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STRONG CONVERGENCE PROPERTIES FOR ARRAYS OF ROWWISE NEGATIVELY ORTHANT DEPENDENT RANDOM VARIABLES

Aiting Shen* — Yu Zhang* — Andrei Volodin**

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ABSTRACT. Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise negatively orthant dependent random variables which is stochastically dominated by a random variable X. Wang et al. [15. *Complete convergence for arrays of rowwise negatively orthant dependent random variables*, RACSAM, **106** (2012), 235–245] studied the complete convergence for arrays of rowwise negatively orthant dependent random variables under the condition that X has an exponential moment, which seems too strong. We will further study the complete convergence for arrays of rowwise negatively orthant dependent random variables under the condition that X has a moment, which is weaker than exponential moment. Our results improve the corresponding ones of Wang et al. [15].

1. Introduction

Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed random variables and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ an array of constants. The strong convergence results for weighted sums $\sum_{i=1}^{n} a_{ni}X_i$ have been studied by many authors, see for example, Bai and Cheng [2], Cai [3], Chen and Gan [6], Cuzick [7], Sung [13], Wang et al. [15–18], Wu [19–21], Zhou et al. [27], Xu and Tang [25, 26], Wu et al. [22], Tang [14] and so forth. Many useful linear statistics are these weighted sums. Examples include least squares estimators, nonparametric regression function estimators and jackknife estimates among others. Bai and Cheng [2] proved the strong law of large numbers for weighted sums

$$\frac{1}{b_n}\sum_{i=1}^{n} a_{ni}X_i \to 0, \quad a.s.$$

when $\{X, X_n, n \geq 1\}$ is a sequence of independent and identically distributed random variables with $EX = 0$ and $E\exp(h|X|^{\gamma}) < \infty$ for some $h > 0$ and $\gamma > 0$, and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ is an

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array of constants satisfying

$$A_\alpha \doteq \limsup_{n\to\infty} A_{\alpha,n} < \infty, \qquad A^\alpha_{\alpha,n} \doteq \frac{1}{n}\sum_{i=1}^{n} |a_{ni}|^\alpha$$

for some $1 < \alpha < 2$, where $b_n = n^{1/\alpha}(\log n)^{1/\gamma+\gamma(\alpha-1)/\alpha(1+\gamma)}$.

The concept of complete convergence was introduced by Hsu and Robbins [10] as follows: a sequence of random variables $\{U_n, n \ge 1\}$ is said to converge completely to a constant C if $\sum\limits_{n=1}^{\infty} P(|U_n - C| > \varepsilon) < \infty$ for all $\varepsilon > 0$. In view of the Borel-Cantelli lemma, this implies that $U_n \to C$ almost surely (a.s.). The converse is true if the $\{U_n, n \ge 1\}$ are independent. Hsu and Robbins [10] proved that the sequence of arithmetic means of independent and identically distributed (i.i.d.) random variables converges completely to the expected value if the variance of the summands is finite. Erdös [8] proved the converse. The result of Hsu-Robbins-Erdös is a fundamental theorem in probability theory and has been extended in several directions by many authors. One of the most important generalizations is the Baum-Katz-Spitzer type result. For more details about the Baum-Katz-Spitzer type results, one can refer to Spitzer [12], Baum and Katz [4] and Gut [9], and so forth.

A finite collection of random variables $X_1, X_2, \ldots, X_n$ is said to be negatively associated (NA) if for every pair of disjoint subsets A_1, A_2 of $\{1, 2, \ldots, n\}$,

$$\mathrm{Cov}\{f(X_i : i \in A_1), g(X_j : j \in A_2)\} \le 0,$$

whenever f and g are coordinatewise nondecreasing such that this covariance exists. An infinite sequence $\{X_n, n \ge 1\}$ is NA if every finite subcollection is NA.

An array of random variables $\{X_{ni}, i \ge 1, n \ge 1\}$ is called rowwise NA random variables if for every $n \ge 1$, $\{X_{ni}, i \ge 1\}$ is a sequence of NA random variables.

A finite collection of random variables $X_1, X_2, \ldots, X_n$ is said to be negatively orthant dependent (NOD) if

$$P(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n) \le \prod_{i=1}^{n} P(X_i > x_i)$$

and

$$P(X_1 \le x_1, X_2 \le x_2, \ldots, X_n \le x_n) \le \prod_{i=1}^{n} P(X_i \le x_i)$$

hold for all $x_1, x_2, \ldots, x_n \in \mathbb{R}$. An infinite sequence $\{X_n, n \ge 1\}$ is said to be NOD if every finite subcollection is NOD.

An array of random variables $\{X_{ni}, i \ge 1, n \ge 1\}$ is called rowwise NOD if for every $n \ge 1$, $\{X_{ni}, i \ge 1\}$ is a sequence of NOD random variables.

The concepts of NA and NOD sequences were introduced by Joag-Dev and Proschan [11]. Obviously, independent random variables are NOD. Joag-Dev and Proschan [11] pointed out that NOD is weaker than NA.

Recently, Cai [3] obtained the following complete convergence result for weighted sums of identically distributed NA random variables.

Theorem 1.1. *Let $\{X, X_n, n \ge 1\}$ be a sequence of NA random variables with identical distribution, and $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be a triangular array of constants satisfying $\sum\limits_{i=1}^{n} |a_{ni}|^\alpha = O(n)$ for $0 < \alpha \le 2$. Let $b_n = n^{1/\alpha}\log^{1/\gamma} n$ for some $\gamma > 0$. Furthermore, assume that $EX = 0$ when*

$1 < \alpha \le 2$. *If* $E \exp\big(h|X|^\gamma\big) < \infty$ *for some* $h > 0$, *then for any* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \frac{1}{n} P\bigg(\max_{1\le j\le n} \bigg|\sum_{i=1}^{j} a_{ni} X_i\bigg| > \varepsilon b_n\bigg) < \infty.$$

Wang et al. [15] extended the result of Cai [3] for sequences of NA random variables to the case of arrays of rowwise NOD random variables and obtained the following result.

Theorem 1.2. *Let* $\{X_{ni} : i \ge 1, n \ge 1\}$ *be an array of rowwise NOD random variables which is stochastically dominated by a random variable* X *and* $\{a_{ni} : i \ge 1, n \ge 1\}$ *be an array of real numbers. Assume that there exist some* δ *with* $0 < \delta < 1$ *and some* α *with* $0 < \alpha \le 2$ *such that* $\sum_{i=1}^{n} |a_{ni}|^\alpha = O(n^\delta)$ *and assume further that* $EX_{ni} = 0$ *when* $1 < \alpha \le 2$. *If for some* $h > 0$ *and* $\gamma > 0$ *such that* $E \exp(h|X|^\gamma) < \infty$, *then for any* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{p\alpha-2} P\bigg(\max_{1\le j\le n} \bigg|\sum_{i=1}^{j} a_{ni} X_{ni}\bigg| > \varepsilon b_n\bigg) < \infty,$$

where $p \ge 1/\alpha$ *and* $b_n \doteq n^{1/\alpha} \log^{1/\gamma} n$.

All the results above are based on the condition that $E \exp(h|X|^\gamma) < \infty$ for some $h > 0$ and $\gamma > 0$ (or for all $h > 0$ and some $\gamma > 0$). The exponential moment seems too strong. The question is that whether the exponential moment can be replaced by a moment, i.e., there exists a constant $\beta > 0$ such that $E|X|^\beta < \infty$. Our answer is positive.

Our goal in this paper is to further study the complete convergence for arrays of rowwise NOD random variables under the condition that X has a moment, which is weaker than exponential moment. The results presented in this paper are inspired by Wang et al. [15]. The techniques used in the paper are the truncated method and the Rosenthal type inequality for NOD random variables.

Definition 1.1. An array of random variables $\{X_{ni}, i \ge 1, n \ge 1\}$ is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$P(|X_{ni}| > x) \le CP(|X| > x)$$

for all $x \ge 0$, $i \ge 1$ and $n \ge 1$.

The following lemmas are useful for the proofs of the main results. The first one is a basic property for NOD random variables, which was given by Bozorgnia et al. [5].

Lemma 1.1. ([5]) *Let random variables* $X_1, X_2, \dots, X_n$ *be NOD,* $f_1, f_2, \dots, f_n$ *be all nondecreasing (or all nonincreasing) functions, then random variables* $f_1(X_1), f_2(X_2), \dots, f_n(X_n)$ *are NOD.*

The next one is the Rosenthal type inequality for NOD random variables. For the proofs, one can refer to Asadian et al. [1] and Wu [24].

Lemma 1.2. ([1, 24]) *Let* $p \ge 2$ *and* $\{X_n, n \ge 1\}$ *be a sequence of NOD random variables with* $EX_n = 0$ *and* $E|X_n|^p < \infty$ *for every* $n \ge 1$. *Then there exists a positive constant* C *depending only on* p *such that for every* $n \ge 1$,

$$E\bigg|\sum_{i=1}^{n} X_i\bigg|^p \le C\bigg\{\sum_{i=1}^{n} E|X_i|^p + \bigg(\sum_{i=1}^{n} EX_i^2\bigg)^{p/2}\bigg\},$$

$$E\bigg(\max_{1\le j\le n} \bigg|\sum_{i=1}^{j} X_i\bigg|^p\bigg) \le C \log^p 2n \bigg\{\sum_{i=1}^{n} E|X_i|^p + \bigg(\sum_{i=1}^{n} EX_i^2\bigg)^{p/2}\bigg\}.$$

The last one is a basic property for stochastic domination. For the proof, one can refer to Wu [24] or Wang et al. [23].

Lemma 1.3. *Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X. Then for any $\alpha > 0$ and $b > 0$,*

$$E|X_n|^\alpha I\big(|X_n| \leq b\big) \leq C_1\big[E|X|^\alpha I\big(|X| \leq b\big) + b^\alpha P\big(|X| > b\big)\big],$$
$$E|X_n|^\alpha I\big(|X_n| > b\big) \leq C_2 E|X|^\alpha I\big(|X| > b\big),$$

where C_1 and C_2 are positive constants.

Throughout the paper, let $I(A)$ be the indicator function of the set A. C denotes a positive constant which may be different in various places and $a_n = O(b_n)$ stands for $a_n \leq C b_n$.

2. Main results and their proofs

Our main results are as follows.

Theorem 2.1. *Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise NOD random variables which is stochastically dominated by a random variable X and $\{a_{ni}, i \geq 1, n \geq 1\}$ be an array of real numbers. Assume that the following two conditions are satisfied:*

(i) *There exist some δ with $0 < \delta < 1$ and some α with $0 < \alpha \leq 2$ such that $\sum_{i=1}^{n} |a_{ni}|^\alpha = O(n^\delta)$ and assume further that $EX_{ni} = 0$ when $1 < \alpha \leq 2$;*

(ii) *$p \geq 1/\alpha$. For some $\beta > \max\{p\alpha^2, \alpha + \frac{\alpha(p\alpha-1)}{1-\delta}, \alpha + 2, \alpha(p\alpha - 1) + 2\delta\}$, $E|X|^\beta < \infty$.*

Then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| > \varepsilon b_n\right) < \infty, \tag{2.1}$$

where $b_n \doteq n^{1/\alpha}\log^{1/\gamma} n$ for some $\gamma > 0$.

P r o o f. For fixed $n \geq 1$, define

$$X_i^{(n)} = -b_n I(X_{ni} < -b_n) + X_{ni} I(|X_{ni}| \leq b_n) + b_n I(X_{ni} > b_n), \quad i \geq 1,$$
$$T_j^{(n)} = \sum_{i=1}^{j} a_{ni}\left(X_i^{(n)} - EX_i^{(n)}\right), \quad j = 1, 2, \dots, n.$$

It is easy to check that for any $\varepsilon > 0$,

$$\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| > \varepsilon b_n\right) \subset \left(\max_{1\leq i\leq n}|X_{ni}| > b_n\right) \cup \left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_i^{(n)}\right| > \varepsilon b_n\right), \tag{2.2}$$

which implies that

$$\begin{aligned} &P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| > \varepsilon b_n\right) \\ &\quad\leq P\left(\max_{1\leq i\leq n}|X_{ni}| > b_n\right) + P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_i^{(n)}\right| > \varepsilon b_n\right) \\ &\quad\leq \sum_{i=1}^{n} P\Big(|X_{ni}| > b_n\Big) + P\left(\max_{1\leq j\leq n}\left|T_j^{(n)}\right| > \varepsilon b_n - \max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}EX_i^{(n)}\right|\right). \end{aligned} \tag{2.3}$$

Firstly, we will show that

$$b_n^{-1} \max_{1\le j\le n} \left| \sum_{i=1}^{j} a_{ni} E X_i^{(n)} \right| \to 0, \quad \text{as} \quad n\to\infty. \tag{2.4}$$

By $\sum\limits_{i=1}^{n} |a_{ni}|^\alpha = O(n^\delta)$ and Hölder's inequality, we have for $1 \le k < \alpha$ that

$$\sum_{i=1}^{n} |a_{ni}|^k \le \left(\sum_{i=1}^{n} \left(|a_{ni}|^k \right)^{\frac{\alpha}{k}} \right)^{\frac{k}{\alpha}} \left(\sum_{i=1}^{n} 1 \right)^{\frac{\alpha-k}{\alpha}} \le Cn. \tag{2.5}$$

Hence, when $1 < \alpha \le 2$, we have by $EX_{ni} = 0$, Lemma 1.3, (2.5) (Taking $k = 1$), Markov's inequality and condition (ii) that

$$\begin{aligned}
& b_n^{-1} \max_{1\le j\le n} \left| \sum_{i=1}^{j} a_{ni} E X_i^{(n)} \right| \\
& \le \sum_{i=1}^{n} |a_{ni}| P(|X_{ni}| > b_n) + b_n^{-1} \max_{1\le j\le n} \left| \sum_{i=1}^{j} a_{ni} E X_{ni} I(|X_{ni}| > b_n) \right| \\
& \le C \sum_{i=1}^{n} |a_{ni}| P(|X| > b_n) + b_n^{-1} \sum_{i=1}^{n} |a_{ni}| E|X_{ni}| I(|X_{ni}| > b_n) \\
& \le Cn \frac{E|X|^\beta}{b_n^\beta} + C b_n^{-1} \sum_{i=1}^{n} |a_{ni}| E|X| I(|X| > b_n) \\
& \le \frac{Cn}{n^{\beta/\alpha} \log^{\beta/\gamma} n} + C b_n^{-1} n E|X| I(|X| > b_n) \\
& = \frac{Cn}{n^{\beta/\alpha} \log^{\beta/\gamma} n} + C b_n^{-1} n \sum_{k=n}^{\infty} E|X| I(b_k < |X| \le b_{k+1}) \qquad (2.6) \\
& \le \frac{Cn}{n^{\beta/\alpha} \log^{\beta/\gamma} n} + C b_n^{-1} n \sum_{k=n}^{\infty} b_{k+1} P(|X| > b_k) \\
& \le \frac{Cn}{n^{\beta/\alpha} \log^{\beta/\gamma} n} + C b_n^{-1} n \sum_{k=n}^{\infty} b_{k+1} \frac{E|X|^\beta}{b_k^\beta} \\
& \le \frac{Cn}{n^{\beta/\alpha} \log^{\beta/\gamma} n} + C b_n^{-1} n \sum_{k=n}^{\infty} \frac{(k+1)^{1/\alpha} \log^{1/\gamma}(k+1)}{k^{\beta/\alpha} \log^{\beta/\gamma} k} \\
& \le \frac{Cn}{n^{\beta/\alpha} \log^{\beta/\gamma} n} + C b_n^{-1} \sum_{k=n}^{\infty} k^{1/\alpha + 1 - \beta/\alpha} \\
& \le \frac{Cn}{n^{\beta/\alpha} \log^{\beta/\gamma} n} + \frac{C n^{1/\alpha + 2 - \beta/\alpha}}{n^{1/\alpha} \log^{1/\gamma} n} \to 0, \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Elementary Jensen's inequality implies that for any $0 < s < t$,

$$\left(\sum_{i=1}^{n} |a_{ni}|^t \right)^{1/t} \le \left(\sum_{i=1}^{n} |a_{ni}|^s \right)^{1/s}. \tag{2.7}$$

Therefore, when $0 < \alpha \leq 1$, we have by Lemma 1.3, (2.7), Markov's inequality and condition (ii) that

$$
\begin{aligned}
& b_n^{-1} \max_{1\leq j\leq n} \left| \sum_{i=1}^{j} a_{ni} E X_i^{(n)} \right| \\
& \leq \sum_{i=1}^{n} |a_{ni}| P(|X_{ni}| > b_n) + b_n^{-1} \sum_{i=1}^{n} |a_{ni}| E|X_{ni}| I(|X_{ni}| \leq b_n) \\
& \leq C \sum_{i=1}^{n} |a_{ni}| P(|X| > b_n) + C b_n^{-1} \sum_{i=1}^{n} |a_{ni}| \Big(E|X| I(|X| \leq b_n) + b_n P(|X| > b_n) \Big) \\
& \leq C b_n^{-1} n^{\delta/\alpha} E|X| I(|X| \leq b_n) + C n^{\delta/\alpha} P(|X| > b_n) \\
& \leq C b_n^{-1} n^{\delta/\alpha} \sum_{k=2}^{n} E|X| I(b_{k-1} < |X| \leq b_k) + \frac{C n^{\delta/\alpha} E|X|^\beta}{b_n^\beta} \\
& \leq C b_n^{-1} n^{\delta/\alpha} \sum_{k=2}^{n} b_k P(|X| > b_{k-1}) + \frac{C n^{\delta/\alpha} E|X|^\beta}{n^{\beta/\alpha} \log^{\beta/\gamma} n} \\
& \leq C b_n^{-1} n^{\delta/\alpha} \sum_{k=2}^{n} b_k \frac{E|X|^\beta}{b_{k-1}^\beta} + \frac{C n^{\delta/\alpha} E|X|^\beta}{n^{\beta/\alpha} \log^{\beta/\gamma} n} \\
& \leq C b_n^{-1} n^{\delta/\alpha} \sum_{k=2}^{n} \frac{k^{1/\alpha} \log^{1/\gamma} k}{(k-1)^{\beta/\alpha} \log^{\beta/\gamma}(k-1)} + \frac{C n^{\delta/\alpha} E|X|^\beta}{n^{\beta/\alpha} \log^{\beta/\gamma} n} \\
& \leq C b_n^{-1} n^{\delta/\alpha} \sum_{k=2}^{n} k^{1/\alpha - \beta/\alpha} + \frac{C n^{\delta/\alpha} E|X|^\beta}{n^{\beta/\alpha} \log^{\beta/\gamma} n} \\
& \leq \frac{C n^{\delta/\alpha} n^{1/\alpha - \beta/\alpha + 1}}{n^{1/\alpha} \log^{1/\gamma} n} + \frac{C n^{\delta/\alpha} E|X|^\beta}{n^{\beta/\alpha} \log^{\beta/\gamma} n} \\
& = \frac{C}{n^{\beta/\alpha - \delta/\alpha - 1} \log^{1/\gamma} n} + \frac{C n^{\delta/\alpha} E|X|^\beta}{n^{\beta/\alpha} \log^{\beta/\gamma} n} \to 0, \quad \text{as} \quad n \to \infty.
\end{aligned}
\tag{2.8}
$$

By (2.6) and (2.8), we can get (2.4) immediately. Hence, for n large enough,

$$
P\Big(\max_{1\leq j\leq n} \Big| \sum_{i=1}^{j} a_{ni} X_{ni} \Big| > \varepsilon b_n \Big) \leq \sum_{i=1}^{n} P\big(|X_{ni}| > b_n\big) + P\Big(\max_{1\leq j\leq n} \big|T_j^{(n)}\big| > \frac{\varepsilon}{2} b_n \Big).
$$

To prove (2.1), we only need to show that

$$
I \doteq \sum_{n=1}^{\infty} n^{p\alpha-2} \sum_{i=1}^{n} P\Big(|X_{ni}| > b_n\Big) < \infty \tag{2.9}
$$

and

$$
J \doteq \sum_{n=1}^{\infty} n^{p\alpha-2} P\Big(\max_{1\leq j\leq n} \big|T_j^{(n)}\big| > \frac{\varepsilon}{2} b_n \Big) < \infty. \tag{2.10}
$$

By Definition 1.1, Markov's inequality and condition (ii), we can see that

$$\begin{aligned} I &\doteq \sum_{n=1}^{\infty} n^{p\alpha-2} \sum_{i=1}^{n} P\Big(|X_{ni}| > b_n\Big) \\ &\le C \sum_{n=1}^{\infty} n^{p\alpha-2} \sum_{i=1}^{n} P\Big(|X| > b_n\Big) \ \le\ C \sum_{n=2}^{\infty} n^{p\alpha-1} \frac{E|X|^{\beta}}{b_n^{\beta}} \\ &\le C \sum_{n=2}^{\infty} \frac{n^{p\alpha-1}}{n^{\beta/\alpha} \log^{\beta/\gamma} n} \ <\ \infty, \qquad (\text{since } \beta > p\alpha^2). \end{aligned} \tag{2.11}$$

For fixed $n \ge 1$, it is easily seen that $\{X_i^{(n)}, 1 \le i \le n\}$ are still NOD by Lemma 1.1. For $q > 2$, it follows from Lemma 1.2, C_r inequality and Jensen's inequality that

$$\begin{aligned} J &\doteq \sum_{n=1}^{\infty} n^{p\alpha-2} P\Big(\max_{1\le j\le n} \Big|T_j^{(n)}\Big| > \frac{\varepsilon}{2} b_n\Big) \ \le\ C \sum_{n=2}^{\infty} n^{p\alpha-2} b_n^{-q} E\Big(\max_{1\le j\le n} \Big|T_j^{(n)}\Big|^q\Big) \\ &\le C \sum_{n=2}^{\infty} n^{p\alpha-2} b_n^{-q} (\log n)^q \Big[\sum_{i=1}^{n} |a_{ni}|^q E\Big|X_i^{(n)}\Big|^q + \Big(\sum_{i=1}^{n} |a_{ni}|^2 E\Big|X_i^{(n)}\Big|^2\Big)^{q/2}\Big] \\ &\doteq J_1 + J_2. \end{aligned} \tag{2.12}$$

Taking a suitable constant q such that $\max\{2, \alpha(p\alpha-1)/(1-\delta)\} < q < \min\{\beta-\alpha, \frac{\beta - p\alpha^2+\alpha}{\delta}\}$, which implies that

$$\beta > \alpha + q, \quad \frac{\beta}{\alpha} - \frac{q}{\alpha} > 1, \quad \beta > p\alpha^2 - \alpha + q\delta, \quad \frac{\beta}{\alpha} - p\alpha + 2 - q\frac{\delta}{\alpha} > 1$$

and

$$p\alpha - 2 + q\frac{\delta}{\alpha} - \frac{q}{\alpha} < -1, \qquad q > \alpha.$$

It follows from C_r inequality, Lemma 1.3, (2.7), Markov's inequality and condition (ii) that

$$\begin{aligned} J_1 &\doteq C \sum_{n=2}^{\infty} n^{p\alpha-2} b_n^{-q} (\log n)^q \sum_{i=1}^{n} |a_{ni}|^q E\Big|X_i^{(n)}\Big|^q \\ &\le C \sum_{n=2}^{\infty} n^{p\alpha-2} b_n^{-q} (\log n)^q \sum_{i=1}^{n} |a_{ni}|^q \Big[E\Big|X_{ni}\Big|^q I(|X_{ni}| \le b_n) + b_n^q P(|X_{ni}| > b_n)\Big] \\ &\le C \sum_{n=2}^{\infty} n^{p\alpha-2} b_n^{-q} (\log n)^q \sum_{i=1}^{n} |a_{ni}|^q \Big[E|X|^q I(|X| \le b_n) + b_n^q P(|X| > b_n)\Big] \\ &\le C \sum_{n=2}^{\infty} n^{p\alpha-2+q\delta/\alpha} b_n^{-q} (\log n)^q E|X|^q I(|X| \le b_n) \\ &\quad + C \sum_{n=2}^{\infty} n^{p\alpha-2+q\delta/\alpha} (\log n)^q P(|X| > b_n) \\ &\le C \sum_{n=2}^{\infty} n^{p\alpha-2+q\delta/\alpha} b_n^{-q} (\log n)^q \sum_{k=2}^{n} E|X|^q I(b_{k-1} < |X| \le b_k) \\ &\quad + C \sum_{n=2}^{\infty} n^{p\alpha-2+q\delta/\alpha} (\log n)^q \frac{E|X|^{\beta}}{b_n^{\beta}} \end{aligned}$$

$$
\begin{aligned}
&\le C\sum_{k=2}^{\infty}\sum_{n=k}^{\infty} n^{p\alpha-2+q\delta/\alpha-q/\alpha}(\log n)^{q-q/\gamma} b_k^q P(|X|>b_{k-1}) \\
&\quad + C\sum_{n=2}^{\infty}\frac{n^{p\alpha-2+q\delta/\alpha}\log^q n}{n^{\beta/\alpha}\log^{\beta/\gamma} n} \\
&\le C\sum_{k=3}^{\infty} b_k^q\frac{E|X|^\beta}{b_{k-1}^\beta} + C\sum_{n=2}^{\infty}\frac{1}{n^{\beta/\alpha-p\alpha+2-q\delta/\alpha}\log^{\beta/\gamma-q} n} \qquad (2.13)\\
&\le C\sum_{k=3}^{\infty}\frac{k^{q/\alpha}\log^{q/\gamma} k}{(k-1)^{\beta/\alpha}\log^{\beta/\gamma}(k-1)} + C\sum_{n=2}^{\infty}\frac{1}{n^{\beta/\alpha-p\alpha+2-q\delta/\alpha}\log^{\beta/\gamma-q} n} \\
&\le C\sum_{k=3}^{\infty}\frac{1}{k^{\beta/\alpha-q/\alpha}} + C\sum_{n=2}^{\infty}\frac{1}{n^{\beta/\alpha-p\alpha+2-q\delta/\alpha}\log^{\beta/\gamma-q} n} \ < \ \infty.
\end{aligned}
$$

By C_r inequality, Lemma 1.3, (2.7) and Jensen's inequality, we can get that

$$
\begin{aligned}
J_2 &\doteq C\sum_{n=2}^{\infty} n^{p\alpha-2} b_n^{-q}(\log n)^q\left(\sum_{i=1}^{n}|a_{ni}|^2 E\left|X_i^{(n)}\right|^2\right)^{q/2} \\
&\le C\sum_{n=2}^{\infty} n^{p\alpha-2} b_n^{-q}(\log n)^q\left(\sum_{i=1}^{n}|a_{ni}|^2\left[E|X_{ni}|^2 I(|X_{ni}|\le b_n)+b_n^2 P(|X_{ni}|>b_n)\right]\right)^{q/2} \\
&\le C\sum_{n=2}^{\infty} n^{p\alpha-2} b_n^{-q}(\log n)^q\left(\sum_{i=1}^{n}|a_{ni}|^2\left[EX^2 I(|X|\le b_n)+b_n^2 P(|X|>b_n)\right]\right)^{q/2} \\
&\le C\sum_{n=2}^{\infty} n^{p\alpha-2+q\delta/\alpha} b_n^{-q}(\log n)^q\left[EX^2 I(|X|\le b_n)+b_n^2 P(|X|>b_n)\right]^{q/2} \qquad (2.14)\\
&\le C\sum_{n=2}^{\infty} n^{p\alpha-2+q\delta/\alpha} b_n^{-q}(\log n)^q\left[EX^2 I(|X|\le b_n)\right]^{q/2} \\
&\quad + C\sum_{n=2}^{\infty} n^{p\alpha-2+q\delta/\alpha}(\log n)^q\left[P(|X|>b_n)\right]^{q/2} \\
&\le C\sum_{n=2}^{\infty} n^{p\alpha-2+q\delta/\alpha} b_n^{-q}(\log n)^q E|X|^q I(|X|\le b_n) \\
&\quad + C\sum_{n=2}^{\infty} n^{p\alpha-2+q\delta/\alpha}(\log n)^q P(|X|>b_n) < \infty. \quad \text{(by (2.13))}
\end{aligned}
$$

Therefore, the desired result (2.1) follows from (2.11)–(2.14) immediately. This completes the proof of the theorem. □

Similar to the proof of Theorem 2.1 above and Theorems 2.3–2.6 of Wang et al. [15], we can get the following results.

Theorem 2.2. *Let $\{X_n, n\ge 1\}$ be a sequence of NOD random variables which is stochastically dominated by a random variable X and $\{a_n, n\ge 1\}$ be a sequence of real numbers. Assume that there exist some δ with $0<\delta<1$ and some α with $0<\alpha\le 2$ such that $\sum_{i=1}^{n}|a_i|^\alpha = O(n^\delta)$ and assume further that $EX_n = 0$ when $1<\alpha\le 2$. If condition* (ii) *of* Theorem 2.1 *holds, then for any $\varepsilon>0$,*

$$
\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le j\le n}\left|S_j\right| > \varepsilon b_n\right) < \infty
$$

and

$$\lim_{n\to\infty} \frac{|S_n|}{b_n} = 0 \quad a.s.,$$

where $b_n \doteq n^{1/\alpha} \log^{1/\gamma} n$ *for some* $\gamma > 0$ *and* $S_n = \sum_{i=1}^{n} a_i X_i$ *for* $n \geq 1$.

Theorem 2.3. *Let* $\{X_{ni} : i \geq 1, n \geq 1\}$ *be an array of rowwise NOD random variables which is stochastically dominated by a random variable* X *and* $\{a_{ni} : i \geq 1, n \geq 1\}$ *be an array of real numbers. Assume that there exists some* α *with* $0 < \alpha \leq 2$ *such that* $\sum_{i=1}^{n} |a_{ni}|^\alpha = O(n)$ *and assume further that* $EX_{ni} = 0$ *when* $1 < \alpha \leq 2$. *If there exists some* $\beta > \alpha + 2$ *such that* $E|X|^\beta < \infty$, *then for any* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} P\bigg(\max_{1\leq j\leq n} \bigg|\sum_{i=1}^{j} a_{ni} X_{ni}\bigg| > \varepsilon b_n\bigg) < \infty,$$

where $b_n \doteq n^{1/\alpha} \log^{1/\gamma} n$ *for some* $\gamma > 0$.

Theorem 2.4. *Let* $\{X_n, n \geq 1\}$ *be a sequence of NOD random variables which is stochastically dominated by a random variable* X *and* $\{a_{ni}, i \geq 1, n \geq 1\}$ *be an array of real numbers. Assume that there exists some* α *with* $0 < \alpha \leq 2$ *such that* $\sum_{i=1}^{n} |a_{ni}|^\alpha = O(n)$ *and assume further that* $EX_n = 0$ *when* $1 < \alpha \leq 2$. *If there exists some* $\beta > \alpha + 2$ *such that* $E|X|^\beta < \infty$, *then for any* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} P\bigg(\max_{1\leq j\leq n} \bigg|\sum_{i=1}^{j} a_{ni} X_{i}\bigg| > \varepsilon b_n\bigg) < \infty,$$

where $b_n \doteq n^{1/\alpha} \log^{1/\gamma} n$ *for some* $\gamma > 0$.

Theorem 2.5. *Let* $\{X_n, n \geq 1\}$ *be a sequence of NOD random variables which is stochastically dominated by a random variable* X *and* $\{a_n, n \geq 1\}$ *be a sequence of real numbers. Assume that there exists some* α *with* $0 < \alpha \leq 2$ *such that* $\sum_{i=1}^{n} |a_i|^\alpha = O(n)$ *and assume further that* $EX_n = 0$ *when* $1 < \alpha \leq 2$. *If there exists some* $\beta > \alpha + 2$ *such that* $E|X|^\beta < \infty$, *then for any* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} P\bigg(\max_{1\leq j\leq n} \Big|S_j\Big| > \varepsilon b_n\bigg) < \infty$$

and

$$\lim_{n\to\infty} \frac{|S_n|}{b_n} = 0 \quad a.s.,$$

where $b_n \doteq n^{1/\alpha} \log^{1/\gamma} n$ *for some* $\gamma > 0$ *and* $S_n = \sum_{i=1}^{n} a_i X_i$ *for* $n \geq 1$.

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* *School of Mathematical Sciences*
Anhui University
Hefei 230601
Jiulong Road 111
P.R.CHINA
E-mail: empress201010@126.com
1311476707@126.com

** *Department of Mathematics and Statistics*
University of Regina
Regina
Saskatchewan S4S 0A2
CANADA
E-mail: volodin@math.uregina.ca