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A NOTE ON THE RATE OF COMPLETE CONVERGENCE FOR MAXIMUS OF PARTIAL SUMS FOR MOVING AVERAGE PROCESSES IN RADEMACHER TYPE BANACH SPACES

(submitted by D. Kh. Mushtari)

Abstract. We obtain the complete convergence rates for maximums of partial sums of Banach space valued random elements consisting of a moving average process. The corresponding almost sure convergence results for partial sums are derived, too.

1. Introduction

The concept of complete convergence was first introduced by Hsu and Robbin (1947) as follows. A sequence of random variables $\{U_n, n \geq\}$ is said to *converge completely* to a constant c if $\sum_{n=1}^{\infty} P\{|U_n - c| > \varepsilon\} < \infty$ for all $\varepsilon > 0$. By the Borel-Cantelli lemma, this implies $U_n \to c$ almost surely (a.s.) and the converse implication is true if the $\{U_n, n \geq 1\}$ are independent. Hsu and Robbin (1947) proved that the sequence of arithmetic means of independent and identically distributed random

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variables converges completely to the excepted value if the variance of the summands is finite. Their research was continued by Erdös (1949,1950), and Baum and Katz(1965) among others.

The following generalization of the Hsu and Robbin (1947) result was obtained in Baum and Katz(1965).

Theorem A. *Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed random variables, $S_n = \sum_{k=1}^{n} X_k$, $\beta \geq -1$, and $0 < \nu < 2$. Then the conditions $E|X_1|^{(\beta+2)\nu} < \infty$ and $EX_1 = 0$ for the case $\nu \geq 1$, are necessary and sufficient for*

$$\sum_{n=1}^{\infty} n^{\beta} P\left\{|S_n| > \varepsilon n^{1/\nu}\right\} < \infty \text{ for all } \varepsilon > 0.$$

It is an interesting problem to investigate the rate of complete convergence for dependent random variables. One of the first results in this direction, that is the rate of complete convergence for moving average sequences was in Li, Rao, and Wang (1992). This gives a partial solution for the sufficiency part of the Baum-Katz statement for $\beta = 0$.

Theorem B. *Let $\{Y_n, -\infty < n < \infty\}$ denote a double infinite sequence of independent identically distributed random variables, and let $V_k = \sum_{i=-\infty}^{\infty} a_{i+k} Y_i$ for $k \geq 1$ and $S_n = \sum_{k=1}^{n} V_k$ for $n \geq 1$. If $EY_1 = 0, E|Y_1|^{2\nu} < \infty, 1 \leq \nu < 2$, then*

$$\sum_{n=1}^{\infty} P\left\{|S_n| > \varepsilon n^{1/\nu}\right\} < \infty \text{ for all } \varepsilon > 0.$$

The question of the rate of convergence of the moving average process for other values of parameter β and in Banach space setting was discussed in Ahmed, Giuliano Antonini, and Volodin (2002) and Chen, Sung, and Volodin (2006). In this paper we are interested only in the moving average process taking values in Banach space of Rademacher type (technical definitions will be discussed in the next section), and hence we will present only the following result. It contains the case $\beta > -1$ from Corollary 4.2 of Ahmed, Giuliano Antonini, and Volodin (2002) and the special case $\beta = -1$ from Corollary 3 of Chen, Sung, and Volodin (2006).

Theorem C. *Assume that $\{Y_i, -\infty < i < \infty\}$ is a doubly infinite sequence of independent mean zero random elements taking values in a separable real Rademacher type p, $1 < p < 2$, Banach space and is stochastically dominated by a random variable X. Let $\{a_i, -\infty < i <$*

$\infty\}$ be an absolutely summable sequence of real numbers and set $V_i = \sum_{k=-\infty}^{\infty} a_{i+k} Y_k, i \geq 1$ and $S_n = \sum_{k=1}^{n} V_k$ for $n \geq 1$. If $E|X|^{(\beta+2)\nu} < \infty$ where $\beta \geq -1$ and $1 \leq \nu < p$, then

$$\sum_{n=1}^{\infty} n^{\beta} P\{||S_n|| > \varepsilon n^{1/\nu}\} < \text{ for all } \varepsilon > 0.$$

We should mention that the proofs of the cases $\beta > -1$ from Corollary 4.2 of Ahmed, Giuliano Antonini, and Volodin (2002) and the special case $\beta = -1$ from Corollary 3 of Chen, Sung, and Volodin (2006) are completely different. The initial goal of the present investigation was to find unified proof of Theorem C as it is stated, but it appears that a stronger result can be obtained. Namely, in this paper we consider the rate of complete convergence of *maximums* of partial sums for moving average process.

The plan of the paper is as follows. In Section 2, we recall some well known definitions relevant to the current work. In Section 3, we prove Theorem D which presents a sufficient condition for the rate of complete convergence of *maximums* of partial sums for moving average process. As in Theorems B and C, Theorem D contains an assumption concerning the geometry of the underlying Banach space, namely it is assumed that it is of the Rademacher type p. In Section 4, we present a necessary and sufficient result for almost sure convergence of the moving average process. Finally, in Section 5, we provide an additional result for the rate of complete convergence of *supremums* of normed partial sums for moving average process.

2. Preliminaries

Let B be a real separable Banach space with norm $\|\cdot\|$ and $\{\Omega, \mathcal{F}, P\}$ be a probability space. A random element X taking values in B is defined as a Borel measurable function from $\{\Omega, \mathcal{F}\}$ into B with Borel sigma-algebra. The expected value of a B-valued random variable X is defined to be Bochner integral and is denoted by EX.

A Banach space is said to be of *Rademacher type* p, $1 \leq p \leq 2$ if there is a constant $C > 0$ such that

$$E\|\sum_{i=1}^{n} X_i\|^p \leq C \sum_{i=1}^{n} E\|X_i\|^p$$

for all $n \geq 1$ and each sequence $\{X_n, n \geq 1\}$ of independent mean zero random elements taking values in B with finite pth moments.

We know if B is of Rademacher type $p > 1$, then for each $r, 1 \le r \le p$, B is of Rademacher type r. Every separable Hilbert space and finite dimensional Banach space is of Rademacher type 2.

The interested reader can find the complete discussion of this and subsequent notions connected with the geometry of Banach spaces in the book by Ledoux and Talagrand (1991).

A double infinite sequence of random elements $\{Y_i, -\infty < i < \infty\}$ is said to be *stochastically dominated* by a random variable X if there exists a constant C such that

$$\sup_{-\infty<i<\infty} P\{\|Y_i\| > x\} \le CP\{|CX| > x\}$$

for all $x > 0$. In the following, C will be used to denote various positive constants.

When B is of Rademacher type p, $1 < p \le 2$, Shao (1988) showed the following inequality for each sequence $\{X_n, n \ge 1\}$ of independent,mean zero random elements taking values in B with finite qth moments ($q \ge p$)

$$\begin{aligned} &E \max_{1\le m\le n} \|\textstyle\sum_{i=1}^m X_i\|^q \\ &\quad\le (96q)^q \left((C\textstyle\sum_{i=1}^n E\|X_i\|^p)^{q/p} + E\max_{1\le i\le n} ||X_i||^q\right), \end{aligned} \tag{2.1}$$

where C is as in the definition of Rademacher type p.

The following lemma (see Lemma 3 of Chow and Lai (1973)) is important for the proof of our second result.

Lemma. *Let $\{W_n\}$ and $\{Z_n\}$ be two sequences of random variables such that $W_n + Z_n \to 0$, a.s. Assume that $\{\mathcal{F}_n\}$ is a monotone increasing sequence of σ-fields. For each $n \ge 1$, $W_1, \cdots, W_n$ are adapted to $\{\mathcal{F}_n\}$, and Z_n and $\{\mathcal{F}_n\}$ are independent. If $Z_n \to 0$ in probability, then both W_n and Z_n converge to zero almost surely.*

3. Main Results

Theorem D. *Let $\{Y_i, -\infty < i < \infty\}$ be a doubly infinite sequence of independent means 0 random elements taking values in a separable real Rademacher type p $(1 < p \le 2)$ Banach space B. Assume that $\{Y_i, -\infty < i < \infty\}$ is stochastically dominated by a real valued random variable X. Let $\{a_i, -\infty < i < \infty\}$ be an absolutely summable sequence of real numbers and set $V_i = \sum_{k=-\infty}^{\infty} a_{i+k} Y_k, i \ge 1$ and $S_n = \sum_{k=1}^n V_k$ for $n \ge 1$. If*

$$E|X|^{(\beta+2)\nu} < \infty, \text{ where } 1 \le \nu < p, \quad \nu(\beta+2) \ne 1 \text{ and } \beta \ge -1,$$

then

$$\sum_{n=1}^{\infty} n^{\beta} P\{\max_{1\le m\le n} \|S_m\| > \varepsilon n^{1/\nu}\} < \infty \text{ for all } \varepsilon > 0.$$

Proof. Let $b = \sum_{i=-\infty}^{\infty} a_i$. Note that

$$S_n = \sum_{k=1}^{n} \sum_{i=-\infty}^{\infty} a_i Y_{i+k} = \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_j$$

and

$$\begin{aligned} & n^{-1/\nu} \|E \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_j I(\|Y_j\| \\ \le\ & n^{1/\nu})\| \le n^{-1/\nu} \sum_{i=-\infty}^{\infty} |a_i| \|E \sum_{j=i+1}^{i+n} Y_j I(\|Y_j\| > n^{1/\nu})\| \\ \le\ & n^{-1/\nu} \sum_{i=-\infty}^{\infty} |a_i| \sum_{j=i+1}^{i+n} \|EY_j\| I(\|Y_j\| > n^{1/\nu}) \\ \le\ & b n^{1-1/\nu} E|X| I(|X| > n^{1/\nu}) \\ \le\ & b E|X|^{\nu} I(|X| > n^{1/\nu}) \to 0, \text{ as } n \to \infty. \end{aligned}$$

Hence for sufficiently large n we have

$$n^{1/\nu} \| \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} EY_j I(\|Y_j\| \le n^{1/\nu})\| < \varepsilon/4.$$

Let $Y_{nj} = Y_j I(\|Y_j\| \le n^{1/\nu}) - EY_j I(\|Y_j\| \le n^{1/\nu})$. Then according to the inequality above, in order to prove the theorem it is enough to prove that

$$I_1 = \sum_{n=1}^{\infty} n^{\beta} P\{\max_{1\le k\le n} \| \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_j I(\|Y_j\| > n^{1/\nu})\| \ge \varepsilon n^{1/\nu}/2\} < \infty$$

and

$$I_2 = \sum_{n=1}^{\infty} n^{\beta} P\{\max_{1\le k\le n} \| \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_{nj}\| \ge \varepsilon n^{1/\nu}/4\} < \infty.$$

For I_1 by Chebyshev inequality

$$\begin{aligned}
I_1 &\leq C\sum_{n=1}^{\infty} n^{\beta}n^{-1/\nu}E\max_{1\leq k\leq n}\|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+k}Y_jI(\|Y_j\|>n^{1/\nu})\| \\
&\leq C\sum_{n=1}^{\infty} n^{\beta-1/\nu+1}E|X|I(|X|>n^{1/\nu}) \\
&= C\sum_{n=1}^{\infty} n^{\beta-1/\nu+1}\sum_{m=n}^{\infty}E|X|I(m<|X|^{\nu}\leq m+1) \\
&= C\sum_{m=1}^{\infty}E|X|I(m<|X|^{\nu}\leq m+1)\sum_{n=1}^{m}n^{\beta-1/\nu+1} \\
&\leq C\sum_{m=1}^{\infty} m^{\beta+2-1/\nu}E|X|I(m<|X|^{\nu}\leq m+1) \\
&\leq CE|X|^{(\beta+2)\nu}<\infty.
\end{aligned}$$

For I_2, by Chebyshev and Hölder inequalities we have for $q\geq p$

$$\begin{aligned}
I_2 &\leq C\sum_{n=1}^{\infty} n^{\beta-q/\nu}E\max_{1\leq k\leq n}\|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+k}Y_{nj}\|^q \\
&\leq C\sum_{n=1}^{\infty} n^{\beta-q/\nu}E\left(\sum_{i=-\infty}^{\infty}|a_i|\max_{1\leq k\leq n}\|\sum_{j=i+1}^{i+k}Y_{nj}\|\right)^q \\
&\leq C\sum_{n=1}^{\infty} n^{\beta-q/\nu}(\sum_{i=-\infty}^{\infty}|a_i|)^{q-1}\sum_{i=-\infty}^{\infty}|a_i|E\max_{1\leq k\leq n}\|\sum_{j=i+1}^{i+k}Y_{nj}\|^q.
\end{aligned}$$

For the case $(\beta+2)\nu<p$, let $q=p$. By (2.1)

$$\begin{aligned}
&I_2\leq C\sum_{n=1}^{\infty} n^{\beta-p/\nu}\sum_{i=-\infty}^{\infty}|a_i|\sum_{k=i+1}^{i+n}E\|Y_{nk}\|^p \\
&\leq C\sum_{n=1}^{\infty} n^{\beta-p/\nu+1}E|X|^pI(|X|\leq n^{1/\nu}) \\
&= C\sum_{n=1}^{\infty} n^{\beta-p/\nu+1}\sum_{m=1}^{n}E|X|^pI(m-1<|X|^{\nu}\leq m)
\end{aligned}$$

$$
\begin{aligned}
&= C\sum_{m=1}^{\infty} E|X|^p I(m-1<|X|^\nu \le m)\sum_{n=m}^{\infty} n^{\beta-q/\nu+1} \\
&\le C\sum_{m=1}^{\infty} m^{\beta-p/\nu+2} E|X|^p I(m-1<|X|^\nu \le m) \\
&\le C\sum_{m=1}^{\infty} E|X|^{(\beta+2)\nu} I(m-1<|X|^\nu \le m) \\
&\le CE|X|^{(\beta+2)\nu} < \infty.
\end{aligned}
$$

For the case $(\beta+2)\nu \ge p$, let $q > \frac{\beta-1}{(1/\nu)-(1/p)}$. By (2.1)

$$
\begin{aligned}
I_2 &\le C\sum_{n=1}^{\infty} n^{\beta-q/\nu}\sum_{i=-\infty}^{\infty}|a_i|\left\{\left(\sum_{k=i+1}^{i+n} E\|Y_{nk}\|^p\right)^{q/p} + \sum_{k=i+1}^{i+n} E\|Y_{nk}\|^q\right\} \\
&\le C\sum_{n=1}^{\infty} n^{\beta-q/\nu+q/p}(E|X|^p I(|X|\le n^{1/\nu}))^{q/p} \\
&+ C\sum_{n=1}^{\infty} n^{\beta-q/\nu+1} E|X|^q I(|X|\le n^{1/\nu}).
\end{aligned}
$$

Since $E|X|^p I(|X|\le n^{1/\nu}) < \infty$, we have

$$
\sum_{n=1}^{\infty} n^{\beta-q/\nu+q/p}(E|X|^p I(|X|\le n^{1/\nu}))^{q/p} \le C\sum_{n=1}^{\infty} n^{\beta-q/\nu+q/p} < \infty
$$

and by the same argument as $I_2<\infty$ in the case $(\beta+2)\nu<2$,

$$
\sum_{n=1}^{\infty} n^{\beta-q/\nu+1} E|X|^q I(|X|\le n^{1/\nu}) < CE|X|^{(\beta+2)\nu} < \infty.
$$

The proof of the theorem is completed. □

Corollary. *Assume that* $\{Y_i, -\infty<i<\infty\}$ *is a doubly infinite sequence of independent means 0 random elements taking values in a separable real Rademacher type* p $(1<p\le 2)$ *Banach space* B *and is stochastically dominated by a real valued random variable* X*. Let* $\{a_i, -\infty<i<\infty\}$ *be an absolutely summable sequence of real numbers and set* $V_i = \sum_{k=-\infty}^{\infty} a_{i+k}Y_k, i\ge 1$ *and* $S_n=\sum_{k=1}^{n} V_k$ *for* $n\ge 1$*. If* $E|X|^\nu<\infty$*, where* $1<\nu<p$*, then*

$$
n^{-1/\nu}S_n \to 0 \ a.s.
$$

Proof. If $E|X|^\nu < \infty$, then by Theorem D with $\beta = -1$

$$\sum_{n=1}^{\infty} n^{-1} P\{\max_{1\le m\le n} \|S_m\| > \varepsilon n^{1/\nu}\} < \infty, \quad \text{for all } \varepsilon > 0.$$

Hence for all $\varepsilon > 0$

$$\begin{aligned} \infty &> \sum_{n=1}^{\infty} n^{-1} P\{\max_{1\le m\le n} \|S_m\| > \varepsilon n^{1/\nu}\} \\ &= \sum_{k=1}^{\infty} \sum_{n=2^{k-1}}^{2^k} n^{-1} P\{\max_{1\le m\le n} \|S_m\| > \varepsilon n^{1/\nu}\} \\ &\ge 1/2 \sum_{k=1}^{\infty} P\{\max_{1\le m\le 2^{k-1}} \| S_m\| > \varepsilon 2^{k/\nu}\}. \end{aligned}$$

By Borel-Cantelli Lemma,

$$2^{-k/\nu} \max_{1\le m\le 2^k} \|S_m\| \to 0 \ \text{ a.s.}$$

which implies that $n^{-1/\nu} S_n \to 0$ a.s. $\square$

4. Necessary and Sufficient Condition

The following theorem gives us the necessary and sufficient for the almost sure convergence of partial sums of moving average process.

Theorem E. *Let $\{Y, Y_i, -\infty < i < \infty\}$ be a double infinite sequence of independent identically distributed random elements taking values in a separable real Rademacher type p $(1 < p \le 2)$ Banach space B and $\{a_i, -\infty < i < \infty\}$ be an absolutely summable sequence of real numbers with $\sum_{i=-\infty}^{\infty} a_i \neq 0$ and $1 < \nu < p$. Let $V_i = \sum_{k=-\infty}^{\infty} a_{i+k} Y_k, i \ge 1$ and $S_n = \sum_{k=1}^{n} V_k$ for $n \ge 1$. Then $n^{-1/\nu} S_n \to 0$ a.s. if and only if*

$$EY = 0 \ \textit{and}\ E||Y||^\nu < \infty.$$

Proof. Note that the sufficiency was proved in the corollary. Hence, we should prove only the necessity part.

Assume that $n^{-1/\nu} S_n \to 0$ a.s. Then

$$n^{-1/\nu} V_n = n^{-1/\nu} S_n - \left(\frac{n-1}{n}\right)^{1/\nu} (n-1)^{-1/\nu} S_{n-1} \to 0, \ \text{a.s., too.}$$

Without loss of generality, we assume that $a_0 \neq 0$.

Let Y' and Y_i' be independent copies of Y and $Y_i, -\infty < i < \infty$, which are also independent of each other. Set $V_i' = \sum_{k=-\infty}^{\infty} a_{i+k} Y_k', i \ge 1$, then

$n^{-1/\nu}V_n' \to 0$ a.s., and hence

$$n^{-1/\nu}(V_n - V_n') = n^{-1/\nu} \sum_{k=-\infty}^{\infty} a_{n+k}(Y_k - Y_k') \to 0 \text{ a.s.}$$

Set

$$W_n = n^{-1/\nu} \sum_{k=-n+1}^{\infty} a_{n+k}(Y_k - Y_k'), \quad Z_n = n^{-1/\nu} \sum_{k=-\infty}^{-n} a_{n+k}(Y_k - Y_k').$$

Then $W_n + Z_n \to 0$ a.s., hence $W_n + Z_n \to 0$ in probability. By Lévy inequality, $Z_n \to 0$ in probability and it is easy to show that $\sigma(W_1, \cdots, W_n)$ and Z_n are independent. By Lemma we have that $Z_n \to 0$ a.s. Repeating the argument again, we have

$$n^{-1/\nu}a_0(Y_n - Y_n') \to 0 \text{ a.s.}$$

Since $a_0 \neq 0$, by Borel-Cantelli lemma $E||Y - Y'||^\nu < \infty$, that is $E||Y||^\nu < \infty$.

Because $E||Y||^\nu < \infty$, by Theorem D (sufficiency part of the current result), we obtain

$$n^{-1/\nu}S_n - n^{1-1/\nu}(\sum_{i=-\infty}^{\infty} a_i)EY \to 0 \text{ a.s.}$$

By $n^{-1/\nu}S_n \to 0$ a.s. and $\sum_{i=-\infty}^{\infty} a_i \neq 0$, we have that $EY = 0$. $\square$

Remark. It is interesting to find a different proof of the fact that $EY = 0$ in the necessity part of Theorem E that is not based on the sufficiency part. We expect that a geometry of the underlying Banach space does not play any role in the necessity part.

5. One additional result.

The following theorem was proved in Baum and Katz (1965).

Theorem F. *Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed random variables, $S_n = \sum_{k=1}^n X_k$, and $0 < q < 2$. Then the conditions $E|X_1|^{(\beta+2)\nu} < \infty$ and $EX_1 = 0$ for the case $\nu \geq 1$, and $\beta > -1$ are necessary and sufficient for*

$$\sum_{n=1}^{\infty} n^\beta P\left\{\sup_{k \geq n} |S_k|/k^{1/\nu} > \varepsilon\right\} < \infty \textit{ for all } \varepsilon > 0.$$

For $\beta > -1$, Theorem D provides the following extension of Theorem F for the moving average process.

Theorem G. *Assume that $\{Y_i, -\infty < i < \infty\}$ is a doubly infinite sequence of independent means 0 random elements taking values in a* *separable real Rademacher type p $(1 < p \leq 2)$ Banach space B and is stochastically dominated by a real valued random variable X. Let $\{a_i, -\infty < i < \infty\}$ be an absolutely summable sequence of real numbers and set $V_i = \sum_{k=-\infty}^{\infty} a_{i+k}Y_k, i \geq 1$ and $S_n = \sum_{k=1}^{n} V_k$ for $n \geq 1$. If*

$$E|X|^{(\beta+2)\nu} < \infty, \text{ where } 1 \leq \nu < p, \quad \nu(\beta+2) \neq 1 \text{ and } \beta > -1,$$

then

$$\sum_{n=1}^{\infty} n^{\beta} P\left\{\sup_{k\geq n} ||S_k||/k^{1/\nu} > \varepsilon\right\} < \infty \text{ for all } \varepsilon > 0.$$

Proof. By Theorem D

$$\sum_{n=1}^{\infty} n^{\beta} P\{\max_{1\leq m\leq n} \|S_m\| > \varepsilon n^{1/\nu}\} < \infty, \quad \text{for all } \varepsilon > 0.$$

Next, we have the following estimations:

$$\begin{aligned}
& \sum_{n=1}^{\infty} n^{\beta} P\left\{\sup_{k\geq n} ||S_k||/k^{1/\nu} > \varepsilon\right\} \\
= \ & \sum_{m=1}^{\infty} \sum_{n=2^{m-1}}^{2^m-1} n^{\beta} P\left\{\sup_{k\geq n} ||S_k||/k^{1/\nu} > \varepsilon\right\} \\
\leq \ & C \sum_{m=1}^{\infty} \sum_{n=2^{m-1}}^{2^m-1} 2^{m\beta} P\left\{\sup_{k\geq 2^{m-1}} ||S_k||/k^{1/\nu} > \varepsilon\right\} \\
\leq \ & C \sum_{m=1}^{\infty} 2^{m(\beta+1)} P\left\{\sup_{k\geq 2^{m-1}} ||S_k||/k^{1/\nu} > \varepsilon\right\} \\
= \ & C \sum_{m=1}^{\infty} 2^{m(\beta+1)} P\left\{\sup_{l\geq m} \max_{2^{l-1}<k\leq 2^l} ||S_k||/k^{1/\nu} > \varepsilon\right\} \\
\leq \ & C \sum_{m=1}^{\infty} 2^{m(\beta+1)} \sum_{l=m}^{\infty} P\left\{\max_{1\leq k\leq 2^l} ||S_k|| > \varepsilon 2^{(l-1)/\nu}\right\} \\
= \ & C \sum_{l=1}^{\infty} P\left\{\max_{1\leq k\leq 2^l} ||S_k|| > \varepsilon 2^{(l-1)/\nu}\right\} \sum_{m=1}^{l} 2^{m(\beta+1)} \\
= \ & C \sum_{l=1}^{\infty} 2^{l\beta} P\left\{\max_{1\leq k\leq 2^l} ||S_k|| > \varepsilon 2^{(l-1)/\nu}\right\} \quad (\text{since} \beta > -1) \\
\leq \ & C \sum_{n=1}^{\infty} n^{\beta} P\left\{\max_{1\leq k\leq n} ||S_k|| > (\varepsilon/2^{1/\nu}) n^{1/\nu}\right\} < \infty. \square
\end{aligned}$$

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