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A Probabilistic Model of Growth for Two-Sided Cracks Based on the Physical Description of the Phenomenon

Kamon Budsaba*[a], Poonchayar Patthanangkoor [a] and Andrei Volodin [b]
[a] Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University, Rangsit Center, Pathum Thani, 12120, Thailand.
[b] Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan, Canada.
*Corresponding author; e-mail: 2555kamon@gmail.com

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Abstract

There is detailed, consistent, and rigorous general probability model of growth of a one-sided crack for a metallic block presented below. So far there is no such model for a development of a fatigue cracks from upper and bottom sides of the block. Our goal is to present such detailed, consistent, and rigorous general probability model of growth of a two-sided cracks. The significance of this research follows from the fact that such type of investigations have numerous applications in physics, engineering, statistics, environment studies and economics.

Keywords: Failure time models, Birnbaum-Saunders distribution, physical model of a phenomenon.

1. Introduction

We start with a brief literature survey on the Birnbaum-Saunders distribution, explaining what has been done in the area, and describe what new contribution to the field can be made in our paper.

The two-parameter Birnbaum-Saunders distribution (in this paper we call it one-sided Birnbaum-Saunders distribution) was introduced by Birnbaum and Saunders (1969a) as a failure-time distribution for fatigue failure caused under cyclic loading. This distribution is widely used as a lifetime distribution in the various models of reliability theory in the case when a failure of the object under consideration appears to be due to the development of fatigue cracks. Desmond (1985, 1986) provided a more general derivation based on a biological model and strengthened the physical justification for the use of this distribution. This derivation follows from considerations of renewal theory for the number of cycles needed to force a fatigue crack extension to exceed a critical value. Birnbaum and Saunders (1969b) presented a comprehensive review, both theoretical and practical, of the fitting of this family of distributions to the solution of the problem of crack development.

Desmond (1986) considered estimation of the parameters for censored data. Ahmad (1988) considered the estimation of the scale parameter (which overestimates the median life) by the jackknife method to eliminate first-order bias. This estimate has the same limiting behavior as that of

Birnbaum and Saunders (1969b). Rieck (1995) derived an asymptotically optimal linear estimator for symmetrically type II censored samples. We refer to the monograph by Bogdanoff and Kozin (1985) for motivating examples of probabilistic models of cumulative damage. A more recent view on the problem of fatigue crack damages based on stochastic differential equations is suggested by Singpurwalla (1995). For the most recent publications on Birnbaum-Saunders distribution we refer to Xie and Wei (2007), Lemonte et al. (2007), Ng et al. (2006) Balakrishnan et al. (2007), From and Li (2006), Rieck (1999), Dupuis and Mills (1998), Chang and Tang (1993, 1994), and a review of these developments can be found in Johnson et al. (1995).

The maximum likelihood estimators were first discussed by Birnbaum and Saunders (1969b), who suggested some iterative schemes to solve the required non-linear equation. Englehardt et al. (1981) established the asymptotic distribution of the maximum likelihood estimators. Conventional moment estimators have a drawback in that they may not always exist and, if they do exist, they may not be unique. Ng et al. (2003) considered the modified moment estimators for the parameters to overcome this problem. However, Wu and Wong (2004) reported that those expressions for the intervals of estimators for β suggested by Ng et al. (2003) are presented incorrectly. Furthermore, there is no guarantee that the upper bounds of those intervals are always positive.

Ahmed et al. (2008) introduced the new parametrization of Birnbaum-Saunders distribution based on the recurrence relations which presents a general probability model of growth of a one-sided crack, see the discussion in the next section. Importantly, the physics of the phenomena under the study is fitted by this re-parametrization since the suggested parameters correspond to the thickness of the sample and the nominal treatment loading on the sample, respectively.

The Birnbaum-Saunders distribution is a two-parameter life time distribution originated in modeling material fatigue data (Birnbaum and Saunders 1969a). Due to its mathematical tractability and ability to fit right skewed data, the Birnbaum-Saunders model is also used for many other applications. Recently, the Birnbaum-Saunders distribution has been extended to various classes of distributions. We propose a class of generalized Birnbaum-Saunders distribution families by using elliptically contoured density functions in place of standard normal density function. Ferreira et al. (2012) then discussed the tail behavior of this generalized Birnbaum-Saunders distribution in the context of extreme value theory. The authors show that the tail properties of the generalized Birnbaum-Saunders distribution are essentially governed by that of its auxiliary distribution (i.e., standard elliptically symmetric distribution).

In this paper, we establish and investigate a probability model for two-sided Birnbaum-Saunders distribution. We consider different choices of impulse function that corresponds to the crack development from two sides and establish how does it influence on the resulting probability distribution. This allows us to evaluate the performance of the proposed models for different impulse functions.

A general probability model of two-sided crack development will be constructed based on the new parametrization presented by Ahmed et al. (2008). We provide intense computer simulations for different choices of impulse function that corresponds to the crack development from two sides in order to establish how does it influence on the resulting probability distribution.

2. Definitions of One-Sided Birnbaum-Saunders Distribution

2.1. Definition by physical probabilistic model

The following description of the general probability model of growth of a one-sided crack has been introduced in Ahmed et al. (2008). We present a modified and extended description of the construction in the present paper because it is crucial for a similar construction of two-sided crack development.

Consider a rectangular metal block which is fixed from two sides. A periodic loading is applied to its middle part and this leads to a development of a fatigue crack. Assume that at the beginning the length of the crack was $x_0 \geq 0$, and after each loading we measure the crack length and obtain a sequence of nondecreasing numbers $x_1, x_2, \ldots$. First, we are interested in the prediction of the crack length after the n th loading. After we are interested in finding a distribution of the time when the block breaks down.

It is obvious that the crack development is achieved under several factors, such as the strength of loading, quality of metal from which the block is made, and so on. Therefore we obviously are dealing with a stochastic forecasting problem. Therefore, we should consider the measurements $x_1, x_2, \ldots$ as a realization of a sequence of random variables $X_1, X_2, \ldots$. We formalize mathematically the phenomenon of a crack development in terms of the increments $\Delta_k = X_k - X_{k-1}$ of the crack lengths.

It is natural to assume that the increment $\Delta_k \geq 0$ is achieved by the sum of all values produced by factors of the crack growth that we mentioned above. That is, under some nonnegative "impulse" $\xi_k (\geq 0)$, there exists an approximate linear relationship between Δ_k and ξ_k such as $\Delta_k = \alpha_k \xi_k$, where α_k depends on the previous crack length X_{k-1} that it achieved at $k-1$ loading. Let $\alpha_k = g(X_{k-1})$ with the natural assumption of that the *impulse function* $g(\cdot)$ is nonnegative and continuous. Therefore, we have the following recurrent relations that describe the crack development after each loading:

$$X_k - X_{k-1} = \xi_k \cdot g(X_{k-1}), \quad k = 1, 2, \ldots. \tag{1}$$

Now we make some assumptions on distributions of the random variables $\xi_k, k \geq 1$. Assume that these random variables are nonnegative, independent identically distributed with finite second moment and denote by $a = E(\xi_k)$ their mean value and by $b^2 = Var(\xi_k)$ their variance.

Recall that we are interested in the distribution of the random variable X_n, whose realization x_n gives the length of a particular crack after the n^{th} loading. Rewrite the first n recurrent relations (1) as

$$\xi_k = \frac{X_k - X_{k-1}}{g(X_{k-1})}, \quad k = 1, \ldots, n$$

and add all of them to obtain

$$\sum_{k=1}^{n} \xi_k = \sum_{k=1}^{n} \frac{X_k - X_{k-1}}{g(X_{k-1})}.$$

If each impulse provides an insignificant increase in the crack length, that is, all $\Delta_k = X_k - X_{k-1}$ are small, then we can interpret the right hand side of the summation as an integral sum and obtain the approximate equality

$$\sum_{k=1}^{n} \xi_k \approx \int_{x_0}^{X} \frac{dt}{g(t)}, \tag{2}$$

where $X = X_n$ is the final crack size. By the "$\approx$" sign in (2), we mean that the left hand side of the expression is a pointwise approximation of the right hand side.

Since the function $g(x)$ is positive, the integral in the right hand side of (2) represents some monotone increasing function $h(X)$. By the Central Limit Theorem applied to the left hand side of (2), we obtain the following statement: For some long time $(n >> 1)$ after the crack started to grow, the distribution of its length X is defined from the relations $h(X) : N(\mu, \sigma^2)$, where $\mu = na, \sigma^2 = nb^2$. By the monotonicity of the function $h(\cdot)$, the distribution function of the random variable X is

$$F(x) = P(X < x) = P(h(X) < h(x)) = \Phi\left(\frac{h(x) - \mu}{\sigma}\right),$$

where $\Phi(x)$ is the standard normal distribution function.

It is left to solve a problem with a choice of the function $g(\cdot)$. If we postulate that the increase of the crack is proportional to the length achieved, that is, to assume that $g(t) = t$ (this assumption is the most commonly used in the models of growth), then we obtain the *lognormal* distribution of the random variable X.

In order to define the one-sided Birnbaum-Saunders distribution, consider the following problem. In the framework of the probability model of growth constructed above, we are interested not in the finding the distribution of the crack length X. Let the critical length x of the crack be fixed and we are interested in the distribution of the *moment of time (number of loadings)* at which this length will be achieved. It is interesting that in the framework of our model, this distribution does not depend on a choice of the positive function $g(\cdot)$, the choice of g influences only the concrete values of the parameters. The distribution of the time can be obtained by the following simple observations.

Let τ be a random variable which represents the moment when the length of the crack achieves the critical length x. Then the event $\tau > n$ is equivalent to the event $X_n < x$ (recall that all $\xi_k \geq 0$ and for moment of time n the crack length does not achieve the critical length x). Therefore,

$$P(\tau > n) = P(X_n < x) = P\left(h\left(X_n\right) < h\left(x\right)\right) = \Phi\left(\frac{h\left(x\right) - na}{b\sqrt{n}}\right). \tag{3}$$

Replace the variable n by a "continuous" variable t and introduce new parameters λ and θ, defined as $\lambda = ah\left(x\right)/b^2, \theta = b^2 / a^2$. The chain of equalities (3) helps us to write the distribution function of the random moment of time τ at which the crack achieves the critical length x:

$$F_{BS}\left(t; \theta, \lambda\right) = P(\tau < t) = 1 - \Phi\left(\lambda\sqrt{\frac{\theta}{t}} - \sqrt{\frac{t}{\theta}}\right) = \Phi\left(\sqrt{\frac{t}{\theta}} - \lambda\sqrt{\frac{\theta}{t}}\right), \quad t \geq 0.$$

This distribution called the *one-sided Birnbaum-Saunders distribution*.

The density function for this distribution is

$$f_{BS}\left(t; \theta, \lambda\right) = \frac{1}{2\sqrt{2\pi\theta}}\left[\lambda\left(\frac{\theta}{t}\right)^{3/2} + \left(\frac{t}{\theta}\right)^{-1/2}\right]\exp\left\{-\frac{1}{2}\left(\lambda\sqrt{\frac{\theta}{t}} - \sqrt{\frac{t}{\theta}}\right)^2\right\}, \quad t \geq 0.$$

2.2. Formal definition by Brownian motion

In this section, we would like to present a formal definition of the Birnbaum-Saunders distribution. This construction will not be used in the following, but it is interesting because most probably this is the way how Birnbaum-Saunders have been arguing when they introduced their distribution in 1969.

Let $\{B_t\}$ be a standard Brownian motion with zero drift. We define a process $X_t = \mu t + \sigma B_t$, $\mu > 0, \sigma > 0,$ which quantifies the level of stress or fatigue accumulated to a subject of interest up to time t. The failure event occurs when the accumulated stress hits a critical threshold $w > 0$. Then the failure time T is the first hitting time of X_t to the threshold w. By using quite advanced technique of Stochastic Analysis, namely Girsanov's Theorem and the reflection principle, the distribution of T is given as follows:

$$P[T \le t] = P[M(t) > w] = 1 - \Phi\left(\frac{w - \mu t}{\sigma\sqrt{t}}\right) + e^{2\mu w/\sigma^2}\Phi\left(\frac{-w - \mu t}{\sigma\sqrt{t}}\right), \tag{4}$$

where

$$M(t) = \max_{0 \le s \le t}\{X_s\}.$$

The Birnbaum-Saunders distribution is obtained as an approximation to (4) by ignoring the last term in the formula. Specifically, the distribution function of the Birnbaum-Saunders distribution, respectively, are

$$F_{BS}(t) = 1 - \Phi\left(\frac{1}{\alpha}\left(\sqrt{\frac{\beta}{t}} - \sqrt{\frac{t}{\beta}}\right)\right),$$

where $\alpha = \sigma / \sqrt{\mu w} > 0, \beta = w / \mu > 0.$ The parameters α and β are the shape and scale parameters, respectively.

Note that the new parametrization of Birnbaum-Saunders distribution presented in the previous section develops the new parameters which are meaningful in a practical setting. Importantly, this re-parametrization fits the physics of the phenomena under study since the proposed parameters λ and θ correspond to the thickness of the sample and the nominal treatment loading on the sample, respectively.

3. Formal Definition of Two-Sided Birnbaum-Saunders Distribution

3.1. Definition by physical probabilistic model

Similar to case of one-sided Birnbaum-Saunders distribution, consider a rectangular metal block of size a which is fixed from two sides. A periodic loading is applied first to its upper part, then immediately to its lower part (consider as one period of loading) and this leads to a development of a fatigue cracks from upper and lower sides of the block. Denote by X_i the length of the upper crack and by Y_i the length of the lower crack after the i^{th} loading.

As above, we assume that the increases of the cracks lengths follow the following recurrence relations:

$$X_i = X_{i-1} + \xi_i \cdot g_1(X_{i-1}, Y_{i-1}),$$
$$Y_i = Y_{i-1} + \eta_i \cdot g_2(X_{i-1}, Y_{i-1}). \tag{5}$$

Here each set $\{\xi_i, i \ge 1\}$ and $\{\eta_i, i \ge 1\}$ consists of positive independent identically distributed random variables with the finite second moments, and *impulse functions* $g_1(\cdot,\cdot)$ and $g_2(\cdot,\cdot)$ are positive and continuous. Note that the growth of both cracks depends on the previous cracks lengths in the upper as well as in the bottom parts.

Rewrite the recurrence relations in the following form:

$$\frac{X_i - X_{i-1}}{g_1(X_{i-1}, Y_{i-1})} = \xi_i, \qquad \frac{Y_i - Y_{i-1}}{g_2(X_{i-1}, Y_{i-1})} = \eta_i.$$

Taking sums of all these equalities gives

$$\sum_{i=1}^{n}\left[\frac{X_i - X_{i-1}}{g_1(X_{i-1}, Y_{i-1})} + \frac{Y_i - Y_{i-1}}{g_2(X_{i-1}, Y_{i-1})}\right] = \sum_{i=1}^{n}(\xi_i + \eta_i). \tag{6}$$

Now we can investigate how the choice of functions g_1 and g_2 influences on the distribution of the random variable τ, the moment of the block breaks down under two-sided loading.

Case 1 which reduces to one-sided Birnbaum-Saunders distribution. Consider the following choice of the impulse functions c that is the impulse function depends only on the total length of the crack. Examples of such functions may be $g_1(x,y) = g_2(x,y) = x + y,$ or $g_1(x,y) = g_2(x,y) = \exp(x+y),$ or $g_1(x,y) = g_2(x,y) = (x+y)^2.$

If we let $\Delta X_i = X_i - X_{i-1}, \Delta Y_i = Y_i - Y_{i-1}$ and assume that these increments are sufficiently small, then we obtain the integral sum

$$\sum_{i=1}^{n}\frac{\Delta(X_i + Y_i)}{g(X_{i-1} + Y_{i-1})} = \sum_{i=1}^{n}\frac{\Delta(Z_i)}{g(Z_{i-1})},$$

where $Z_i = X_i + Y_i$. Finally exchanging Z_{i-1} by a close to it value Z_i, we obtain

$$\sum_{i=1}^{n}\frac{\Delta(Z_i)}{g(Z_{i-1})} \approx \int_{X_0+Y_0}^{X_n+Y_n}\frac{dx}{g(x)} \approx \sum_{i=1}^{n}(\xi_i + \eta_i).$$

This integral is an increasing function of the total crack length, hence the same arguments are true as in the physical model for one-sided Birnbaum-Saunders distribution presented above. The moment of the break down has the distribution function

$$P(\tau > n) = P(X_n + Y_n < a) = P(h(X_n + Y_n) < h(a)) \approx \Phi\left(\frac{h(a) - 2n\mu}{\sigma\sqrt{n}}\right).$$

If we exchange n by t, then we obtain one sided Birnbaum-Saunders distribution. Hence in the case of the impulse function depends only on the total length of the crack, then two-sided Birnbaum-Saunders distribution is the same as one-sided Birnbaum-Saunders distribution up to parameter values.

Case 2 which apparently reduces to one-sided Birnbaum-Saunders distribution, too. Assume that the length of a crack from each side depends only from its previous length, for example, impulse functions $g_1(x,y) = g(x)$ and $g_2(x,y) = g(y)$, where function $g(\cdot)$ is nonnegative and continuous. Then, formula (6) corresponds to two integrals:

$$\sum_{i=1}^{n}\left[\frac{X_i - X_{i-1}}{g_1(X_{i-1}, Y_{i-1})} + \frac{Y_i - Y_{i-1}}{g_2(X_{i-1}, Y_{i-1})}\right] \approx \int_{X_0}^{X_n}\frac{dx}{g(x)} + \int_{Y_0}^{Y_n}\frac{dy}{g(y)}.$$

Obviously, we cannot state that the sum of these integrals produce a monotone function of the total length $X_n + Y_n$. For example, if we take $g_1(x,y) = x$ and $g_2(x,y) = y$, then sum of integrals is a monotone function of the product $X_n \cdot Y_n$. Hence, the integral that has approximately normal distribution will be monotone increasing function of the product $X_n \cdot Y_n$ and will not be able to say that the events $\tau > a$ and $X_n + Y_n < a$ are the same in order to establish one-sided Birnbaum-Saunders distribution.

Therefore, for the impulse functions $g_1(x,y)$ and $g_2(x,y)$ that are not monotone functions of $x+y$, the theoretical construction of the model is completely unsolved problem. Our goal is to model it by computer simulations for the case $g_1(x,y)=x$ and $g_2(x,y)=y$ and check how it is related to one-sided Birnbaum-Saunders distribution.

Because the problem is not solvable analitically, we simulate the recurrent relation (5) with $g_1(x,y)=x$, $g_2(x,y)=y$ and $\xi_i : Exp(1)$, $\eta_i : Exp(1)$. Let X_0 and Y_0 be initial crack lengths. We are interested in determining the time moment τ, when X_i+Y_i becomes more than the critical length 1, that is, $\tau=\min(i: X_i+Y_i>1)$.

Simulation plan:

1. Fix the values X_0 and Y_0 of the initial crack lengths.

2. Obtain 1,000 simulated values of the random variable τ using the recurrent relation (5) with $g_1(x,y)=x$, $g_2(x,y)=y$ and $\xi_i : Exp(1)$, $\eta_i : Exp(1)$.

3. Estimate the parameters θ and λ of the one-sided Birnbaum-Saunders distribution by the method of maximum likelihood. This is not a simple task, see Ahmed et al. (2008) for a detailed discussion of this problem. It is interesting to try to substitute the estimates of the parameters θ and λ of the one-sided Birnbaum-Saunders distribution by the method of minimum chi-square, see the discussion below.

4. Use the chi-square test for the hypothesis of goodness-of-fit with one-sided Birnbaum-Saunders distribution.

For chi-square testing we divided our data into $r=8$ intervals (groups).

Note that in the case when we know completely the hypothetical distribution (the values of parameters are known) and substitute the theoretical frequencies of falling into intervals into the chi square test statistic, then it has approximately chi-square distribution with $r-1$ degrees of freedom. But in our case when we substitute the estimates of the parameters and hence the distribution of the statistics changes. Fisher proved (see the proof in Cramér 1999) that if we substitute the estimates by the method of minimum chi-square, which are the points of minimum (by the parameter variables in theoretical frequencies) of chi-square statistics, then the statistics will have chi-square distribution with $r-s-1$ degrees of freedom, where s is the number of parameters. In the case when we substitute the estimates by the method of maximum likelihood, then the asymptotic distribution function will be in between chi-square distribution functions with $r-1$ and $r-s-1$ degrees of freedom (see Chernoff and Lehmann 1954). Hence in our case, because we substitute the estimates by the maximum likelihood, $r=8$ and $s=2$, then the critical values should be found using quantiles of chi-square distribution with $8-1=7$ and $8-2-1=5$ degrees of freedom. From tables, the 5% critical values are $\chi^2_{df=7}=14.0671$ and $\chi^2_{df=5}=11.0705$.

We also note that if we substitute the estimates of parameters obtained by other methods (for example, the method of moments is exceptionally simple for the one-sided Birnbaum-Saunders distribution), then we were not able to find any results in literature on a distribution of the resulting chi-square statistics.

The simulation results show that the obtained random numbers for two-sided Birnbaum-Saunders distribution are exceptionally well consistent with one-sided Birnbaum-Saunders distribution by chi-square distribution. Figures 1 to 3, we present the analysis of simulated data with different values X_0 and Y_0 of the intimal crack lengths. We also provide the histograms of the simulated data with fitted density functions by the method of maximum likelihood.

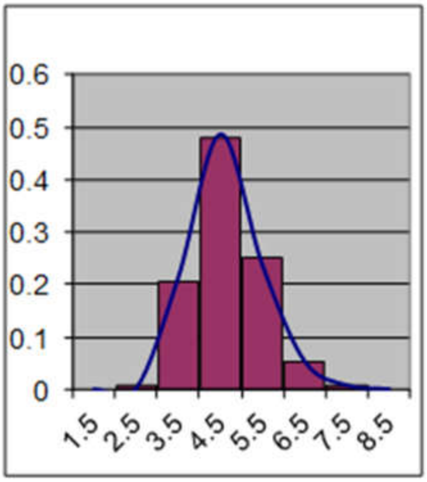

Figure 1 Histogram of the simulated data with values $X_0 = Y_0 = 0.1$ of the intimal crack lengths with fitted density function. The corresponding $\chi^2 = 5.22$. The X-axis represent the time moment τ, when $X_i + Y_i$ becomes more than the critical length 1, that is, $\tau = \min(i : X_i + Y_i > 1)$.
The Y-axis provide the relative frequency.

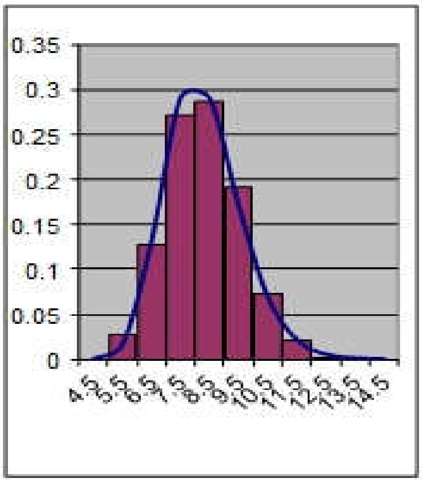

Figure 2 Histogram of the simulated data with values $X_0 = Y_0 = 0.01$ of the intimal crack lengths with fitted density function. The corresponding $\chi^2 = 4.75$. The X-axis represent the time moment τ, when $X_i + Y_i$ becomes more than the critical length 1, that is, $\tau = \min(i : X_i + Y_i > 1)$.
The Y-axis provide the relative frequency.

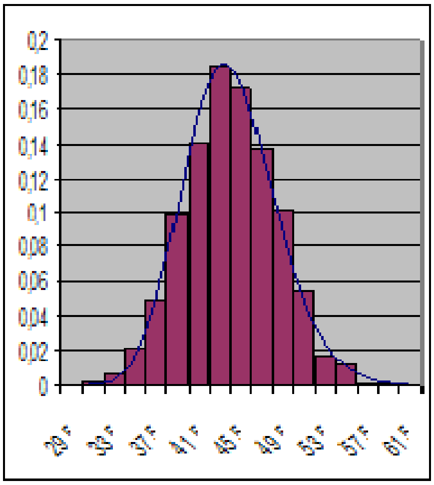

Figure 3 Histogram of the simulated data with values $X_0 = Y_0 = 0.001$ of the intimal crack lengths with fitted density function. The corresponding $\chi^2 = 1.12$. The X-axis represent the time moment τ, when $X_i + Y_i$ becomes more than the critical length 1, that is, $\tau = \min(i : X_i + Y_i > 1)$. The Y-axis provide the relative frequency.

It is still unclear for us why we obtain such perfect fit of two-sided Birnbaum-Saunders distribution with one-sided Birnbaum-Saunders distribution. Maybe this happens because of the aproximate normality of not only one-sided, but also two-sided Birnbaum-Saunders distribution. We are working on this problem, too.

Case 3. Two-sided Birnbaum-Saunders distribution that arises from the above mentioned model with functions different than considered in Cases 1 and 2. We did not consider this case yet, but we expect that we will obtain distributions which are strictly different than a one-sided Birnbaum-Saunders distribution. To observe how two-sided Birnbaum-Saunders distribution changes for different $g_1(x, y)$ and $g_2(x, y)$ is the problem we are working on now.

3.2. Formal definition of two-sided Birnbaum-Saunders distribution

We should also mention the formal definition (not based on a physical model as above) of two-sided Birnbaum-Saunders lifetime distributions presented in Lisawadi (2008). This distribution consider in the case when a crack develops from two sides of a metallic object. Consider a rectangular metallic block of hight a, which is fixed from both edges. To its middle area, a periodic loading is applied which leads to a development of fatigue cracks. Consider the case when a crack develops from two sides from the lower edge of the block and from the upper edge.

Let τ_U be a random variable with one-sided Birnbaum-Saunders distribution, that is, τ_U is the break down time for one–sided loading at the upper side of the block. Then the random variable $Y_U = a / \tau_U$ can be interpreted as a speed of the crack evolution from the upper side. If at the lower side of the block a crack is developing with the same Birnbaum-Saunders distribution, then we have two, assumed to be independent identically distributed random variables τ_U and τ_L.

Let $F_\tau(t)$, $t > 0$, be the distribution function of the random variable τ_U (or τ_L, they are identically distritbuted) and $f(t)$ be its density function. Of course, we should consider the one-sided

Birnbaum-Saunders distribution and density functions for $F_\tau(t)$ and $f(t)$, but these expressions are a little bit cumbersome to present them here in each formula. Then the random variable $Y_U = a/\tau_U$ has the distribution function $F_Y(t) = 1 - F(at^{-1})$ and density function $f_Y(t) = at^{-2} f(at^{-1})$.

The speed of the crack evolution for this two-sided case equals $Y_U + Y_L = a\tau_U^{-1} + a\tau_L^{-1}$ and the random variable $\nu = \frac{a}{Y_U + Y_L} = \left[\tau_U^{-1} + \tau_L^{-1}\right]^{-1}$, corresponds to a moment of the block break down. The distribution function of this random variable is

$$F_\nu(z) = \iint_{t+s>z^{-1}} f\left(\frac{1}{t}\right) f\left(\frac{1}{s}\right) \frac{dtds}{t^2 s^2},$$

and its density function is

$$f_\nu(z) = z^{-2} \int_0^{z^{-1}} f\left(\frac{1}{t}\right) f\left(\frac{1}{z^{-1} - t}\right) \frac{1}{t^2 \left(z^{-1} - t\right)^2} dt.$$

We say that the random variable ν has the *two-sided Birnbaum-Saunders distribution*. Obviously, there is no closed form of this integral when we substitute the one-sided Birnbaum-Saunders density function for $f(t)$.

4. Conclusions

In this paper, we provided the probabilistic model of a crack development in a metalic plate when the crack is developing from two sides. We are interested in a distribution of the time when the total length of the crack reaches the critical value. Importantly, the model is based on the physical description of the phenomenon. Note that one-sided crack development leads to the famous Birnbaum-Saunders distribution and this was the reason why we named the resulting distribution for the crack developing from two sides as two-sided Birnbaum-Saunders distribution. Contrary to the classical one-sided Birnbaum-Saunders distribution, two-sided Birnbaum-Saunders distribution depends on the form of the impulse functions and the main goal was to investigate what is the infuse. It is proven mathematically that in the case when the impulse function depends only on the total length of the crack, the two-sided Birnbaum-Saunders distribution coincided with the classical one-sided Birnbaum-Saunders distribution. The case when the impulse function depends on the previous length of each side length separately, is not trackable mathematically and hence the method of statistical simulations has been used. Surprisingly, we received a high similarity of the classical and two-sided Birnbaum-Saunders distributions in this case again.

We would like to note that this is only a beginning of our investigations for the probabilistic model of two-sided crack development, right now we are working on simulations for other choices of the impulse functions with our graduate students.

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