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ON THE RATE OF COMPLETE CONVERGENCE FOR WEIGHTED SUMS OF ARRAYS OF BANACH SPACE VALUED RANDOM ELEMENTS*

T.-C. HU†, D. LI‡, A. ROSALSKY§, and A. I. VOLODIN¶

Abstract. By applying a recent result of Hu et al. [*Stochastic Anal. Appl.*, 17 (1999), pp. 963–992], we extend and generalize the complete convergence results of Pruitt [*J. Math. Mech.*, 15 (1966), pp. 769–776] and Rohatgi [*Proc. Cambridge Philos. Soc.*, 69 (1971), pp. 305–307] to arrays of row-wise independent Banach space valued random elements. No assumptions are made concerning the geometry of the underlying Banach space. Illustrative examples are provided comparing the various results.

Key words. array of Banach space valued random elements, row-wise independence, weighted sums, complete convergence, rate of convergence, almost sure convergence

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1. Introduction. In this article, new results on complete convergence with corresponding convergence rates are obtained by applying a recent result of Hu et al. [9]. The concept of complete convergence was introduced by Hsu and Robbins [7] as follows. A sequence of random variables $\{U_n,\ n \geqq 1\}$ is said to *converge completely* to a constant c if $\sum_{n=1}^{\infty} \mathbf{P}\{|U_n - c| > \varepsilon\} < \infty$ for all $\varepsilon > 0$. By the Borel–Cantelli lemma, this implies $U_n \to c$ almost surely (a.s.), and the converse implication is true if the $\{U_n,\ n \geqq 1\}$ are independent. Hsu and Robbins [7] proved that the sequence of arithmetic means of independent and identically distributed (i.i.d.) random variables converges completely to the expected value if the variance of the summands is finite. Erdös [4] proved the converse. The Hsu–Robbins–Erdös [7], [4] result may be formulated as follows.

Theorem 1.1 (see [7], [4]). *If $\{X, X_n,\ n \geqq 1\}$ are i.i.d. random variables, then $n^{-1} \sum_{k=1}^{n} X_k$ converges completely to 0 if and only if $\mathbf{E}X = 0$ and $\mathbf{E}X^2 < \infty$.*

This result has been generalized and extended in several directions (see [14], [15], [8], [5], [18], [12], [16], and [9] among others). Some of these articles concern a Banach space setting. A sequence of Banach space valued random elements is said to *converge completely* to the 0 element of the Banach space if the corresponding sequence of norms converges completely to 0.

In [14], weighted sums of i.i.d. random variables were considered, permitting a more general normalization than in Theorem 1.1. Relying heavily on the techniques of [14], the authors of [15] generalized Pruitt's result to the case of independent stochastically dominated random variables. A sequence of random variables

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†Department of Mathematics, Tsing Hua University, Hsinchu, Taiwan 30043, Republic of China.

‡Department of Mathematics and Statistics, Lakehead University, Thunder Bay, Ontario, Canada P7B 5E1.

§Department of Statistics, University of Florida, Gainesville, Florida 32611 (rosalsky@stat.ufl.edu).

¶Research Institute of Mathematics and Mechanics, Kazan State University, Kazan 420008, Russia.

$\{X_n,\ n \geqq 1\}$ is said to be *stochastically dominated* by a random variable X if there exists a constant $D < \infty$ such that $\mathbf{P}\{|X_n| > x\} \leqq D\mathbf{P}\{|DX| > x\}$ for all $x > 0$ and $n \geqq 1$. We recall that an array of real numbers $\{a_{nk},\ k \geqq 1,\ n \geqq 1\}$ is said to be a *Toeplitz array* if $\lim_{n\to\infty} a_{nk} = 0$ for each $k \geqq 1$ and $\sum_{k=1}^{\infty} |a_{nk}| \leqq C$ for all $n \geqq 1$, where $C < \infty$ is a constant.

THEOREM 1.2 (see [14], [15]). *Let $\{X_n,\ n \geqq 1\}$ be a sequence of independent random variables which are stochastically dominated by a random variable X, and let $\{a_{nk},\ k \geqq 1,\ n \geqq 1\}$ be a Toeplitz array such that $\sup_{k\geqq 1} |a_{nk}| = O(n^{-\gamma})$ for some $\gamma > 0$. If $\mathbf{E}|X|^{1+\gamma^{-1}} < \infty$, then $\sum_{k=1}^{\infty} a_{nk}X_k$ converges a.s. for each $n \geqq 1$, and $\sum_{k=1}^{\infty} a_{nk}X_k$ converges completely to 0 (as $n \to \infty$).*

It should be mentioned that Theorems 1.1 and 1.2 are markedly different results with substantially different proofs. This is due to the fact that Theorem 1.1 concerns the sequence of partial sums $\sum_{k=1}^{n} X_k$ (each with a finite number of terms) whereas Theorem 1.2 concerns the sequence of infinite series $\sum_{k=1}^{\infty} a_{nk}X_k$.

Hu et al. [9] presented a very general result establishing complete convergence for the row sums of an array of row-wise independent but not necessarily identically distributed Banach space valued random elements. Their result also specified the corresponding rate of convergence. No geometric conditions were imposed on the underlying Banach space. The result of Hu et al. [9] unifies and extends previously obtained results in the literature in that many of them (for example, the results of Hsu and Robbins [7], Hu, Móricz, and Taylor [8], Gut [5], Wang et al. [18], Kuczmaszewska and Szynal [12], and Sung [16]) follow from it.

THEOREM 1.3 (see [9]). *Let $\{V_{nk},\ 1 \leqq k \leqq k_n \leqq \infty,\ n \geqq 1\}$ be an array of row-wise independent random elements in a separable real Banach space and let $\{c_n,\ n \geqq 1\}$ be a sequence of positive constants. Suppose that*

$$(1.1)\qquad \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \mathbf{P}\{\|V_{nk}\| > \varepsilon\} < \infty \qquad \text{for all} \quad \varepsilon > 0,$$

$$(1.2)\qquad \sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} \mathbf{E}\|V_{nk}\|^q \right)^J < \infty \qquad \text{for some} \quad 0 < q \leqq 2 \text{ and } J \geqq 2,$$

$$S_n \equiv \sum_{k=1}^{k_n} V_{nk} \xrightarrow{\mathbf{P}} 0,$$

and

$$(1.3)\qquad \text{if} \quad \liminf_{n\to\infty} c_n = 0, \ \text{then} \ \sum_{k=1}^{k_n} \mathbf{P}\big\{\|V_{nk}\| > \delta\big\} = o(1) \qquad \text{for some} \quad \delta > 0.$$

Then

$$\sum_{n=1}^{\infty} c_n \mathbf{P}\big\{\|S_n\| > \varepsilon\big\} < \infty \qquad \text{for all} \quad \varepsilon > 0.$$

It is implicitly assumed in Theorem 1.3 that if $k_n = \infty$ for any $n \geqq 1$, then for that n the series S_n converges a.s. The pertinent devices employed in the proof of Theorem 1.3 are

(i) an iterated version of the Hoffmann-Jørgensen [6] inequality due to Jain [10];

(ii) a Banach space version of the classical Marcinkiewicz–Zygmund inequality due to de Acosta [1];

(iii) a modified version of a result of Kuelbs and Zinn [13] concerning the relationship between convergence in probability and mean convergence for sums of independent bounded random variables.

The current work is devoted to presenting applications of Theorem 1.3 to obtain new complete convergence results. The plan of the paper is as follows. In section 2, we recall some well-known definitions pertaining to the current work. In section 3, we apply Theorem 1.3 to obtain complete convergence for row sums with corresponding rates of convergence. As in Theorem 1.3, no assumptions are made concerning the geometry of the underlying Banach space. Finally, in section 4, we present some examples which compare the results.

2. Preliminaries. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and let $\mathcal{X}$ be a separable real Banach space with norm $\|\cdot\|$. A *random element* is defined to be an $\mathcal{F}$-measurable mapping of Ω into $\mathcal{X}$ equipped with the Borel σ-algebra (that is, the σ-algebra generated by the open sets determined by $\|\cdot\|$). A detailed account of basic properties of random elements in separable real Banach spaces can be found in the book by Taylor [17].

Let $\{V_{nk},\ k \geqq 1,\ n \geqq 1\}$ be an array of row-wise independent, but not necessarily identically distributed, random elements taking values in $\mathcal{X}$. The array of random elements $\{V_{nk},\ k \geqq 1,\ n \geqq 1\}$ is said to be *stochastically dominated* by a random variable X if there exists a constant $D < \infty$ such that $\mathbf{P}\{\|V_{nk}\| > x\} \leqq D\mathbf{P}\{|DX| > x\}$ for all $x > 0$ and for all $n \geqq 1$ and $k \geqq 1$.

Let $\{a_{nk},\ k \geqq 1,\ n \geqq 1\}$ be an array of constants (called *weights*) and consider the sequence of *weighted sums* $S_n \equiv \sum_{k=1}^{\infty} a_{nk}V_{nk},\ n \geqq 1$. We assume without explicit mention that each series S_n converges a.s. if such almost sure convergence is not automatic from the hypotheses.

Let $f(t)$ be a real function of bounded variation on $[a,b]$, where $-\infty < a < b < \infty$. Denote the total variation of $f(t)$ on $[a,b]$ by $\mathcal{V}_{f(t)}(a,b)$. The following simple properties of total variation and Lebesgue–Stieltjes integration are well known and may be found, for example, in the book by Apostol [2, Chaps. 6 and 7]:

(2.1) If $f(t)$ is nondecreasing, then $\mathcal{V}_{f(t)}(a,b) = f(b) - f(a)$;

(2.2) if $f(t) = f_1(t) + f_2(t)$, then $\mathcal{V}_{f(t)}(a,b) \leqq \mathcal{V}_{f_1(t)}(a,b) + \mathcal{V}_{f_2(t)}(a,b)$;

(2.3) for any constant $c, \mathcal{V}_{f(t)+c}(a,b) = \mathcal{V}_{f(t)}(a,b)$;

(2.4) $\mathcal{V}_{-f(t)}(a,b) = \mathcal{V}_{f(t)}(a,b)$;

(2.5) $\left| \int_a^b f(x)\, dg(x) \right| \leqq \int_a^b |f(x)|\, d\mathcal{V}_{g(t)}(a,x)$;

(2.6) for $a < c < b, \mathcal{V}_{f(t)}(a,b) = \mathcal{V}_{f(t)}(a,c) + \mathcal{V}_{f(t)}(c,b)$.

Finally, the symbol C denotes throughout a generic constant ($0 < C < \infty$) which is not necessarily the same in each appearance, and for $x \geqq 0$ the symbol $[x]$ denotes the greatest integer in x.

3. Main results. With the preliminaries accounted for, the main results may now be established. The first main result, Theorem 3.1, generalizes Theorem 1.2 in three directions, namely,

(i) we consider Banach space valued random elements instead of random variables;

(ii) we consider an array rather than a sequence;

(iii) we obtain the rate of convergence.

THEOREM 3.1. *Let $\{V_{nk},\ k \geqq 1,\ n \geqq 1\}$ be an array of row-wise independent random elements taking values in a separable real Banach space $\mathcal{X}$. Suppose that $\{V_{nk},\ k \geqq 1,\ n \geqq 1\}$ is stochastically dominated by a random variable X. Let $\{a_{nk},\ k \geqq 1,\ n \geqq 1\}$ be an array of constants such that*

$$\sup_{k\geqq 1} |a_{nk}| = O(n^{-\gamma}) \qquad \textit{for some} \quad \gamma > 0, \tag{3.1}$$

$$\sum_{k=1}^{\infty} |a_{nk}| = O(n^{\alpha}) \qquad \textit{for some} \quad \alpha \in [0,\gamma). \tag{3.2}$$

If

$$\mathbf{E}|X|^{1+(1+\alpha+\beta)/\gamma} < \infty \qquad \textit{for some} \quad \beta \in (-1, \gamma-\alpha-1] \tag{3.3}$$

and

$$S_n \equiv \sum_{k=1}^{\infty} a_{nk}V_{nk} \xrightarrow{\mathbf{P}} 0, \tag{3.4}$$

then

$$\sum_{n=1}^{\infty} n^{\beta}\mathbf{P}\big\{\|S_n\| > \varepsilon\big\} < \infty \qquad \textit{for all} \quad \varepsilon > 0. \tag{3.5}$$

Proof. Note at the outset that the stochastic domination hypothesis ensures that $\mathbf{E}\|V_{nk}\| \leqq C\mathbf{E}|X|$, $k \geqq 1$, $n \geqq 1$, and hence for all $n \geqq 1$, by the Beppo–Levi theorem, (3.2), and (3.3),

$$\mathbf{E}\sum_{k=1}^{\infty} \|a_{nk}V_{nk}\| = \sum_{k=1}^{\infty} \mathbf{E}\|a_{nk}V_{nk}\| \leqq C\mathbf{E}|X| \sum_{k=1}^{\infty} |a_{nk}| \leqq C\, n^{\alpha} < \infty.$$

Thus for all $n \geqq 1$, $\sum_{k=1}^{\infty} \|a_{nk}V_{nk}\| < \infty$ a.s., and so for all $n \geqq 1$ and all $K \geqq 1$,

$$\begin{aligned}\sup_{L>K} \left\| \sum_{k=1}^{L} a_{nk}V_{nk} - \sum_{k=1}^{K} a_{nk}V_{nk} \right\| &= \sup_{L>K} \left\| \sum_{k=K+1}^{L} a_{nk}V_{nk} \right\| \\ \leqq \sup_{L>K} \sum_{k=K+1}^{L} \|a_{nk}V_{nk}\| &= \sum_{k=K+1}^{\infty} \|a_{nk}V_{nk}\| \xrightarrow[K\to\infty]{} 0 \quad \text{a.s.}\end{aligned}$$

Thus for all $n \geqq 1$, with probability 1, $\{\sum_{k=1}^{K} a_{nk}V_{nk},\ K \geqq 1\}$ is a Cauchy sequence in $\mathcal{X}$, whence $\sum_{k=1}^{\infty} a_{nk}V_{nk}$ converges a.s.

Let $c_n = n^{\beta}$, $n \geqq 1$. Then we only need to verify that conditions (1.1), (1.2), and (1.3) (if $\beta < 0$) of Theorem 1.3 hold, with $a_{nk}V_{nk}$ playing the role of V_{nk} in the formulation of that theorem. Set $q = 1 + \gamma^{-1}(1 + \alpha + \beta)$. Then by (3.3) we have $\mathbf{E}|X|^q < \infty$, where $1 < q \leqq 2$. Without loss of generality, ε can be taken to be 1 in (3.5) and, in view of (3.1) and (3.2), we can assume that

$$\sup_{k\geqq 1} |a_{nk}| = n^{-\gamma}, \tag{3.6}$$

$$\sum_{k=1}^{\infty} |a_{nk}| = n^{\alpha}. \tag{3.7}$$

Now to verify (1.1), we will proceed as in Lemma 1 of [14]. Let $b_{nk} = 1/|a_{nk}|$, $k \geqq 1$, $n \geqq 1$, and $G(x) = \mathbf{P}\{|DX| > x\}$, $x \geqq 0$. To estimate the series

$$\begin{aligned} A_n &= \sum_{k=1}^{\infty} \mathbf{P}\big\{\|a_{nk}V_{nk}\| > 1\big\} = \sum_{k=1}^{\infty} \mathbf{P}\big\{\|V_{nk}\| > b_{nk}\big\} \\ &\leqq D \sum_{k=1}^{\infty} \mathbf{P}\big\{|DX| > b_{nk}\big\} \quad \text{(by stochastic domination)} \\ &= D \sum_{k=1}^{\infty} G(b_{nk}), \end{aligned} \tag{3.8}$$

we reformulate the problem as one of estimating a Lebesgue–Stieltjes integral. Let us introduce the functions

$$N_n(x) = \sum_{\{k\colon\ b_{nk} \leqq x\}} |a_{nk}|, \qquad x > 0, \quad n \geqq 1.$$

Then each $N_n(x)$ is a step function with jumps at the points b_{nk}. Moreover, for $n \geqq 1$,

$$N_n(x) = 0 \ \text{ for } x < n^{\gamma} \quad \big(\text{since } \{k\colon\ b_{nk} \leqq x < n^{\gamma}\} = \varnothing \text{ recalling (3.6)}\big), \tag{3.9}$$

$$\sup_{x>0} N_n(x) = \sum_{k=1}^{\infty} |a_{nk}| = n^{\alpha} \quad \big(\text{recalling (3.7)}\big). \tag{3.10}$$

Then by (3.8) and expressing its right-hand side as a Lebesgue–Stieltjes integral and employing integration by parts,

$$\begin{aligned} A_n &\leqq D \sum_{k=1}^{\infty} G(b_{nk}) = D \int_0^{\infty} xG(x)\, dN_n(x) \\ &= D\bigg[\lim_{t\to\infty} t\, G(t) N_n(t) - \int_0^{\infty} N_n(x)\, d\big(xG(x)\big)\bigg]. \end{aligned} \tag{3.11}$$

Now $\mathbf{E}|X| < \infty$ and (3.10) ensure that $\lim_{t\to\infty} tG(t)\, N_n(t) = 0$ whence, by (3.11), (3.9), and (2.5), we have

$$\begin{aligned} A_n &\leqq D \int_{n^{\gamma}}^{\infty} N_n(x)\, d\mathcal{V}_{tG(t)}(0,x) \\ &\overset{(3.10)}{=} Dn^{\alpha} \int_{n^{\gamma}}^{\infty} d\mathcal{V}_{tG(t)}(0,x) = Dn^{\alpha} \sum_{j=n}^{\infty} \int_{j^{\gamma}}^{(j+1)^{\gamma}} d\mathcal{V}_{tG(t)}(0,x) \\ &= Dn^{\alpha} \sum_{j=n}^{\infty} \Big(\mathcal{V}_{tG(t)}\big(0,(j+1)^{\gamma}\big) - \mathcal{V}_{tG(t)}\big(0,j^{\gamma}\big)\Big) \\ &\overset{(2.6)}{=} Dn^{\alpha} \sum_{j=n}^{\infty} \mathcal{V}_{tG(t)}\big(j^{\gamma},(j+1)^{\gamma}\big) \\ &\overset{(2.3)}{=} Dn^{\alpha} \sum_{j=n}^{\infty} \mathcal{V}_{tG(t)-j^{\gamma}G(j^{\gamma})}\big(j^{\gamma},(j+1)^{\gamma}\big). \end{aligned} \tag{3.12}$$

We will estimate each total variation in (3.12) separately. Define the functions $f_1(t)$ and $f_2(t)$ by

$$f_1(t) = (t - j^{\gamma})\, G(j^{\gamma}), \quad f_2(t) = t\big(G(j^{\gamma}) - G(t)\big), \qquad t \geqq j^{\gamma}.$$

Note that $f_1(t)$ and $f_2(t)$ are nondecreasing and $f_1(t)-f_2(t) = tG(t)-j^\gamma G(j^\gamma)$. Thus

$$\begin{aligned}
&\mathcal{V}_{tG(t)-j^\gamma G(j^\gamma)}\big(j^\gamma,(j+1)^\gamma\big) = \mathcal{V}_{f_1(t)-f_2(t)}\big(j^\gamma,(j+1)^\gamma\big)\\
&\overset{(2.2)}{\leqq} \mathcal{V}_{f_1(t)}\big(j^\gamma,(j+1)^\gamma\big) + \mathcal{V}_{-f_2(t)}\big(j^\gamma,(j+1)^\gamma\big)\\
&\overset{(2.4)}{=} \mathcal{V}_{f_1(t)}\big(j^\gamma,(j+1)^\gamma\big) + \mathcal{V}_{f_2(t)}\big(j^\gamma,(j+1)^\gamma\big)\\
&\overset{(2.1)}{=} f_1\big((j+1)^\gamma\big) - f_1(j^\gamma) + f_2\big((j+1)^\gamma\big) - f_2(j^\gamma)\\
&= \big((j+1)^\gamma - j^\gamma\big)\, G(j^\gamma) + (j+1)^\gamma\big(G(j^\gamma) - G((j+1)^\gamma)\big),
\end{aligned}$$

implying by (3.12) that

$$A_n \leqq Dn^\alpha\left[\sum_{j=n}^{\infty}\big((j+1)^\gamma - j^\gamma\big)\, G(j^\gamma) + \sum_{j=n}^{\infty}(j+1)^\gamma\Big(G(j^\gamma) - G\big((j+1)^\gamma\big)\Big)\right]. \tag{3.13}$$

Applying the Abel summation by parts lemma (see, e.g., Lemma 5.1.1(5) of [3, p. 114]) to the first series of (3.13) yields

$$\begin{aligned}
\sum_{j=n}^{\infty}\big((j+1)^\gamma - j^\gamma\big)\, G(j^\gamma) &= \sum_{j=n}^{\infty}(j+1)^\gamma\Big(G(j^\gamma) - G\big((j+1)^\gamma\big)\Big) - (n+1)^\gamma\, G(n^\gamma)\\
&\leqq 2^\gamma\sum_{j=n}^{\infty} j^\gamma\Big(G(j^\gamma) - G\big((j+1)^\gamma\big)\Big).
\end{aligned}$$

Then by (3.13),

$$A_n \leqq Cn^\alpha \sum_{j=n}^{\infty} j^\gamma\Big(G(j^\gamma) - G\big((j+1)^\gamma\big)\Big). \tag{3.14}$$

Thus,

$$\begin{aligned}
\sum_{n=1}^{\infty} n^\beta \sum_{k=1}^{\infty}\mathbf{P}\big\{\|a_{nk}V_{nk}\| > 1\big\} &= \sum_{n=1}^{\infty} n^\beta A_n\\
&\overset{(3.14)}{\leqq} C\sum_{n=1}^{\infty} n^\beta n^\alpha \sum_{j=n}^{\infty} j^\gamma\Big(G(j^\gamma) - G\big((j+1)^\gamma\big)\Big)\\
&= C\sum_{j=1}^{\infty} j^\gamma\Big(G(j^\gamma) - G\big((j+1)^\gamma\big)\Big)\sum_{n=1}^{j} n^{\alpha+\beta}\\
&\leqq C\sum_{j=1}^{\infty} j^\gamma j^{\alpha+\beta+1}\Big(G(j^\gamma) - G\big((j+1)^\gamma\big)\Big) \quad (\text{since } \alpha+\beta > -1). \qquad (3.15)
\end{aligned}$$

Now observe that (3.3) is equivalent to the convergence of the series in (3.15). Thus, (1.1) is proved.

To verify (1.2), note that for any $J \geqq 2$,

$$\sum_{n=1}^{\infty} n^{\beta}\left(\sum_{k=1}^{\infty}\mathbf{E}\|a_{nk}V_{nk}\|^{q}\right)^{J} = \sum_{n=1}^{\infty} n^{\beta}\left(\sum_{k=1}^{\infty}|a_{nk}|^{q}\mathbf{E}\|V_{nk}\|^{q}\right)^{J}$$
$$\leqq \sum_{n=1}^{\infty} n^{\beta}\left(\sup_{k\geqq 1}|a_{nk}|^{(\alpha+\beta+1)/\gamma}\sum_{k=1}^{\infty}|a_{nk}|\mathbf{E}\|V_{nk}\|^{q}\right)^{J}$$
$$\leqq C\sum_{n=1}^{\infty} n^{\beta}\left(n^{-\gamma((\alpha+\beta+1)/\gamma)}n^{\alpha}\mathbf{E}|X|^{q}\right)^{J} = C\sum_{n=1}^{\infty} n^{\beta}(n^{-\beta-1})^{J} < \infty.$$

Here the second inequality is valid by (3.6), (3.7), and stochastic domination.

Finally, to verify (1.3) if $\beta < 0$, note that for any $\delta > 0$,

$$\sum_{k=1}^{\infty}\mathbf{P}\{\|a_{nk}V_{nk}\| > \delta\} \leqq \sum_{k=1}^{\infty}\delta^{-q}\mathbf{E}\|a_{nk}V_{nk}\|^{q} \quad \text{(by the Markov inequality)}$$
$$\leqq C\sup_{k\geqq 1}|a_{nk}|^{q-1}\sum_{k=1}^{\infty}|a_{nk}|\mathbf{E}\|V_{nk}\|^{q}$$
$$\leqq Cn^{-\gamma(q-1)}n^{\alpha}\mathbf{E}|X|^{q} \quad \text{(by (3.6), (3.7), and stochastic domination)}$$
$$= Cn^{-\beta-1} = o(1) \quad (\text{since } \beta > -1).$$

Remarks. (i) Verification of (1.1) is substantially simpler if (3.2) is strengthened to

$$\sum_{k=1}^{\infty}|a_{nk}| = O\left(\frac{n^{\alpha}}{(\ln n)^{1+\delta}}\right) \qquad \text{for some} \quad \alpha \in [0,\gamma) \text{ and some } \delta > 0.$$

The details are left to the reader.

(ii) Theorem 1.1 (sufficiency half), Theorem 1.2, and some results of Hu, Móricz, and Taylor [8] and Wang et al. [18] are immediate corollaries of Theorem 3.1.

(iii) In Theorem 3.1 we consider general weighted sums $\sum_{k=1}^{\infty} a_{nk}V_{nk}$, whereas Wang et al. [18] considered $\sum_{k=1}^{n} V_{nk}/n^{1/t}$ $(1 \leqq t < 2)$, which is the standard partial sum with a Marcinkiewicz–Zygmund normalization. Moreover, we assume (3.4), whereas Wang et al. [18] assume

$$\max_{1\leqq i\leqq n}\mathbf{P}\left\{\frac{\|\sum_{k=1}^{i}V_{nk}\|}{n^{1/t}} > \varepsilon\right\} = o(1) \qquad \text{for all} \quad \varepsilon > 0.$$

However, it must be pointed out that the work of Wang et al. [18] includes the interesting case $\beta = -1$. Our method of proving Theorem 3.1 does not work for this case. But the case $\beta = -1$ is included in the next two theorems.

The differences between Theorem 3.1 and the next theorem are that (i) (3.2) is strengthened to (3.17); (ii) β can be -1; (iii) (3.4) does not need to be stated as an assumption.

THEOREM 3.2. *Let $\{V_{nk},\ k \geqq 1,\ n \geqq 1\}$ be as in Theorem* 3.1 *and let $\{a_{nk}$, $k \geqq 1$, $n \geqq 1\}$ be an array of constants such that*

$$\sup_{k\geqq 1}|a_{nk}| = O(n^{-\gamma}) \qquad \textit{for some} \quad \gamma > 0, \tag{3.16}$$

$$\sum_{k=1}^{\infty}|a_{nk}| = O(n^{-\alpha}) \qquad \textit{for some} \quad \alpha \in (0,\gamma). \tag{3.17}$$

If

$$\mathbf{E}|X|^{1+(1+\alpha+\beta)/\gamma} < \infty \qquad \textit{for some} \quad \beta \in [-1, \gamma-\alpha-1], \tag{3.18}$$

then $S_n \equiv \sum_{k=1}^{\infty} a_{nk} V_{nk}$ *converges a.s. for each* $n \geqq 1$ *and*

$$\sum_{n=1}^{\infty} n^{\beta} \mathbf{P}\{\|S_n\| > \varepsilon\} < \infty \qquad \textit{for all} \quad \varepsilon > 0.$$

Proof. As in Theorem 3.1, the hypotheses ensure that S_n converges a.s. for each $n \geqq 1$. Note at the outset that for arbitrary $\varepsilon > 0$,

$$\begin{aligned} \mathbf{P}\{\|S_n\| > \varepsilon\} &\leqq \varepsilon^{-1} \mathbf{E}\|S_n\| \quad \text{(by the Markov inequality)} \\ &\leqq C \sum_{k=1}^{\infty} |a_{nk}|\, \mathbf{E}\|V_{nk}\| \quad \text{(by the Beppo–Levi theorem)} \\ &\leqq C n^{-\alpha} \mathbf{E}|X| \quad \text{(by (3.17) and stochastic domination)} \\ &= o(1) \quad \text{(by (3.18) and } \alpha > 0). \end{aligned}$$

Thus $S_n \xrightarrow{\mathbf{P}} 0$. Let $c_n = n^{\beta}$, $n \geqq 1$. As in the proof of Theorem 3.1, we only need to verify that conditions (1.1), (1.2), and (1.3) (if $\beta < 0$) of Theorem 1.3 hold with $a_{nk} V_{nk}$ playing the role of V_{nk} in the formulation of that theorem.

We now verify (1.1). Let $q = 1 + \gamma^{-1}(1 + \alpha + \beta)$. For arbitrary $\varepsilon > 0$,

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} \mathbf{P}\{\|a_{nk} V_{nk}\| > \varepsilon\} \\ &\quad \leqq D \sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} \mathbf{P}\left\{|X| > \frac{\varepsilon}{D|a_{nk}|}\right\} \quad \text{(by stochastic domination)} \\ &\quad \leqq C \sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} |a_{nk}|^{q} \mathbf{E}|X|^{q} \quad \text{(by the Markov inequality)} \\ &\quad \overset{(3.18)}{=} C \sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} |a_{nk}|\, |a_{nk}|^{(1+\alpha+\beta)/\gamma} \overset{(3.17),\,(3.16)}{\leqq} C \sum_{n=1}^{\infty} n^{\beta} n^{-\alpha} n^{-(1+\alpha+\beta)} \\ &\quad = C \sum_{n=1}^{\infty} n^{-1-2\alpha} < \infty \quad \text{(since } \alpha > 0), \end{aligned}$$

thereby establishing (1.1).

Verification of (1.2) and (1.3) (if $\beta < 0$) follows as in the proof of Theorem 3.1 mutatis mutandis.

The next result was obtained by Hu et al. [9, Corollary 4.5] when $\delta > 1$.

THEOREM 3.3. *Let* $\{V_{nk},\ k \geqq 1,\ n \geqq 1\}$ *be as in Theorem* 3.1 *and let* $\{a_{nk},\ k \geqq 1,\ n \geqq 1\}$ *be an array of constants such that for some* $0 < q \leqq 2$ *and* $\delta > 0$

$$\sum_{k=1}^{\infty} |a_{nk}|^{q} = O(n^{-\delta}).$$

If $\mathbf{E}|X|^q < \infty$ *and* $S_n \equiv \sum_{k=1}^{\infty} a_{nk} V_{nk} \xrightarrow{\mathbf{P}} 0$, *then for all* $-1 \leqq \beta < \delta - 1$,

$$\sum_{n=1}^{\infty} n^{\beta} \mathbf{P}\big\{\|S_n\| > \varepsilon\big\} < \infty \qquad \textit{for all} \quad \varepsilon > 0.$$

Proof. The argument is a slight modification of Theorem 3.2. The details are left to the reader.

The ensuing corollary is in effect a special case of Theorem 3.3 and is presented for purposes of comparison with our other results. Corollary 3.1 replaces conditions (3.16) and (3.17) of Theorem 3.2 by the weaker single condition (3.19). However, in contrast to Theorem 3.2, it needs to be assumed in Corollary 3.1 that $S_n = \sum_{k=1}^{\infty} a_{nk} V_{nk} \xrightarrow{\mathbf{P}} 0$.

COROLLARY 3.1. *Let* $\{V_{nk},\ k \geqq 1,\ n \geqq 1\}$ *be as in Theorem* 3.1 *and let* $\{a_{nk},$ $k \geqq 1, n \geqq 1\}$ *be an array of constants such that for some constants* γ, α, *and* β *with* $\gamma > 0$, $-1 - \beta - \gamma < \alpha \leqq \gamma$, *and* $-1 \leqq \beta \leqq \gamma - \alpha - 1$,

$$\sum_{k=1}^{\infty} |a_{nk}|^{1+(1+\alpha+\beta)/\gamma} = O(n^{-\delta}) \qquad \textit{for some} \quad \delta > \beta + 1. \tag{3.19}$$

If $\mathbf{E}|X|^{1+(1+\alpha+\beta)/\gamma} < \infty$ *and*

$$S_n \equiv \sum_{k=1}^{\infty} a_{nk} V_{nk} \xrightarrow{\mathbf{P}} 0, \tag{3.20}$$

then $\sum_{n=1}^{\infty} n^{\beta} \mathbf{P}\{\|S_n\| > \varepsilon\} < \infty$ *for all* $\varepsilon > 0$.

The next theorem handles the situation where $\delta > 0$ satisfying (3.19) is taken arbitrarily small, and this will be transparent from Corollary 3.2. The proof of Theorem 3.4 again uses Theorem 1.2 but the overall argument is substantially different from Theorems 3.1 and 3.2.

THEOREM 3.4. *Let* $\{V_{nk},\ k \geqq 1,\ n \geqq 1\}$ *be as in Theorem* 3.1, *and let* $\{a_{nk},$ $k \geqq 1, n \geqq 1\}$ *be an array of constants. Suppose that*

$$\sup_{k \geqq 1} |a_{nk}| = O(n^{-\gamma}) \qquad \textit{for some} \quad \gamma > 0, \tag{3.21}$$

$$\sup_{n \geqq 1} |a_{nk}| = O(k^{-\lambda}) \qquad \textit{for some} \quad \lambda > \frac{1}{2}, \tag{3.22}$$

$$\mathbf{E}|X|^{1/\lambda+(\beta+1)/\gamma} < \infty \qquad \textit{for some} \quad \beta \in \Big(-1,\ 2\gamma - \frac{\gamma}{\lambda} - 1\Big], \tag{3.23}$$

$$\sum_{k=1}^{\infty} |a_{nk}|^{1/\lambda+(\beta+1)/\gamma} = O(n^{-\delta}) \qquad \textit{for some} \quad \delta > 0. \tag{3.24}$$

If $S_n \equiv \sum_{k=1}^{\infty} a_{nk} V_{nk} \xrightarrow{\mathbf{P}} 0$, *then*

$$\sum_{n=1}^{\infty} n^{\beta} \mathbf{P}\big\{\|S_n\| > \varepsilon\big\} < \infty \qquad \textit{for all} \quad \varepsilon > 0.$$

Proof. Let $c_n = n^{\beta}$, $n \geqq 1$. Again, we only need to verify that conditions (1.1), (1.2), and (1.3) (if $\beta < 0$) of Theorem 1.3 hold, with $a_{nk} V_{nk}$ playing the role of

V_{nk}, $k \geqq 1$, $n \geqq 1$, in the formulation of that theorem. Let $q = 1/\lambda + (\beta+1)/\gamma$. Then by (3.23), $\mathbf{E}|X|^q < \infty$ and $0 < q \leqq 2$.

To verify (1.1), without loss of generality suppose, by (3.21), that $\sup_{k\geqq 1} |a_{nk}| \leqq n^{-\gamma}$, $n \geqq 1$, and that $\varepsilon = 1$. Define $b_{nk} = 1/|a_{nk}|$, $k \geqq 1$, $n \geqq 1$. Then $\inf_{k\geqq 1} b_{nk} \geqq n^{\gamma}$, $n \geqq 1$. For $n \geqq 1$ and $j \geqq 1$, let $b_n^{-1}(j) = \max\{k \geqq 1\colon\ [b_{nk}] \leqq j\}$, where $\max \varnothing = 0$. For $j \geqq 1$, let $n(j) = \max\{n \geqq 1\colon\ [n^{\gamma}] \leqq j\}$. Then

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} \mathbf{P}\big\{\|a_{nk}V_{nk}\| > 1\big\} \\
&\quad \leqq D \sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} \mathbf{P}\big\{|DX| > b_{nk}\big\} \quad \text{(by stochastic domination)} \\
&\quad \leqq D \sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} \sum_{j=[b_{nk}]}^{\infty} \mathbf{P}\big\{j < |DX| \leqq j+1\big\} \\
&\quad \leqq D \sum_{n=1}^{\infty} n^{\beta} \sum_{j=[n^{\gamma}]}^{\infty} \sum_{k=1}^{b_n^{-1}(j)} \mathbf{P}\big\{j < |DX| \leqq j+1\big\} \\
&\quad = D \sum_{n=1}^{\infty} n^{\beta} \sum_{j=[n^{\gamma}]}^{\infty} b_n^{-1}(j)\, \mathbf{P}\big\{j < |DX| \leqq j+1\big\} \\
&\quad \overset{(3.22)}{\leqq} C \sum_{n=1}^{\infty} n^{\beta} \sum_{j=[n^{\gamma}]}^{\infty} j^{1/\lambda} \mathbf{P}\big\{j < |DX| \leqq j+1\big\} \\
&\quad = C \sum_{j=1}^{\infty} \sum_{n=1}^{n(j)} n^{\beta} j^{1/\lambda} \mathbf{P}\big\{j < |DX| \leqq j+1\big\} \\
&\quad \leqq C \sum_{j=1}^{\infty} j^{1/\lambda} \mathbf{P}\big\{j < |DX| \leqq j+1\big\} \sum_{n=1}^{[2j^{1/\gamma}]} n^{\beta} \\
&\quad \leqq C \sum_{j=1}^{\infty} j^{1/\lambda} j^{(\beta+1)/\gamma} \mathbf{P}\big\{j < |DX| \leqq j+1\big\} \\
&\quad \leqq C\mathbf{E}|X|^{1/\lambda + (\beta+1)/\gamma} \overset{(3.23)}{<} \infty.
\end{aligned}$$

Next, let us verify (1.2). Note that for any $J \geqq 2$

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\beta} \left(\sum_{k=1}^{\infty} \mathbf{E}\|a_{nk}V_{nk}\|^q \right)^J \\
&\qquad \leqq C \sum_{n=1}^{\infty} n^{\beta} \left(\sum_{k=1}^{\infty} |a_{nk}|^q \mathbf{E}|X|^q \right)^J \quad \text{(by stochastic domination)} \\
&\qquad \overset{(3.23),(3.24)}{\leqq} C \sum_{n=1}^{\infty} n^{\beta - J\delta} < \infty
\end{aligned}$$

provided $J > (\beta+1)/\delta$.

Finally, to verify (1.3) if $\beta < 0$, note that

$$\begin{aligned}
&\sum_{k=1}^{\infty} \mathbf{P}\{\|a_{nk}V_{nk}\| > 1\} \\
&\quad \leqq D \sum_{k=1}^{\infty} \mathbf{P}\{|a_{nk}\|DX| > 1\} \quad \text{(by stochastic domination)} \\
&\quad \leqq D^{q+1} \sum_{k=1}^{\infty} |a_{nk}|^q \mathbf{E}|X|^q \quad \text{(by the Markov inequality)} \\
&\quad \overset{(3.23),\,(3.24)}{\leqq} Cn^{-\delta} = o(1).
\end{aligned}$$

Remark. Let $\lambda > \frac{1}{2}$. If

$$|a_{nk}| \downarrow \text{ as } k \uparrow \text{ for all } n \geqq 1, \tag{3.25}$$

and if the series $\sum_{k=1}^{\infty} |a_{nk}|^{1/\lambda}$ converges uniformly in n, that is,

$$\lim_{K\to\infty} \sup_{n\geqq 1} \sum_{k=K}^{\infty} |a_{nk}|^{1/\lambda} = 0, \tag{3.26}$$

then (3.22) holds.

Indeed, it follows from the triangle inequality and (3.26) that

$$\lim_{\substack{K\to\infty \\ K'\to\infty}} \sup_{n\geqq 1} \left| \sum_{k=1}^{K} |a_{nk}|^{1/\lambda} - \sum_{k=1}^{K'} |a_{nk}|^{1/\lambda} \right| = 0. \tag{3.27}$$

Then by the same argument used to prove the theorem in [11, p. 124], it follows from (3.25) and (3.26) that

$$\lim_{k\to\infty} \sup_{n\geqq 1} k|a_{nk}|^{1/\lambda} = 0,$$

implying (3.22).

The next corollary is in effect a reparametrization of Theorem 3.4 and it will be compared with our other results. Corollary 3.2 weakens condition (3.19) of Corollary 3.1 by permitting δ to be arbitrarily close to 0, but it also imposes additional conditions ((3.28), (3.29), and $\beta > -1$).

COROLLARY 3.2. *Let* $\{V_{nk},\ k \geqq 1,\ n \geqq 1\}$ *be as in Theorem* 3.1 *and let* $\{a_{nk},\ k \geqq 1,\ n \geqq 1\}$ *be an array of constants. Suppose that*

$$\sup_{k\geqq 1} |a_{nk}| = O(n^{-\gamma}) \qquad \textit{for some} \quad \gamma > 0, \tag{3.28}$$

$$\sup_{n\geqq 1} |a_{nk}| = O(k^{-\gamma/(\gamma+\alpha)}) \qquad \textit{for some} \quad \alpha \in [0, \gamma), \tag{3.29}$$

$$\mathbf{E}|X|^{1+(1+\alpha+\beta)/\gamma} < \infty \qquad \textit{for some} \quad \beta \in (-1, \gamma-\alpha-1], \tag{3.30}$$

$$\sum_{k=1}^{\infty} |a_{nk}|^{1+(1+\alpha+\beta)/\gamma} = O(n^{-\delta}) \qquad \textit{for some} \quad \delta > 0. \tag{3.31}$$

If $S_n \equiv \sum_{k=1}^{\infty} a_{nk} V_{nk} \xrightarrow{\mathbf{P}} 0$, *then*

$$\sum_{n=1}^{\infty} n^{\beta} \mathbf{P}\big\{\|S_n\| > \varepsilon\big\} < \infty \qquad \textit{for all} \quad \varepsilon > 0.$$

Proof. Let $\lambda = \gamma/(\gamma + \alpha)$. Then $\lambda > \frac{1}{2}$ since $\gamma > \alpha$ and $\mathbf{E}|X|^{1/\lambda + (\beta+1)/\gamma} = \mathbf{E}|X|^{1+(1+\alpha+\beta)/\gamma} < \infty$ (by (3.30)). Moreover, by (3.30) $-1 < \beta \leqq \gamma - \alpha - 1 = 2\gamma - \gamma/\lambda - 1$. Finally,

$$\sum_{k=1}^{\infty} |a_{nk}|^{1/\lambda + (\beta+1)/\gamma} = \sum_{k=1}^{\infty} |a_{nk}|^{1+(1+\alpha+\beta)/\gamma} \overset{(3.31)}{=} O(n^{-\delta}).$$

Corollary 3.2 now follows immediately from Theorem 3.4.

4. Some examples. In this section, examples will be presented of arrays of weights $\{a_{nk},\ k \geqq 1,\ n \geqq 1\}$ satisfying the conditions of some but not all of the results in section 3. Moreover, an example is given wherein Corollary 3.1 applies but Corollary 4.6 of [9] does not.

Example 4.1. Let $\gamma > 0$ and $0 \leqq \alpha < \gamma$ and set $a_{nk} = n^{-\gamma} k^{-\gamma/(\gamma+\alpha)}$, $k \geqq 1$, $n \geqq 1$. Then conditions (3.19) of Corollary 3.1 and (3.28), (3.29), and (3.31) of Corollary 3.2 hold but conditions (3.2) of Theorem 3.1 and (3.17) of Theorem 3.2 fail.

Example 4.2. Let $\gamma > 0$ and set for $n \geqq 1$

$$a_{nk} = \begin{cases} n^{-\gamma} k^{-1/2}, & k = 2, 2^2, 2^3, 2^4, \dots, \\ n^{-\gamma} k^{-2} & \text{otherwise.} \end{cases}$$

Then conditions (3.1) and (3.2) of Theorem 3.1, (3.16) and (3.17) of Theorem 3.2, and (3.19) (provided $\alpha > -\gamma$) of Corollary 3.1 all hold, but condition (3.29) of Corollary 3.2 fails.

The array $\{a_{nk},\ k \geqq 1,\ n \geqq 1\}$ exhibited in the next example satisfies conditions (3.28), (3.29), and (3.31) of Corollary 3.2 but does not satisfy conditions (3.19) of Corollary 3.1 and (3.17) of Theorem 3.2.

Example 4.3. Let $\frac{1}{2} < \gamma < 1$ and $\alpha = 1 - \gamma$. Then $0 < \alpha < \gamma$. Set

$$a_{nk} = \begin{cases} n^{-\gamma}, & 1 \leqq k \leqq n, \quad n \geqq 1, \\ 0, & k > n, \quad n \geqq 1. \end{cases}$$

Let $\beta = \gamma - \alpha - 1 = 2\gamma - 2$. Now

$$\sup_{k \geqq 1} |a_{nk}| = n^{-\gamma}, \tag{4.1}$$

$$\sup_{n \geqq 1} |a_{nk}| = k^{-\gamma} = k^{-\gamma/(\gamma+\alpha)},$$

and

$$\sum_{k=1}^{\infty} |a_{nk}|^{1+(1+\alpha+\beta)/\gamma} = \sum_{k=1}^{\infty} a_{nk}^2 = n^{-(2\gamma-1)}. \tag{4.2}$$

Thus conditions (3.28), (3.29), and (3.31) (with $\delta = 2\gamma - 1$) of Corollary 3.2 hold. However, condition (3.19) of Corollary 3.1 fails for every $\delta > \beta + 1$ since, then,

$$\frac{\sum_{k=1}^{\infty} |a_{nk}|^{1+(1+\alpha+\beta)/\gamma}}{n^{-\delta}} \overset{(4.2)}{=} \frac{n^{\delta}}{n^{2\gamma-1}} = \frac{n^{\delta}}{n^{\beta+1}} \to \infty.$$

Finally, note that

$$\sum_{k=1}^{\infty} |a_{nk}| = n^{1-\gamma} = n^{\alpha} \tag{4.3}$$

and so condition (3.17) of Theorem 3.2 fails.

Remark. In the preceding example, conditions (3.1) and (3.2) of Theorem 3.1 indeed hold according to (4.1) and (4.3), respectively. It would be particularly interesting to construct an example wherein the conditions of Corollary 3.2 are satisfied but those of Theorem 3.1, Theorem 3.2, and Corollary 3.1 are not satisfied. The authors were not able to accomplish this and hope that this problem will be considered by an interested reader.

The last example shows that Corollary 3.1 can be applied when the conditions of Corollary 4.6 of [9] fail, but barely so.

Example 4.4. Let $\{V_{nk},\ k \geqq 1,\ n \geqq 1\}$ be an array of row-wise independent random elements taking values in a separable real Banach space. Suppose that $\{V_{nk}$, $k \geqq 1,\ n \geqq 1\}$ is stochastically dominated by a random variable X with $\mathbf{E}X^2 < \infty$. Let $1 < p < 2$, $\gamma > 1$, $\alpha = \gamma - 1$, and $\beta = 0$. Set

$$a_{nk} = n^{-1/p}k^{-1}, \qquad k \geqq 1, \quad n \geqq 1.$$

Suppose that $\mathbf{E}|X|^{p+1} = \infty$. Note that

$$\sum_{k=1}^{\infty} |a_{nk}|^{1+(1+\alpha+\beta)/\gamma} = \sum_{k=1}^{\infty} a_{nk}^2 = O(n^{-2/p}) = O(n^{-\delta}),$$

where $\delta = 2/p > \beta + 1$. Also

$$\mathbf{E}|X|^{1+(1+\alpha+\beta)/\gamma} = \mathbf{E}X^2 < \infty.$$

Thus, by Corollary 3.1, if (3.20) is satisfied, then $S_n = \sum_{k=1}^{\infty} a_{nk}V_{nk}$ converges completely to 0. However, Corollary 4.6 of [9] does not apply solely because $\mathbf{E}|X|^{p+1} = \infty$.

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