \leq t^{1/q}\right)\right]^{\eta} \mathrm{d}t \\
&\quad + C\sum_{n=1}^{\infty} c_n \int_{\delta^q}^{\infty} \left[\sum_{k=1}^{k_n} P\left(|X_{nk}| > t^{1/q}\right)\right]^{\eta} \mathrm{d}t \\
&\doteq CH_{81} + CH_{82} + CH_{83}.
\end{aligned}
$$

Noting $\eta > q$, we have by (3.2) that

$$
\begin{aligned}
H_{81} &= C\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} EX_{nk}^2 I(|X_{nk}| \leq \delta)\right)^{\eta} \int_{\delta^q}^{\infty} t^{-2\eta/q}\mathrm{d}t \\
&\leq C\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} EX_{nk}^2 I(|X_{nk}| \leq \delta)\right)^{\eta} < \infty.
\end{aligned}
$$

Noting that

$$
\begin{aligned}
\sum_{k=1}^{k_n} E|X_{nk}|I(|X_{nk}| > \delta) &\leq \delta^{1-q}\sum_{k=1}^{k_n} E|X_{nk}|^q I(|X_{nk}| > \delta) \\
&\leq \delta^{1-q}\sum_{k=1}^{k_n} E|X_{nk}|^q I\left(|X_{nk}| > \frac{\delta}{128\eta}\right) \\
&\to 0, \quad \text{as} \quad n \to \infty,
\end{aligned}
$$

which together with (3.3) yields that for all n large enough,

$$
\sum_{k=1}^{k_n} E|X_{nk}|I(|X_{nk}| > \delta) < 1. \tag{3.10}
$$

Hence, we have by $\eta > q \geq 1$, (3.10) and (3.1) that

$$\begin{aligned} H_{82} &\leq C \sum_{n=1}^{\infty} c_n \int_{\delta^q}^{\infty} \left[t^{-1/q} \sum_{k=1}^{k_n} E|X_{nk}| I \left(\delta < |X_{nk}| \leq t^{1/q} \right) \right]^{\eta} \mathrm{d}t \\ &\leq C \sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} E|X_{nk}| I(|X_{nk}| > \delta) \right)^{\eta} \int_{\delta^q}^{\infty} t^{-\eta/q} \mathrm{d}t \\ &\leq C \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} E|X_{nk}| I(|X_{nk}| > \delta) \\ &\leq C \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} E|X_{nk}|^q I(|X_{nk}| > \delta) < \infty. \end{aligned}$$

For $t \geq \delta^q$, it follows by (3.10) that for all n large enough,

$$\sum_{k=1}^{k_n} P(|X_{nk}| > t^{1/q}) \leq \sum_{k=1}^{k_n} P(|X_{nk}| > \delta) \leq \sum_{k=1}^{k_n} E|X_{nk}| I(|X_{nk}| > \delta) < 1. \tag{3.11}$$

By (3.1) again and (3.11), we have

$$\begin{aligned} H_{83} &\leq C \sum_{n=1}^{\infty} c_n \int_{\delta^q}^{\infty} \sum_{k=1}^{k_n} P(|X_{nk}| > t^{1/q}) \mathrm{d}t \\ &\leq C \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} E|X_{nk}| I(|X_{nk}| > \delta) < \infty. \end{aligned}$$

This completes the proof of the theorem. □

Remark 3.1: For fixed $n \geq 1$, denote $S_m = \sum_{k=1}^{m} (X_{nk} - EX_{nk} I(|X_{nk}| \leq \delta))$ for $m = 1, 2, \ldots, k_n$. Under the conditions of Theorem 3.1, we have for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n P\left(\max_{1 \leq m \leq k_n} |S_m| > \varepsilon \right) < \infty. \tag{3.12}$$

This can be obtained by the following inequality:

$$\begin{aligned} \sum_{n=1}^{\infty} c_n E\left\{ \max_{1 \leq m \leq k_n} |S_m| - \varepsilon \right\}_+^q &= \sum_{n=1}^{\infty} c_n \int_0^{\infty} P\left(\max_{1 \leq m \leq k_n} |S_m| > \varepsilon + t^{1/q} \right) \mathrm{d}t \\ &\geq \sum_{n=1}^{\infty} c_n \int_0^{\varepsilon^q} P\left(\max_{1 \leq m \leq k_n} |S_m| > \varepsilon + t^{1/q} \right) \mathrm{d}t \\ &\geq \varepsilon^q \sum_{n=1}^{\infty} c_n P\left(\max_{1 \leq m \leq k_n} |S_m| > 2\varepsilon \right). \end{aligned}$$

Hence, we can see that the complete moment convergence (3.4) is stronger than complete convergence (3.12).

With Theorem 3.1 accounted for, we can get the following important corollaries.

Corollary 3.1: *Let $q \geq 1$, $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ be an array of rowwise NSD random variables with mean zero, and $\{c_n, n \geq 1\}$ be a sequence of positive constants. Suppose that for all $\varepsilon > 0$ and some $\delta > 0$, $\eta > q$, conditions (3.1), (3.2) and (3.3) hold. Then for all $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} c_n E\left\{\max_{1\leq m\leq k_n}\left|\sum_{k=1}^{m} X_{nk}\right| - \varepsilon\right\}_{+}^{q} < \infty, \tag{3.13}$$

and

$$\sum_{n=1}^{\infty} c_n P\left(\max_{1\leq m\leq k_n}\left|\sum_{k=1}^{m} X_{nk}\right| > \varepsilon\right) < \infty. \tag{3.14}$$

Proof: It follows by $EX_{nk} = 0$, (3.1) and (3.3) that

$$\begin{aligned}\max_{1\leq m\leq k_n}\left|\sum_{k=1}^{m} EX_{nk}I(|X_{nk}| \leq \delta)\right| &= \max_{1\leq m\leq k_n}\left|\sum_{k=1}^{m} EX_{nk}I(|X_{nk}| > \delta)\right| \\ &\leq \sum_{k=1}^{k_n} E|X_{nk}|I(|X_{nk}| > \delta) \\ &\leq \delta^{1-q}\sum_{k=1}^{k_n} E|X_{nk}|^q I(|X_{nk}| > \delta). \end{aligned}\tag{3.15}$$

Note that for any real numbers a, b, c, the following inequality holds:

$$(|a+b| - |c|)^+ \leq (|a| - |c|)^+ + |b|.$$

The desired result (3.13) follows by the inequality above, C_r-inequality, (3.15), (3.1), (3.3) and (3.4) immediately.

Noting that

$$\begin{aligned}\sum_{n=1}^{\infty} c_n E\left\{\max_{1\leq m\leq k_n}\left|\sum_{k=1}^{m} X_{nk}\right| - \varepsilon\right\}_{+}^{q} &= \sum_{n=1}^{\infty} c_n \int_0^{\infty} P\left(\max_{1\leq m\leq k_n}\left|\sum_{k=1}^{m} X_{nk}\right| > \varepsilon + t^{1/q}\right) \mathrm{d}t \\ &\geq \sum_{n=1}^{\infty} c_n \int_0^{\varepsilon^q} P\left(\max_{1\leq m\leq k_n}\left|\sum_{k=1}^{m} X_{nk}\right| > \varepsilon + t^{1/q}\right) \mathrm{d}t \\ &\geq \varepsilon^q \sum_{n=1}^{\infty} c_n P\left(\max_{1\leq m\leq k_n}\left|\sum_{k=1}^{m} X_{nk}\right| > 2\varepsilon\right),\end{aligned}$$

which together with (3.13) yields (3.14). This completes the proof of the corollary. □

By using Corollary 3.1, we can get the following corollary for sequences of NSD random variables.

Corollary 3.2: *Let $\alpha > 0$, $\alpha p > 0$, $q \geq 1$ and $\{X_k, k \geq 1\}$ be a sequence of NSD random variables with mean zero. Let $l(x) > 0$ be a slowly varying function as $x \to \infty$. Suppose that the following conditions are satisfied:*

(i) *for every $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q} l(n) \sum_{k=1}^{n} E|X_k|^q I(|X_k| > \varepsilon n^{\alpha}) < \infty, \tag{3.16}$$

(ii) *for some $\delta > 0$, there exists $\eta > q$ such that*

$$\sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha\eta} l(n) \left(\sum_{k=1}^{n} EX_k^2 I(|X_k| \leq \delta n^{\alpha}) \right)^{\eta} < \infty, \tag{3.17}$$

and

$$n^{-\alpha q} \sum_{k=1}^{n} E|X_k|^q I(|X_k| > n^{\alpha}\delta/128\eta) \to 0, \quad as \quad n \to \infty. \tag{3.18}$$

Then for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q} l(n) E \left\{ \max_{1 \leq m \leq n} \left| \sum_{k=1}^{m} X_k \right| - \varepsilon n^{\alpha} \right\}_{+}^{q} < \infty, \tag{3.19}$$

and

$$\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P \left(\max_{1 \leq m \leq n} \left| \sum_{k=1}^{m} X_k \right| > \varepsilon n^{\alpha} \right) < \infty. \tag{3.20}$$

Proof: Taking $c_n = n^{\alpha p-2} l(n)$, $k_n = n$, and replacing X_{nk} by X_k/n^{α} for $1 \leq k \leq n$ and $n \geq 1$, we can get (3.19) and (3.20) by (3.13) and (3.14), respectively. The proof is completed. □

By using Corollary 3.2, we can get the following result for sequences of NSD random variables which are stochastically dominated by a random variable X.

Corollary 3.3: *Let $\{X_k, k \geq 1\}$be a sequence of NSD random variables with mean zero, which is stochastically dominated by a random variable X. Let $l(x) > 0$ be a slowly varying function as $x \to \infty$. If for $\alpha > 1/2$, $\alpha p > 1$ and $1 \leq q < p$, $E|X|^p l(|X|^{1/\alpha}) < \infty$, then for all $\varepsilon > 0$, (3.19) and (3.20) hold.*

Proof: We will show that the conditions of Corollary 3.2 are satisfied. Take $\eta > \max\{q, \frac{\alpha p-1}{2\alpha-1}\}$.

Firstly, we will show that (3.16) holds for any $\varepsilon > 0$. It follows by Lemma 2.1, Lemma 2.5 and $E|X|^p l(|X|^{1/\alpha}) < \infty$ that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q} l(n) \sum_{k=1}^{n} E|X_k|^q I(|X_k| > \varepsilon n^{\alpha}) \\
&\quad \le C \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha q} l(n) E|X|^q I(|X| > \varepsilon n^{\alpha}) \\
&\quad \le C \sum_{s=1}^{\infty} (2^s)^{\alpha p-\alpha q} l(2^s) E|X|^q I(|X| > \varepsilon (2^s)^{\alpha}) \\
&\quad \le C \sum_{m=1}^{\infty} E|X|^q I(\varepsilon 2^{m\alpha} \le |X| < \varepsilon 2^{(m+1)\alpha}) \sum_{s=1}^{m} (2^s)^{\alpha p-\alpha q} l(2^s) \\
&\quad \le C \sum_{m=1}^{\infty} (2^m)^{\alpha p-\alpha q} l(2^m) E|X|^q I(\varepsilon 2^{m\alpha} \le |X| < \varepsilon 2^{(m+1)\alpha}) \\
&\quad \le C \sum_{m=1}^{\infty} l(2^m) E|X|^p I(\varepsilon 2^{m\alpha} \le |X| < \varepsilon 2^{(m+1)\alpha}) \\
&\quad \le C E|X|^p l(|X|^{1/\alpha}) < \infty,
\end{aligned}$$

which implies (3.16).

Next, we will prove (3.17). It follows by Lemma 2.5 again and C_r-inequality that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha\eta} l(n) \left(\sum_{k=1}^{n} EX_k^2 I(|X_k| \le \delta n^{\alpha}) \right)^{\eta} \\
&\quad \le C \sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha\eta+\eta} l(n) \left[EX^2 I(|X| \le \delta n^{\alpha}) \right]^{\eta} \\
&\qquad + C \sum_{n=1}^{\infty} n^{\alpha p-2+\eta} l(n) [P(|X| > \delta n^{\alpha})]^{\eta} \\
&\quad \doteq CJ_1 + CJ_2.
\end{aligned} \tag{3.21}$$

If $1 \le p < 2$, we have

$$[EX^2 I(|X| \le \delta n^{\alpha})]^{\eta} \le C n^{(2-p)\alpha\eta} (E|X|^p)^{\eta},$$

and thus,

$$J_1 \le C \sum_{n=1}^{\infty} n^{-1-(\alpha p-1)(\eta-1)} l(n) (E|X|^p)^{\eta} < \infty. \tag{3.22}$$

If $p > 2$, then $[EX^2I(|X| \le \delta n^\alpha)]^\eta < \infty$. Noting that $\alpha p - 2 - (2\alpha - 1)\eta < -1$, we have

$$J_1 = \sum_{n=1}^{\infty} n^{\alpha p-2-(2\alpha-1)\eta} l(n) \left[EX^2I(|X| \le \delta n^\alpha)\right]^\eta < \infty. \tag{3.23}$$

It follows by Markov's inequality that

$$\begin{aligned} J_2 &\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha p\eta+\eta} l(n)(E|X|^p)^\eta \\ &= C\sum_{n=1}^{\infty} n^{-1-(\alpha p-1)(\eta-1)} l(n)(E|X|^p)^\eta < \infty. \end{aligned} \tag{3.24}$$

By (3.21)–(3.24), we can see that (3.17) holds.

Finally, we will prove that (3.18) holds. Noting that $q < p, \alpha p > 1$ and $E|X|^p < \infty$, we have by Lemma 2.5 that

$$\begin{aligned} n^{-\alpha q}\sum_{k=1}^{n} E|X_k|^q I(|X_k| > n^\alpha \delta/128\eta) &\le Cn^{1-\alpha q}E|X|^q I(|X| > n^\alpha \delta/128\eta) \\ &\le Cn^{1-\alpha p}E|X|^p \to 0, \quad \text{as} \quad n \to \infty, \end{aligned}$$

which implies (3.18). The desired result follows by Corollary 3.2 immediately. This completes the proof of the corollary. □

Remark 3.2: We point out that the conditions of Corollary 3.2 are much weaker than those in Corollary 3.3. Hence, the results of Corollaries 3.1 and 3.2 generalize and improve the corresponding one of Theorem 1.3.

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