STRONG CONVERGENCE PROPERTIES FOR ARRAYS OF ROWWISE NEGATIVELY ORTHANT DEPENDENT RANDOM VARIABLES

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(Communicated by Gejza Wimmer)

ABSTRACT. Let \( \{X_{ni}, i \geq 1, n \geq 1\} \) be an array of rowwise negatively orthant dependent random variables which is stochastically dominated by a random variable \( X \). Wang et al. [15. Complete convergence for arrays of rowwise negatively orthant dependent random variables, RACSAM, 106 (2012), 235–245] studied the complete convergence for arrays of rowwise negatively orthant dependent random variables under the condition that \( X \) has an exponential moment, which seems too strong. We will further study the complete convergence for arrays of rowwise negatively orthant dependent random variables under the condition that \( X \) has a moment, which is weaker than exponential moment. Our results improve the corresponding ones of Wang et al. [15].

1. Introduction

Let \( \{X, X_n, n \geq 1\} \) be a sequence of identically distributed random variables and \( \{a_{ni}, 1 \leq i \leq n, n \geq 1\} \) an array of constants. The strong convergence results for weighted sums \( \frac{1}{b_n} \sum_{i=1}^{n} a_{ni}X_i \) have been studied by many authors, see for example, Bai and Cheng [2], Cai [3], Chen and Gan [6], Cuzick [7], Sung [13], Wang et al. [15–18], Wu [19–21], Zhou et al. [27], Xu and Tang [25, 26], Wu et al. [22], Tang [14] and so forth. Many useful linear statistics are these weighted sums. Examples include least squares estimators, nonparametric regression function estimators and jackknife estimates among others. Bai and Cheng [2] proved the strong law of large numbers for weighted sums

\[
\frac{1}{b_n} \sum_{i=1}^{n} a_{ni}X_i \rightarrow 0, \quad a.s.
\]

when \( \{X, X_n, n \geq 1\} \) is a sequence of independent and identically distributed random variables with \( EX = 0 \) and \( E \exp(h|X|^\gamma) < \infty \) for some \( h > 0 \) and \( \gamma > 0 \), and \( \{a_{ni}, 1 \leq i \leq n, n \geq 1\} \) is an...
array of constants satisfying

\[ A_\alpha \doteq \limsup_{n \to \infty} A_{\alpha,n} < \infty, \quad A_{\alpha,n}^2 \doteq \frac{1}{n} \sum_{i=1}^n |a_{ni}|^\alpha \]

for some \( 1 < \alpha < 2 \), where \( b_n = n^{1/\alpha}(\log n)^{1/\gamma+\gamma(\alpha-1)/\alpha(1+\gamma)} \).

The concept of complete convergence was introduced by Hsu and Robbins [10] as follows: a sequence of random variables \( \{U_n, n \geq 1\} \) is said to converge completely to a constant \( C \) if

\[ \sum_{n=1}^{\infty} P(|U_n - C| > \varepsilon) < \infty \quad \text{for all } \varepsilon > 0. \]

In view of the Borel-Cantelli lemma, this implies that \( U_n \to C \) almost surely (a.s.). The converse is true if the \( \{U_n, n \geq 1\} \) are independent. Hsu and Robbins [10] proved that the sequence of arithmetic means of independent and identically distributed (i.i.d.) random variables converges completely to the expected value if the variance of the summands is finite. Erdős [8] proved the converse. The result of Hsu-Robbins-Erdős is a fundamental theorem in probability theory and has been extended in several directions by many authors. One of the most important generalizations is the Baum-Katz-Spitzer type result. For more details about the Baum-Katz-Spitzer type results, one can refer to Spitzer [12], Baum and Katz [4] and Gut [9], and so forth.

A finite collection of random variables \( X_1, X_2, \ldots, X_n \) is said to be negatively associated (NA) if for every pair of disjoint subsets \( A_1, A_2 \) of \( \{1, 2, \ldots, n\} \),

\[ \text{Cov}(f(X_i : i \in A_1), g(X_j : j \in A_2)) \leq 0, \]

whenever \( f \) and \( g \) are coordinatewise nondecreasing such that this covariance exists. An infinite sequence \( \{X_n, n \geq 1\} \) is NA if every finite subcollection is NA.

An array of random variables \( \{X_{ni}, i \geq 1, n \geq 1\} \) is called rowwise NA random variables if for every \( n \geq 1 \), \( \{X_{ni}, i \geq 1\} \) is a sequence of NA random variables.

A finite collection of random variables \( X_1, X_2, \ldots, X_n \) is said to be negatively orthant dependent (NOD) if

\[ P(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n) \leq \prod_{i=1}^n P(X_i > x_i) \]

and

\[ P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n) \leq \prod_{i=1}^n P(X_i \leq x_i) \]

hold for all \( x_1, x_2, \ldots, x_n \in \mathbb{R} \). An infinite sequence \( \{X_n, n \geq 1\} \) is said to be NOD if every finite subcollection is NOD.

An array of random variables \( \{X_{ni}, i \geq 1, n \geq 1\} \) is called rowwise NOD if for every \( n \geq 1 \), \( \{X_{ni}, i \geq 1\} \) is a sequence of NOD random variables.

The concepts of NA and NOD sequences were introduced by Joag-Dev and Proschan [11]. Obviously, independent random variables are NOD. Joag-Dev and Proschan [11] pointed out that NOD is weaker than NA.

Recently, Cai [3] obtained the following complete convergence result for weighted sums of identically distributed NA random variables.

**Theorem 1.1.** Let \( \{X, X_n, n \geq 1\} \) be a sequence of NA random variables with identical distribution, and \( \{a_{ni}, 1 \leq i \leq n, n \geq 1\} \) be a triangular array of constants satisfying \( \sum_{i=1}^n |a_{ni}|^\alpha = O(n) \) for \( 0 < \alpha \leq 2 \). Let \( b_n = n^{1/\alpha} \log^{1/\gamma} n \) for some \( \gamma > 0 \). Furthermore, assume that \( EX = 0 \) when
1 < \alpha \leq 2. If \( E \exp (h |X|^\gamma) < \infty \) for some \( h > 0 \), then for any \( \varepsilon > 0 \),
\[
\sum_{n=1}^{\infty} \frac{1}{n} P \left( \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} a_{ni} X_i \right| > \varepsilon b_n \right) < \infty.
\]

Wang et al. [15] extended the result of Cai [3] for sequences of NA random variables to the case of arrays of rowwise NOD random variables and obtained the following result.

**Theorem 1.2.** Let \( \{X_{ni} : i \geq 1, n \geq 1\} \) be an array of rowwise NOD random variables which is stochastically dominated by a random variable \( X \) and \( \{a_{ni} : i \geq 1, n \geq 1\} \) be an array of real numbers. Assume that there exist some \( \delta \) with \( 0 < \delta < 1 \) and some \( \alpha \) with \( 0 < \alpha \leq 2 \) such that \( \sum_{i=1}^{n} |a_{ni}|^\alpha = O(n^\delta) \) and assume further that \( E X_{ni} = 0 \) when \( 1 < \alpha \leq 2 \). If for some \( h > 0 \) and \( \gamma > 0 \) such that \( E \exp(h |X|^\gamma) < \infty \), then for any \( \varepsilon > 0 \),
\[
\sum_{n=1}^{\infty} n^{\alpha - 2} P \left( \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} a_{ni} X_{ni} \right| > \varepsilon b_n \right) < \infty,
\]
where \( p \geq 1/\alpha \) and \( b_n = n^{1/\alpha} \log^{1/\gamma} n \).

All the results above are based on the condition that \( E \exp(h |X|^\gamma) < \infty \) for some \( h > 0 \) and \( \gamma > 0 \) (or for all \( h > 0 \) and some \( \gamma > 0 \)). The exponential moment seems too strong. The question is whether the exponential moment can be replaced by a moment, i.e., there exists a constant \( \beta > 0 \) such that \( E |X|^\beta < \infty \). Our answer is positive.

Our goal in this paper is to further study the complete convergence for arrays of rowwise NOD random variables under the condition that \( X \) has a moment, which is weaker than exponential moment. The results presented in this paper are inspired by Wang et al. [15]. The techniques used in the paper are the truncated method and the Rosenthal type inequality for NOD random variables.

**Definition 1.1.** An array of random variables \( \{X_{ni}, i \geq 1, n \geq 1\} \) is said to be stochastically dominated by a random variable \( X \) if there exists a positive constant \( C \) such that
\[
P(\{|X_{ni}| > x\}) \leq CP(|X| > x)
\]
for all \( x \geq 0, i \geq 1 \) and \( n \geq 1 \).

The following lemmas are useful for the proofs of the main results. The first one is a basic property for NOD random variables, which was given by Bozorgnia et al. [5].

**Lemma 1.1.** [5] Let random variables \( X_1, X_2, \ldots, X_n \) be NOD, \( f_1, f_2, \ldots, f_n \) be all nondecreasing (or all nonincreasing) functions, then random variables \( f_1(X_1), f_2(X_2), \ldots, f_n(X_n) \) are NOD.

The next one is the Rosenthal type inequality for NOD random variables. For the proofs, one can refer to Asadian et al. [1] and Wu [24].

**Lemma 1.2.** [1, 24] Let \( p \geq 2 \) and \( \{X_n, n \geq 1\} \) be a sequence of NOD random variables with \( EX_n = 0 \) and \( E|X_n|^p < \infty \) for every \( n \geq 1 \). Then there exists a positive constant \( C \) depending only on \( p \) such that for every \( n \geq 1 \),
\[
E \left( \sum_{i=1}^{n} X_i \right)^p \leq C \left\{ \sum_{i=1}^{n} E|X_i|^p + \left( \sum_{i=1}^{n} EX_i^2 \right)^{p/2} \right\},
\]
\[
E \left( \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} X_i \right|^p \right) \leq C \log^p 2n \left\{ \sum_{i=1}^{n} E|X_i|^p + \left( \sum_{i=1}^{n} EX_i^2 \right)^{p/2} \right\}.
\]
The last one is a basic property for stochastic domination. For the proof, one can refer to Wu [24] or Wang et al. [23].

**Lemma 1.3.** Let \( \{X_n, n \geq 1\} \) be a sequence of random variables which is stochastically dominated by a random variable \( X \). Then for any \( \alpha > 0 \) and \( b > 0 \),

\[
E|X_n|^\alpha I(|X_n| \leq b) \leq C_1 [E|X|^\alpha I(|X| \leq b) + b^\alpha P(|X| > b)],
\]

\[
E|X_n|^\alpha I(|X_n| > b) \leq C_2 E|X|^\alpha I(|X| > b),
\]

where \( C_1 \) and \( C_2 \) are positive constants.

Throughout the paper, let \( I(A) \) be the indicator function of the set \( A \). \( C \) denotes a positive constant which may be different in various places and \( a_n = O(b_n) \) stands for \( a_n \leq Cb_n \).

## 2. Main results and their proofs

Our main results are as follows.

**Theorem 2.1.** Let \( \{X_{ni}, i \geq 1, n \geq 1\} \) be an array of rowwise NOD random variables which is stochastically dominated by a random variable \( X \) and \( \{a_{ni}, i \geq 1, n \geq 1\} \) be an array of real numbers. Assume that the following two conditions are satisfied:

(i) There exist some \( \delta \) with \( 0 < \delta < 1 \) and some \( \alpha \) with \( 0 < \alpha < 2 \) such that \( \sum_{i=1}^{n} |a_{ni}|^\alpha = O(n^{\delta}) \)

and assume further that \( EX_{ni} = 0 \) when \( 1 < \alpha \leq 2 \);

(ii) \( p \geq 1/\alpha \). For some \( \beta \geq \max\{pa^2, \alpha + \frac{p(\alpha - 1)}{1 - \delta}, \alpha + 2, \alpha(pa - 1) + 2\delta\} \), \( E|X|^\beta < \infty \).

Then for any \( \varepsilon > 0 \),

\[
\sum_{n=1}^{\infty} n^{p\alpha - 2} \left( \max_{1 \leq j \leq n} \sum_{i=1}^{j} a_{ni}X_{ni} \right) > \varepsilon b_n < \infty, \tag{2.1}
\]

where \( b_n = n^{1/\alpha} \log^{1/\gamma} n \) for some \( \gamma > 0 \).

**Proof.** For fixed \( n \geq 1 \), define

\[
X_{i}^{(n)} = -b_n(I(X_{ni} < -b_n) + X_{ni}I(|X_{ni}| \leq b_n) + b_nI(X_{ni} > b_n), \quad i \geq 1,
\]

\[
T_{j}^{(n)} = \sum_{i=1}^{j} a_{ni}(X_{i}^{(n)} - EX_{i}^{(n)}), \quad j = 1, 2, \ldots, n.
\]

It is easy to check that for any \( \varepsilon > 0 \),

\[
\left( \max_{1 \leq j \leq n} \sum_{i=1}^{j} a_{ni}X_{ni} \right) > \varepsilon b_n \subset \left( \max_{1 \leq j \leq n} |X_{ni}| > b_n \right) \cup \left( \max_{1 \leq j \leq n} \sum_{i=1}^{j} a_{ni}X_{i}^{(n)} > \varepsilon b_n \right), \tag{2.2}
\]

which implies that

\[
P\left( \max_{1 \leq j \leq n} \sum_{i=1}^{j} a_{ni}X_{ni} \right) > \varepsilon b_n \right)
\]

\[
\leq P\left( \max_{1 \leq i \leq n} |X_{ni}| > b_n \right) + P\left( \max_{1 \leq j \leq n} \sum_{i=1}^{j} a_{ni}X_{i}^{(n)} \right) > \varepsilon b_n \tag{2.3}
\]

\[
\leq \sum_{i=1}^{n} P\left( |X_{ni}| > b_n \right) + P\left( \max_{1 \leq j \leq n} T_{j}^{(n)} \right) > \varepsilon b_n - \max_{1 \leq j \leq n} \sum_{i=1}^{j} a_{ni}EX_{i}^{(n)} \right).
\]
FIRSTLY, WE WILL SHOW THAT

\[ b_n^{-1} \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} a_{ni} E X_{i}^{(n)} \right| \to 0, \text{ as } n \to \infty. \]  

(2.4)

By \( \sum_{i=1}^{n} |a_{ni}|^\alpha = O(n^\delta) \) and Hölder’s inequality, we have for \( 1 \leq k < \alpha \) that

\[ \sum_{i=1}^{n} |a_{ni}|^k \leq \left( \sum_{i=1}^{n} |a_{ni}|^k \right)^{\frac{k}{\alpha}} \left( \sum_{i=1}^{n} 1 \right)^{\frac{\alpha-k}{\alpha}} \leq Cn. \]  

(2.5)

Hence, when \( 1 < \alpha \leq 2 \), we have by \( EX_{ni} = 0 \), Lemma 1.3, (2.5) (Taking \( k = 1 \)), Markov’s inequality and condition (ii) that

\[
\begin{align*}
\left( b_n^{-1} \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} a_{ni} E X_{i}^{(n)} \right| \right) \leq & \sum_{i=1}^{n} |a_{ni}|P(|X_{ni}| > b_n) + b_n^{-1} \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} a_{ni} E X_{ni} I(|X_{ni}| > b_n) \right| \\
\leq & C \sum_{i=1}^{n} |a_{ni}|P(|X| > b_n) + b_n^{-1} \sum_{i=1}^{n} |a_{ni}|E|X_{ni}|I(|X_{ni}| > b_n) \\
\leq & C \frac{E|X|^\beta}{b_n^\beta} + Cb_n^{-1} \sum_{i=1}^{n} |a_{ni}|E|X|I(|X| > b_n) \\
\leq & \frac{Cn}{n^{\beta/\alpha} \log^{2\gamma/\gamma} n} + Cb_n^{-1} nE|X|I(|X| > b_n) \\
= & \frac{Cn}{n^{\beta/\alpha} \log^{2\gamma/\gamma} n} + Cb_n^{-1} \sum_{k=1}^{\infty} E|X|I(b_k < |X| \leq b_{k+1}) \\
\leq & \frac{Cn}{n^{\beta/\alpha} \log^{2\gamma/\gamma} n} + Cb_n^{-1} \sum_{k=1}^{\infty} b_{k+1} P(|X| > b_k) \\
\leq & \frac{Cn}{n^{\beta/\alpha} \log^{2\gamma/\gamma} n} + Cb_n^{-1} \sum_{k=1}^{\infty} b_{k+1} E|X|^\beta \\
\leq & \frac{Cn}{n^{\beta/\alpha} \log^{2\gamma/\gamma} n} + Cb_n^{-1} \sum_{k=1}^{\infty} \frac{(k+1)^{1/\alpha} \log^{1/\gamma} (k+1)}{k^{\beta/\alpha} \log^{2\gamma/\gamma} k} \\
\leq & \frac{Cn}{n^{\beta/\alpha} \log^{2\gamma/\gamma} n} + Cb_n^{-1} \sum_{k=1}^{\infty} k^{1/\alpha + 1 - \beta/\alpha} \\
\leq & \frac{Cn}{n^{\beta/\alpha} \log^{2\gamma/\gamma} n} + Cn^{1/\alpha + 2 - \beta/\alpha} \to 0, \text{ as } n \to \infty.
\end{align*}
\]

(2.6)

Elementary Jensen’s inequality implies that for any \( 0 < s < t \),

\[
\left( \sum_{i=1}^{n} |a_{ni}|^t \right)^{1/t} \leq \left( \sum_{i=1}^{n} |a_{ni}|^s \right)^{1/s}.
\]

(2.7)
To prove (2.1), we only need to show that
\[ \max_{1 \leq j \leq n} \frac{1}{n} \sum_{i=1}^{j} a_{ni} E \mathbf{1}_{X_i}^n \leq C. \]

By (2.6) and (2.8), we can get (2.4) immediately. Hence, for \( n \) large enough,
\[
P \left( \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} a_{ni} X_i \right| > \varepsilon b_n \right) \leq \sum_{i=1}^{n} P \left( |X_n| > b_n \right) + P \left( \max_{1 \leq j \leq n} T_j^{(n)} > \frac{\varepsilon}{2} b_n \right).
\]

To prove (2.1), we only need to show that
\[ I \leq \sum_{n=1}^{\infty} n^{\alpha - 2} \sum_{i=1}^{n} P \left( |X_n| > b_n \right) < \infty \] (2.9)
and
\[ J \leq \sum_{n=1}^{\infty} n^{\alpha - 2} P \left( \max_{1 \leq j \leq n} T_j^{(n)} > \frac{\varepsilon}{2} b_n \right) < \infty. \] (2.10)
By Definition 1.1 Markov’s inequality and condition (ii), we can see that

\[
I \doteq \sum_{n=1}^{\infty} n^{p\alpha-2} \sum_{i=1}^{n} P(|X_{ni}| > b_n)
\leq C \sum_{n=1}^{\infty} n^{p\alpha-2} \sum_{i=1}^{n} P(|X| > b_n) \leq C \sum_{n=2}^{\infty} n^{p\alpha-1} E|X|^\beta
\frac{b_n^\beta}{E|X|^{\beta/\gamma}} \leq C \sum_{n=2}^{\infty} \frac{n^{p\alpha-1}}{n^{\beta/\alpha} \log^{\beta/\gamma} n} < \infty, \quad \text{(since } \beta > p\alpha^2). \tag{2.11}
\]

For fixed \(n \geq 1\), it is easily seen that \(\{X_{i}^{(n)}, 1 \leq i \leq n\}\) are still NOD by Lemma 1.1. For \(q > 2\), it follows from Lemma 1.2, C\(r\) inequality and Jensen’s inequality that

\[
J \doteq \sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1 \leq i \leq n} T_{j}^{(n)} \geq \frac{\varepsilon}{2} b_n\right) \leq C \sum_{n=2}^{\infty} n^{p\alpha-2} b_n^{-q} E\left(\max_{1 \leq i \leq n} |T_{j}^{(n)}|^q\right)
\leq C \sum_{n=2}^{\infty} n^{p\alpha-2} b_n^{-q} (\log n)^q \left[\sum_{i=1}^{n} |a_{ni}|^q E|X_i^{(n)}|^q + \left(\sum_{i=1}^{n} |a_{ni}|^2 E|X_i^{(n)}|^2\right)^{q/2}\right] \tag{2.12}
\]

\[
\leq J_1 + J_2.
\]

Taking a suitable constant \(q\) such that \(\max\{2, \alpha(p\alpha - 1)/(1 - \delta)\} < q < \min\{\beta - \alpha, \frac{\beta - p\alpha^2 + \alpha}{\delta}\}\), which implies that

\[
\beta > \alpha + q, \quad \frac{\beta}{\alpha} - \frac{q}{\alpha} > 1, \quad \beta > p\alpha^2 - \alpha + q\delta, \quad \frac{\beta}{\alpha} - p\alpha + 2 - q\delta > 1
\]

and

\[
p\alpha - 2 + q\delta - q > 1, \quad q > \alpha.
\]

It follows from C\(r\) inequality, Lemma 1.3, (2.7), Markov’s inequality and condition (ii) that

\[
J_1 \doteq C \sum_{n=2}^{\infty} n^{p\alpha-2} b_n^{-q} (\log n)^q \sum_{i=1}^{n} |a_{ni}|^q E|X_i^{(n)}|^q
\leq C \sum_{n=2}^{\infty} n^{p\alpha-2} b_n^{-q} (\log n)^q \sum_{i=1}^{n} |a_{ni}|^q \left[E|X_{ni}^k| \leq b_n + b_n^\beta P(|X_{ni}| > b_n)\right]
\leq C \sum_{n=2}^{\infty} n^{p\alpha-2} b_n^{-q} (\log n)^q \sum_{i=1}^{n} |a_{ni}|^q \left[E|X|^q I(|X| \leq b_n) + b_n^\beta P(|X| > b_n)\right]
\leq C \sum_{n=2}^{\infty} n^{p\alpha-2} b_n^{-q} (\log n)^q E|X|^q I(|X| \leq b_n)
\quad + C \sum_{n=2}^{\infty} n^{p\alpha-2} b_n^{-q} (\log n)^q P(|X| > b_n)
\leq C \sum_{n=2}^{\infty} n^{p\alpha-2} b_n^{-q} (\log n)^q \sum_{k=2}^{n} E|X|^q I(b_{k-1} < |X| \leq b_k)
\quad + C \sum_{n=2}^{\infty} n^{p\alpha-2} b_n^{-q} (\log n)^q \frac{E|X|^b}{b_n^\beta}
\]

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Therefore, the desired result \((2.1)\) follows from \((2.11)–(2.14)\) immediately. This completes the proof of the theorem. \(\square\)

Similar to the proof of Theorem 2.1 above and Theorems 2.3–2.6 of Wang et al. [15], we can get the following results.

**Theorem 2.2.** Let \(\{X_n, n \geq 1\}\) be a sequence of NOD random variables which is stochastically dominated by a random variable \(X\) and \(\{a_n, n \geq 1\}\) be a sequence of real numbers. Assume that there exist some \(\delta\) with \(0 < \delta < 1\) and some \(\alpha\) with \(0 < \alpha \leq 2\) such that \(\sum_{i=1}^{n} |a_i|^{\alpha} = O(n^{\delta})\) and assume further that \(EX_n = 0\) when \(1 < \alpha \leq 2\). If condition (ii) of Theorem 2.1 holds, then for any \(\varepsilon > 0\),

\[
\sum_{n=1}^{\infty} n^{p\alpha - 2} \beta P\left( \max_{1 \leq j \leq n} |S_j| > \varepsilon b_n \right) < \infty
\]
and

\[ \lim_{n \to \infty} \frac{|S_n|}{b_n} = 0 \quad \text{a.s.,} \]

where \( b_n = n^{1/\alpha} \log^{1/\gamma} n \) for some \( \gamma > 0 \) and \( S_n = \sum_{i=1}^{n} a_iX_i \) for \( n \geq 1 \).

**Theorem 2.3.** Let \( \{X_{ni} : i \geq 1, n \geq 1\} \) be an array of rowwise NOD random variables which is stochastically dominated by a random variable \( X \) and \( \{a_{ni} : i \geq 1, n \geq 1\} \) be an array of real numbers. Assume that there exists some \( \alpha \) with \( 0 < \alpha \leq 2 \) such that \( \sum_{i=1}^{n} |a_{ni}|^\alpha = O(n) \) and assume further that \( EX_{ni} = 0 \) when \( 1 < \alpha \leq 2 \). If there exists some \( \beta > \alpha + 2 \) such that \( EX|X|^\beta < \infty \), then for any \( \varepsilon > 0 \),

\[ \sum_{n=1}^{\infty} n^{-1} P \left( \max_{1 \leq j \leq n} | \sum_{i=1}^{j} a_{ni}X_{ni} | > \varepsilon b_n \right) < \infty, \]

where \( b_n = n^{1/\alpha} \log^{1/\gamma} n \) for some \( \gamma > 0 \).

**Theorem 2.4.** Let \( \{X_n, n \geq 1\} \) be a sequence of NOD random variables which is stochastically dominated by a random variable \( X \) and \( \{a_n, n \geq 1\} \) be an array of real numbers. Assume that there exists some \( \alpha \) with \( 0 < \alpha \leq 2 \) such that \( \sum_{i=1}^{n} |a_i|^\alpha = O(n) \) and assume further that \( EX_n = 0 \) when \( 1 < \alpha \leq 2 \). If there exists some \( \beta > \alpha + 2 \) such that \( EX|X|^\beta < \infty \), then for any \( \varepsilon > 0 \),

\[ \sum_{n=1}^{\infty} n^{-1} P \left( \max_{1 \leq j \leq n} | \sum_{i=1}^{j} a_{ni}X_{ni} | > \varepsilon b_n \right) < \infty, \]

where \( b_n = n^{1/\alpha} \log^{1/\gamma} n \) for some \( \gamma > 0 \).

**Theorem 2.5.** Let \( \{X_n, n \geq 1\} \) be a sequence of NOD random variables which is stochastically dominated by a random variable \( X \) and \( \{a_n, n \geq 1\} \) be a sequence of real numbers. Assume that there exists some \( \alpha \) with \( 0 < \alpha \leq 2 \) such that \( \sum_{i=1}^{n} |a_i|^\alpha = O(n) \) and assume further that \( EX_n = 0 \) when \( 1 < \alpha \leq 2 \). If there exists some \( \beta > \alpha + 2 \) such that \( EX|X|^\beta < \infty \), then for any \( \varepsilon > 0 \),

\[ \sum_{n=1}^{\infty} n^{-1} P \left( \max_{1 \leq j \leq n} |S_j | > \varepsilon b_n \right) < \infty \]

and

\[ \lim_{n \to \infty} \frac{|S_n|}{b_n} = 0 \quad \text{a.s.,} \]

where \( b_n = n^{1/\alpha} \log^{1/\gamma} n \) for some \( \gamma > 0 \) and \( S_n = \sum_{i=1}^{n} a_iX_i \) for \( n \geq 1 \).

**Acknowledgement.** The authors are most grateful to the Editor and anonymous referees for careful reading of the manuscript and valuable suggestions which helped in improving an earlier version of this paper.

**REFERENCES**


Received 27. 5. 2014
Accepted 13. 3. 2015

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