Crack Distribution Parameters Estimation by the Method of Moments

Supawadee Kumnadee[1*] ,Andrei Volodin[2] and Supranee Lisawadi[3]

[1,3]Department of Mathematics and Statistics, Faculty Science and Technology, Thammasat University.

[2]School of Mathematics and Statistics (M019)University of Western Australia.

[*] E-mail: aun_haha@hotmail.com

ABSTRACT

In this article we study three parameters crack distribution (CR) which has numerous applications in engineering. This distribution contains as special cases some well known two-parameter distributions, namely, inverse Gaussian (IG), length biased inverse Gaussian (LB), and Birnbaum-Saunders (BS). The purpose of our research is estimate the parameters for crack distribution by the method of moments. Numerical methods are used to solve method of moments equations for three parameters of crack distribution for particular data sets. Computer algebra system, Maple version 11 is implemented and applied to solve numeric solutions for the method of moments. In future studies we could apply this three-parameters estimator to generate the random numbers that follow three-parameter Crack distribution and derive some distribution characteristics and graphs.

Keywords: Crack Distribution , Parameters Estimation, inverse Gaussian distribution , Length biased inverse Gaussian Distribution, Birnbaum-Saunders Distribution , Method of Moments , Numerical Methods.

1. INTRODUCTION

Three parameters crack distribution has many applications in engineering. This distribution contains well known special cases as two-parameter distributions, inverse Gaussian(IG), length biased inverse Gaussian(LB), and Birnbaum-Saunders(BS). We will begin with the literatures survey on the inverse Gaussian, length biased inverse Gaussian, and Birnbaum-Saunders distributions. In the study of crack distribution, we will focus on the derivation of the characteristic function and parameters estimation by the method of moments. In addition, we will explain the up-to-date existing literatures in this area and explain the interesting part of this subject.

Ahmed et al (2008) studied parametric estimation for the parameters of Birnbaum-Saunders lifetime distribution based on a new parametrization.A new parameterization of the two-parameter Birnbaum-Saunders lifetime distribution is considered. Moreover, Leiva et al. (2008) provided a lifetime analysis based on the generalized Birnbaum–Saunders distribution. The estimation method is examined by means of Monte Carlo simulations.

Recently, Besides, Lisawadi(2009), investigated parameter estimation by the method of moments for the two-sided Birnbaum-Saunders distribution. Life-time Birnbaum-Saunders distribution is commonly used in practical applications of the reliability theory for products with failure due to a development of fatigue cracks. Consider a rectangular metal block which is fixed from two sides, a periodic loading is applied to its middle part and this leads to a development of a fatigue crack. The same year, Kundu, Balakrishnan and Jamalizadeh (2009) studied bivariate Birnbaum-Saunders distribution and associated inference procedures. Univariate Birnbaum-Saunders distribution has been used quite effectively in modeling positively skewed data, especially lifetime data and crack growth data.

However, a revision of literature suggests us that nothing was done about the parameter estimation of crack distributions and its characteristics function. Note that the crack distribution contains as special cases Birnbaum-Saunders, length biased inverse Gaussian (LB) distribution, and inverse-Gaussian distribution. Accordingly, it is interesting to investigate the estimation procedure for parameters of the crack distribution. Our research interest is to estimate the parameters by method of moments. The theoretical (true) moments of the crack distribution are derived by the method of Taylor series expansion of the characteristic function. Because crack distribution contains three parameters $\left(P,\lambda,\theta\right)$, we need three central moments.

2.RESEARCH OBJECTIVES

In this research, we will estimate the parameters for crack distribution by the method of moments. We numerically estimate three parameters crack distribution with method of moments for particular data sets.

3. METHODOLOGY

3.1 Characteristic Function of LB, IG and BS Distribution

The length biased inverse Gaussian (LB) distribution has the density function:

$$f_{LB}(x;\lambda,\theta)=\frac{1}{\sqrt{2\pi}\theta}\left(\frac{x}{\theta}\right)^{\frac{1}{2}}\exp\left\{-\frac{1}{2}\left(\lambda\sqrt{\frac{\theta}{x}}-\sqrt{\frac{x}{\theta}}\right)^{2}\right\},\text{ where x > 0}$$

The characteristic function of the LB(λ , θ) distribution is

$$\varphi_{LB}(t;\lambda,\theta)=(1-2i\theta t)^{-1/2}\exp\left\{\lambda\left[1-(1-2i\theta t)^{1/2}\right]\right\}$$

The Inverse-Gaussian IG(λ , θ) distribution has the density function:

$$f_{IG}(x;\theta,\lambda)=\frac{\lambda}{\theta\sqrt{2\pi}}\left(\frac{\theta}{x}\right)^{\frac{3}{2}}\exp\left\{-\frac{1}{2}\left(\lambda\sqrt{\frac{\theta}{x}}-\sqrt{\frac{x}{\theta}}\right)^{2}\right\},\text{ where x>0}$$

The characteristic function of the IG(λ , θ) distribution is

$$\varphi_{IG}(t;\lambda,\theta)=\exp\left\{\lambda\left[1-(1-2i\theta t)^{1/2}\right]\right\}$$

The Birnbaum-Saunders BS(λ , θ) distribution has the density function:

$$f_{BS}(x;\theta,\lambda)=\frac{1}{2\sqrt{2\pi}\theta}\left[\lambda\left(\frac{\theta}{x}\right)^{\frac{3}{2}}+\left(\frac{x}{\theta}\right)^{\frac{1}{2}}\right]\exp\left\{-\frac{1}{2}\left(\lambda\sqrt{\frac{\theta}{x}}-\sqrt{\frac{x}{\theta}}\right)^{2}\right\},\text{ where x > 0}$$

The characteristic function of the BS(λ , θ) distribution is

$$\varphi_{BS}(t;\lambda,\theta)=\frac{1}{2}\left[1+(1-2i\theta t)^{-1/2}\right]\exp\left\{\lambda\left[1-(1-2i\theta t)^{1/2}\right]\right\}$$

3.2 The method of moments

The Moment method is oldest methods for deriving the estimators of distribution parameters. Nevertheless, Moment method is based on the assumption that the sample moments should provide good estimates of the corresponding population moments. Then because the population moments will be functions of population parameters, we will equate corresponding population and sample moments and solve for the desired parameters. the method outlined in

Volodin, A. I. (2002), p.39-42. Although the moment of most distribution can be determined directly by evaluating the necessary integrals or sums, there is an alternative procedure which sometimes provides considerable simplifications. This technique utilizes characteristic functions

Definition: The characteristic function of a random variable X ,where it exists is given by

$$\phi_X(t) = E(e^{itX}) = \sum_x e^{itx} f(x),$$

when X is discrete and

$$\phi_X(t) = E(e^{itX}) = \int_{-\infty}^{\infty} e^{itx} f(x)dx,$$

when X is continuous. The independent variable is t, and we are usually interested in values of t in the neighborhood of 0. To explain why this function could help us to find moments of X , let us substitute for e^{itx} its Maclaurin's series expansion, namely,

$$e^{itx} = 1 + itx + \frac{(itx)^2}{2!} + \frac{(itx)^3}{3!} + \cdots + \frac{(itx)^r}{r!} + \cdots$$

For the discrete case, we thus get

$$\begin{aligned}\phi_X(t) &= \sum_x \left[1 + itx + \frac{(itx)^2}{2!} + \cdots + \frac{(itx)^r}{r!} + \cdots\right] f(x) \\ &= \sum_x f(x) + it\sum_x xf(x) + \frac{(it)^2}{2!}\sum_x x^2 f(x) + \cdots + \frac{(it)^r}{r!}\sum_x x^r f(x) + \cdots \\ &= 1 + (it)\mu + \frac{(it)^2}{2!}\mu_2' + \cdots + \frac{(it)^r}{r!}\mu_r' + \cdots\end{aligned}$$

And it can be seen that in the Maclaurin's series of the characteristic function of X , the coefficient of $\frac{(it)^r}{r!}$ is μ_r' ,the r -th moment about the origin. In the continuous case, the argument is the same.

Guy Lebanon(2006), The Method of Moment Estimator. We have defined some desirable properties of estimators such as efficiency, consistency and sufficiency.However, we have not seen any general purpose method for obtaining good estimators. The method of moment estimator and maximum likelihood estimator are two such general purpose methods. They generally obtain consistent estimators and are usually straightforward to numerically calcualte using computational software. In this note we present the method of moment estimation (MOME) method.

Definition 1. The *k-moment* of *a* RV X is E (X^k).

The motivation behind the mome is that if we have a good estimator $\hat{\theta}$, the distribution that underlies $\hat{\theta}$ should be similar to the distribution of θ - where similarity is compared by equality of moments. However, we do not know the moments of the distribution that corresponds to θ since we don't know the value of θ. For this reason we approximate it by the sample moment - the moment computed by the given sample (that was generated from $\hat{\theta}$). In other words, we would choose $\hat{\theta}$ such that.

$$E_{\theta}(X) = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

In case that θ is a vector of k components, we need more than one equation. Specifically we have *k*-unknown parameters so we need *k* equations. Therefore we require the equality of the first *k* moments.

*Definition 2***.** Let $X_1, \ldots, X_n$ be iid sample from P - *a* distribution with a *k*-dimensional parameter vector θ. The method of moment estimator (MOME) $\hat{\theta}$ is the solution to the following system of equations.

$$E_{\hat{\theta}}(X^j) = \frac{1}{n}\sum_{i=1}^{n} X_i^j \quad , k = 1, 2, \ldots, k.$$

3.3 Taylor and MacLaurin Series

If the given function has derivatives of all orders and $R_n(x) \to 0$ as $n \to \infty$. If a function $f(x)$ has continuous derivatives up to *(n+1)*th order, then this function can be expanded in the following fashion. Taylor's theorem applies to any sufficiently differentiable function *f*, giving an approximation, for x near a point a, of the form

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \ldots + \frac{f^{(n)}(a)(x-a)^n}{n!} + R_n.$$

where R_n, called the *remainder* after n+1 terms, is given by:

$$R_n = \int_a^x f^{(n+1)}(u)\frac{(x-u)^n}{n!}du = \frac{f^{(n+1)}(\xi)(x-a)^{n+1}}{(n+1)!} \quad where \quad a < \xi < x.$$

3.4 Characteristic Function for the Crack Distribution

The density function of the CR(P, λ, θ) distribution is

$$f_{CR}(x; P, \theta, \lambda) = pf_{IG}(x; \lambda, \theta) + (1-p)f_{LB}(x; \lambda, \theta) x > 0; \lambda > 0; \theta > 0; 0 \le p \le 1$$

$$f_{CR}(x;P,\theta,\lambda)=p\left[\frac{\lambda}{\theta\sqrt{2\pi}}\left(\frac{\theta}{x}\right)^{\frac{3}{2}}\exp\left\{-\frac{1}{2}\left(\lambda\sqrt{\frac{\theta}{\lambda}}-\sqrt{\frac{x}{\theta}}\right)^{2}\right\}\right]$$
$$+\left(1-p\right)\left[\frac{1}{\sqrt{2\pi\theta}}\left(\frac{x}{\theta}\right)^{\frac{1}{2}}\exp\left\{-\frac{1}{2}\left(\lambda\sqrt{\frac{\theta}{x}}-\sqrt{\frac{x}{\theta}}\right)^{2}\right\}\right]$$

where x>0 ; $\lambda > 0$; $\theta > 0$; $0 \le p \le 1$.

The characteristic function of the CR(P, λ, θ) distribution is

$$\varphi_{CR}(t;\lambda,\theta)=\left[p+(1-p)(1-2i\theta t)^{-\frac{1}{2}}\right]\exp\left\{\lambda\left[1-(1-2i\theta t)^{\frac{1}{2}}\right]\right\}$$

We use the notation CR (P, λ, θ) for this distribution. It is easy to see that

CR(0, λ, θ)= LB(λ, θ), CR(1, λ, θ)= IG(λ, θ), and CR(1/2, λ, θ)= BS(λ, θ).

This can be derived from the formula

$f_{CR}(x; p, \lambda, \theta) = p f_{IG}(x; \lambda, \theta) + (1-p) f_{LB}(x; \lambda, \theta)$ $x > 0$; $\lambda > 0$; $\theta > 0$; $0 \le p \le 1$.

which gives the density function for the Crack distribution.

3.5 Calculate of the first three moments for the Crack Distribution

In the case of $CR(P, \lambda, \theta)$ distribution, the logarithm of the characteristic function is

$$\ln\left(\varphi_{CR}(t;\lambda,\theta)\right)=\ln\left\{\left[p+(1-p)(1-2i\theta t)^{-\frac{1}{2}}\right]\exp\left\{\lambda\left[1-(1-2i\theta t)^{\frac{1}{2}}\right]\right\}\right\}$$

Here we assume $\theta = 1$. If this is not the case, then instead of t we should write θt. Later we will return to general θ.

For the series expansion we use the formulae below. We note that all the functions of a complex variable we consider here are analytic functions in some neighborhood of zero, if one is careful about choosing branches. If we remove the negative real axis and choose the value of arg(x) on the resulting domain, then all analytic functions have Taylor series with the coeffcients given by Taylor's formula and these series converge on some open disk about the origin. For our functions it will be sufficient to assume that this disk has radius 1, that is $|x|<1$, cf. Ahlfors (1978). Thus,

$$\left(1-x\right)^{-\frac{1}{2}}=1+\frac{1}{2}x+\frac{1\cdot 3}{2\cdot 4}x^{2}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^{3}+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^{4}+O\left(x^{5}\right)$$

And

$$(1-x)^{\frac{1}{2}}=1-\frac{1}{2}x-\frac{1\cdot 1}{2\cdot 4}x^2-\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}x^3-\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^4-O\left(x^5\right)$$

where $|x|<1$

With 0 < t < 1/2 and x = 2it, we obtain from the formulae above

$$\ln\varphi(t)=\ln\left[p+(1-p)\left(1+(1-2i\theta t)^{-\frac{1}{2}}\right)\right]+\lambda\left[1-(1-2i\theta t)^{\frac{1}{2}}\right]$$

$$\ln\varphi(t)=\ln\left[p+(1-p)\left(1+\frac{1}{2}(2i\theta t)+\frac{1\cdot 3}{2\cdot 4}(2i\theta t)^2+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}(2i\theta t)^3+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}(2i\theta t)^4+\ldots\right)\right]$$

$$+\lambda\left[1-\left(1-\frac{1}{2}(2i\theta t)-\frac{1\cdot 1}{2\cdot 4}(2i\theta t)^2-\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}(2i\theta t)^3-\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}(2i\theta t)^4+\ldots\right)\right]$$

$$\ln\varphi(t)\approx\ln\left[1+(1-p)\left((i\theta t)+\frac{3}{2}(i\theta t)^2+\frac{5}{2}(i\theta t)^3\right)\right]+\lambda\left[(i\theta t)+\frac{1}{2}(i\theta t)^2+\frac{1}{2}(i\theta t)^3\right]$$

Now we use the formula

$$\ln(1+x)=1+x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+O(x^5)$$

where again $|x|<1$, and expand $\ln[\cdot]$ in the last representation $\ln\varphi(t)$:

Let $A=\left(i\theta t+\frac{3}{2}(i\theta t)^2+\frac{5}{2}(i\theta t)^3\right)$

$$\ln[\cdot]\approx\left[1+(1-p)A-\frac{(1-p)^2}{2}A^2+\frac{(1-p)^3}{3}A^3\right]+\lambda\left[(i\theta t)+\frac{1}{2}(i\theta t)^2+\frac{1}{2}(i\theta t)^3\right]$$

$$\approx\left[1+(1-p)\left(i\theta t+\frac{3}{2}(i\theta t)^2+\frac{5}{2}(i\theta t)^3\right)\right.$$

$$-\frac{(1-p)^2}{2}\left[(i\theta t)^2+\frac{9}{4}(i\theta t)^4+\frac{25}{4}(i\theta t)^6+2\cdot\frac{3}{2}(i\theta t)^3+2\cdot\frac{5}{2}(i\theta t)^4\right.$$

$$\left.\left.+2\cdot\frac{15}{4}(i\theta t)^5\right]+\frac{(1-p)^3}{3}\left[(i\theta t)^3+\ldots\right]\right]+\lambda\left[(i\theta t)+\frac{1}{2}(i\theta t)^2+\frac{1}{2}(i\theta t)^3\right]$$

$$\approx 1+(1-p+\lambda)(i\theta t)+\left[\frac{3}{2}(1-p)-\frac{(1-p)^2}{2}+\frac{1}{2}\lambda\right](i\theta t)^2$$

$$+\left[\frac{5}{2}(1-p)-\frac{3(1-p)^2}{2}+\frac{(1-p)^3}{3}+\frac{1}{2}\lambda\right](i\theta t)^3$$

$$\ln\varphi(t)\approx 1+(1-p+\lambda)(i\theta t)+\left[\frac{3(1-p)-(1-p)^2+\lambda}{2}\right](i\theta t)^2$$
$$+\left[\frac{15(1-p)-9(1-p)^2+2(1-p)^3+3\lambda}{6}\right](i\theta t)^3$$

Hence, we have the expansion of the logarithm of the characteristic function

$$\ln\varphi(t)\approx 1+(1-p+\lambda)(i\theta t)+\left[\frac{3(1-p)-(1-p)^2+\lambda}{2}\right](i\theta t)^2$$
$$+\left[\frac{15(1-p)-9(1-p)^2+2(1-p)^3+3\lambda}{6}\right](i\theta t)^3$$

$$\ln\varphi(t)\approx(1-p+\lambda)\frac{(i\theta t)}{1!}+\left[3(1-p)-(1-p)^2+\lambda\right]\frac{(i\theta t)^2}{2!}$$
$$+\left[15(1-p)-9(1-p)^2+2(1-p)^3+3\lambda\right]\frac{(i\theta t)^3}{3!}$$

The expansion obtained gives the semi-invariants of a random variable τ having CR($P, \lambda,\ \theta$) -distribution. Here we return to arbitrary θ:

$$k_1(\tau)=\mu(\tau)=(1-p+\lambda)\theta, k_2(\tau)=\sigma^2(\tau)=\left(3(1-p)-(1-p)^2+\lambda\right)\theta^2,$$
$$k_3(\tau)=\mu_3(\tau)=\left(3\lambda+15(1-p)-9(1-p)^2+2(1-p)^3\right)\theta^3$$

The next step is to write the moments**;**

$$\mu(\tau)=k_1(\tau)=\theta(1-p+\lambda), \sigma^2(\tau)=k_2(\tau)=\theta^2\left[3(1-p)-(1-p)^2+\lambda\right] \text{ and}$$
$$\mu_3(\tau)=k_3(\tau)=\theta^3\left[3\lambda+15(1-p)-9(1-p)^2+2(1-p)^3\right]$$

4. RESULTS

Parameters estimate for crack distribution

The density function of the CR(P, $\lambda,\ \theta$) distribution is

$$f_{CR}(x;P,\theta,\lambda)=pf_{IG}(x;\lambda,\theta)+(1-p)f_{LB}(x;\lambda,\theta)x>0;\lambda>0;\theta>0;0\le p\le 1$$

$$f_{CR}(x;P,\theta,\lambda)=p\left[\frac{\lambda}{\theta\sqrt{2\pi}}\left(\frac{\theta}{x}\right)^{\frac{3}{2}}\exp\left\{-\frac{1}{2}\left(\lambda\sqrt{\frac{\theta}{\lambda}}-\sqrt{\frac{x}{\theta}}\right)^2\right\}\right]$$
$$+(1-p)\left[\frac{1}{\sqrt{2\pi\theta}}\left(\frac{x}{\theta}\right)^{\frac{1}{2}}\exp\left\{-\frac{1}{2}\left(\lambda\sqrt{\frac{\theta}{x}}-\sqrt{\frac{x}{\theta}}\right)^2\right\}\right]$$

where x>0 ; $\lambda > 0; \theta > 0; 0 \le p \le 1$.

The characteristic function of the CR(P, λ, θ) distribution is

$$\varphi_{CR}(t;\lambda,\theta)=\left[p+(1-p)(1-2i\theta t)^{-\frac{1}{2}}\right]\exp\left\{\lambda\left[1-(1-2i\theta t)^{\frac{1}{2}}\right]\right\}$$

The results of estimate the parameters for crack distribution by the method of moments following below equations.

Population Value.

$$\mu(\tau)=k_1(\tau)=\theta(1-p+\lambda),\sigma^2(\tau)=k_2(\tau)=\theta^2\left[3(1-p)-(1-p)^2+\lambda\right] \text{ and}$$

$$\mu_3(\tau)=k_3(\tau)=\theta^3\left[3\lambda+15(1-p)-9(1-p)^2+2(1-p)^3\right]$$

Sample Value.

$$\overline{x}=\hat{\theta}\left(1-\hat{p}+\hat{\lambda}\right),\ s^2=\hat{\theta}^2\left[3\left(1-\hat{p}\right)-\left(1-\hat{p}\right)^2+\hat{\lambda}\right] \text{ and}$$

$$m_3=\hat{\theta}^3\left[3\hat{\lambda}+15\left(1-\hat{p}\right)-9(1-\hat{p})^2+2(1-\hat{p})^3\right]$$

The numerical to solve method of moments estimates equations and use Maple Program version 11. The results of estimates equations for parameter estimated values by P = 0.5 For each fixed values of three parameters we run simulations of corresponding random numbers independently, the simulations are repeated 1,000 times for constructing and reporting parameter estimation of Crack distribution by using the program Maple version 11.

5. CONCLUSION AND FUTURE WORK

The summary of the results on the crack distribution are presented below. We have the parameters estimator for Crack distribution by the method of moments**.** We provide a numerical solution of method of moments estimates equations and use Maple Program version 11 for a particular data set. The results of estimates equations for parameter estimated values by p = 0.5. The simulations are repeated 1,000 times for constructing and reporting parameter estimation of Crack distribution. There are the results of good parameter estimated have eight cases. We will find the the results of estimates equations for parameter estimated values by p = 0.5. There are the results of good parameter estimated have eight cases for observation is very large. For small samples method of moments estimates produce not precise results, especially for the estimation of parameter p. For future work we could suggest the following open problems:

1. Use other method to estimate parameters of Crack distribution.

2. Modify estimete equation of Crack distribution to simple form.

3. Compare method of different estimation methods .

4. Apply a generator of random numbers that follow three-parameter Crack distribution to compare the shape of generated distributions for various values of parameters.

REFERENCES

Ahmed, A. N. and Abouammoh. (1992). Characterizations of gamma, inverse Gaussian, and negative binomial distributions via their length-biased distributions. Statistical Papers ,34,167-173.

Ahmed, S. E., Budsaba ,K., Lisawadi ,S.and Volodin, A. (2008). Parametric Estimation for the Birnbaum-saunders Lifetime Distribution Based on a New Parametrization. Thailand Statistician,6(2) , 213-240.

Leiva, V, Riquelme ,M., Balakrishnan, N.and Sanhueza, A. (2008). Lifetime analysis based on the generalized Birnbaum–Saunders distribution. Computational Statistics & Data Analysis ,52 , 2079 – 2097.

Lisawadi,S. (2009). Parameter Estimation for the Two-Sided BS and IG Distributions. VDM Verlag Dr. Müller.

Ng, H. K.T.Kundu,D.Balakrishnan, N. (2006). Point and interval estimation for the two-parameter Birnbaum–Saunders distribution based on Type-II censored samples. Computational Statistics & Data Analysis, 50 , 3222 – 3242.

Wang, Z.Desmond , A.F,and Lu X. (2006). Modified censored moment estimation for the two-parameter Birnbaum–Saunders distribution. Computational Statistics & Data Analysis, 50, 1033 – 1051.