



Chiang Mai J. Sci. 2018; 45(7) : 2826-2837

<http://epg.science.cmu.ac.th/ejournal/>

Contributed Paper

On Two-sided Length Biased Inverse Gaussian Distribution

Teerawat Simmachan* [a], Kamon Budsaba [a] and Andrei Volodin [b]

[a] Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University, Pathum Thani 10121, Thailand.

[b] Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan, Canada.

* Author for correspondence; e-mail: teerawat@mathstat.sci.tu.ac.th

Received: 27 April 2015

Accepted: 31 July 2018

ABSTRACT

The new lifetime distribution based on the non-classical parametrization model called the two-sided length biased inverse Gaussian distribution is introduced. The physical phenomena of this situation can be explained in the case when a crack develops from two sides. Some statistical properties of the distribution such as reciprocal properties and the first four moments are investigated. The conventional point estimation, method of moment, is developed to estimate the parameters of the distribution together with asymptotic analysis of the proposed estimators. In order to evaluate the performance of the suggested estimators, Monte Carlo simulation studies are conducted. Furthermore, real data sets in a practical setting are used to illustrate the presented estimation method. Concluding remarks and discussions are also provided.

Keywords: method of moment estimate, lifetime distribution, parametrization, reciprocal property, asymptotic property

1. INTRODUCTION

Lifetime distributions are frequently studied in reliability aspects. It is easy to consider a lifetime or failure time of physical objects such as coins, electric light bulbs, some pieces of machines, etc. They provide useful information on certain practical problems. Since some machines or systems are very important and extremely expensive, this information motivates practitioners to prevent financial or industrial damages occurring after the failure time terminates.

One of the interesting views of lifetime

distributions in reliability analysis is in the situation when a failure of the object under consideration occurs from a fatigue crack development. The common distributions used in practical applications of this area are Birnbaum-Saunders, inverse Gaussian, and length biased inverse Gaussian [7, 14].

These distributions had been studied in various cases. Birnbaum and Saunders [2] introduced the two-parameter Birnbaum-Saunders (BS) distribution as a lifetime distribution for fatigue failure caused by periodic loading. Ahmed *et. al.* [1] proposed

the new parametrization of BS distribution. Importantly, the physical situation under this study are fitted by this re-parametrization since the suggested parameters correspond to the thickness of the object under study and the nominal treatment loading on the object, respectively. The original parameters of the distribution do not give these characteristics. Several studies regarding on the inverse Gaussian (IG) distribution are often referred to Chhikara and Folks [4], Seshadri [16, 17], Johnson *et. al.* [11] and Tweedi [18, 19]. Recently, Lisawadi [14] presented two new distributions based on the re-parametrization model proposed in [1] and they were called the two-sided Birbaum-Saunders and inverse Gaussian lifetime distributions. These distributions are considered in the situation of a crack develops from two sides. A review of applications of length biased distributions was given in Gupta and Kirmani [10]. Akman and Gupta [2] had studied the length biased inverse Gaussian (LBIG) distribution. They provided comparative simulation studies of different estimators for the mean of data from IG and LBIG distributions. Gupta and Akman [9] offered statistical properties involving the arithmetic and harmonic means of the LBIG distribution.

Essentially, as seen in the reviewed literature, all of them were considered in the term of usual parameters except the studies of Ahmed *et. al.* [1] and Lisawadi [14]. In this article, we introduce a new lifetime distribution based on non-classical parameters presented by Ahmed *et. al.*[1]. The new distribution is called the two-sided length biased inverse Gaussian lifetime distribution denoted as TS-LBIG distribution. Interestingly, our contribution is in the investigation process. The probability model of the TS-LBIG distribution is formed by applying the approach of Lisawadi [14].

The new distribution is considered in the case when a crack develops from two sides. For example, on a metallic rectangular object which is fixed on two sides, a pressure is applied to both upper and lower sides of the object that leads to a crack development from two sides. Some statistical properties of the distribution such as reciprocal properties and the first four moments are investigated. The traditional point estimation, method of moment, is developed together with the asymptotic analysis of the proposed estimators. Monte Carlo simulations are utilized to study the efficiency of the suggested estimators, and illustrative examples for explaining the given estimation method are also provided.

The article is organized as follows. The pdf of IG and LBIG distributions based on non-classical parameters are provided in section 2. Probability model of the TS-LBIG is introduced in section 3. Theoretical results are given in section 4. Numerical results are shown in section 5. Finally, conclusions and discussion are reported in section 6.

2. MATERIALS AND METHODS

2.1 Inverse Gaussian Distribution

The usual pdf of inverse Gaussian distribution of a continuous random variable X is

$$f(x; \mu, \beta) = \sqrt{\frac{\beta}{2\pi}} x^{-3/2} \exp\left\{-\frac{\beta(x-\mu)^2}{2\pi^2 x}\right\}; \quad (1)$$

$x > 0$ and $\mu, \beta > 0$. The parameter μ stands for the mean and β represents the scale parameter.

The proposed parameters provided by Ahmed *et. al.* [1] are $\lambda > 0$ and $\theta > 0$ standing for the thickness of the object under consideration and the nominal treatment loading on the object, correspondingly.

The relationship between the classical and the proposed parameters are the followings.

$$\lambda = \beta/\mu, \theta = \mu^2/\beta, \mu = \lambda\theta \text{ and } \beta = \lambda^2\theta.$$

Hence, the pdf of IG distribution based on the new parametrization is define as

$$f(x; \lambda, \theta) = \frac{\lambda}{\theta\sqrt{2\pi}} \left(\frac{\theta}{x}\right)^{3/2} \times \exp\left\{-\frac{1}{2}\left(\sqrt{\frac{x}{\theta}} - \lambda\sqrt{\frac{\theta}{x}}\right)^2\right\}; \tag{2}$$

where $\lambda > 0$ and $\theta > 0$.

2.2 Length Biased Inverse Gaussian Distribution

We start with the definition of a length biased

pdf presented by Khattree [12].

Definition 2.2.1 Let be a non-negative random variable having an absolutely continuous pdf, $f(x)$. Assuming $\mu = E(X) < \infty$, the length biased random variable Y has a pdf defined as

$$g(\cdot) = \frac{xf(x)}{\mu}; \quad x > 0. \tag{3}$$

It is known that $\mu = \lambda\theta$, and by (2) and (3), finally, the pdf of of the length biased inverse Gaussian (LBIG) distribution is define by

$$f(x; \lambda, \theta) = \frac{1}{\theta\sqrt{2\pi}} \left(\frac{\theta}{x}\right)^{1/2} \times \exp\left\{-\frac{1}{2}\left(\lambda\sqrt{\frac{x}{\theta}} - \sqrt{\frac{\theta}{x}}\right)^2\right\}; \tag{4}$$

where $\lambda > 0$ and $\theta > 0$.

3. PROBABILITY MODEL

The physical phenomena of this situation can be explained in the case when a crack develops from two sides of the object under consideration. Let $F(t)$, $t > 0$, be the distribution function of the breakdown time τ moment for one-sided loading.

We consider $F(t) = F_{LBIG}(t; \lambda, \theta)$. Let $Y = k/\tau$ be the random variable interpreted as a speed of the crack evolution. At the bottom side of the metallic block, a crack is developing with the distribution function of the time to reach the length k . Simultaneously, at the top side of the block, a crack is developing with the same distribution function as the bottom side. Then, we have two random variables τ_1 and τ_2 , and they are assumed to be independent and identically distributed. The speed of the crack development for this two-sided case is $Y_1 + Y_2 = k\tau_1^{-1} + k\tau_2^{-1}$, and the random variable $\tau = \frac{k}{Y_1 + Y_2} = [\tau_1^{-1} + \tau_2^{-1}]^{-1}$ corresponds to a moment of the object under consideration break down. The distribution function and pdf of the two-sided length biased inverse Gaussian distribution are presented in the following theorems.

Theorem 3.1 A random variable τ has a two-sided length biased inverse Gaussian distribution denoted as TS-LBIG (λ, θ) , if it has a distribution function in the form:

$$F_\tau(u) = 1 - \int_0^{u^{-1}} f\left(\frac{1}{t}\right) \frac{dt}{t^2} \int_0^{u^{-1}-t} f\left(\frac{1}{s}\right) \frac{dt}{s^2},$$

and a density function in the following form:

$$F_\tau(u) = u^2 \int_0^{u^{-1}} f\left(\frac{1}{t}\right) f\left(\frac{1}{u^{-1}-t}\right) \frac{dt}{t^2(u^{-1}-t)^2}.$$

Proof Let τ be a random variable defined above, $T = 1/\tau_1$ and $S = 1/\tau_2$. Then

$$F_T(t) = P(T \leq t) = P\left(\tau_1 > \frac{1}{t}\right) = 1 - F_{\tau_1}\left(\frac{1}{t}\right),$$

and $F_T(t) = F_T'(t) = f\left(\frac{1}{t}\right) \frac{1}{t^2}$. Similarly,

$$F_S(s) = 1 - F_{\tau_2}\left(\frac{1}{s}\right) \text{ and } f_S(s) = f\left(\frac{1}{s}\right) \frac{1}{s^2}. \text{ Thus,}$$

$$F_\tau(u) = P(T + S > u^{-1}) = \iint_{t+s > u^{-1}} f_T(t) f_S(s) dt ds$$

$$\begin{aligned}
 &= 1 - \iint_{t+s > u^{-1}} f\left(\frac{1}{t}\right) f\left(\frac{1}{s}\right) \frac{dt ds}{t^2 s^2} \\
 &= 1 - \int_0^{u^{-1}} f\left(\frac{1}{t}\right) \frac{dt}{t^2} \int_0^{u^{-1}-t} f\left(\frac{1}{s}\right) \frac{ds}{s^2}.
 \end{aligned}$$

The density function is obtained by differentiating the $F_{\tau}(u)$. Therefore,

$$f_{\tau}(u) = u^{-2} \int_0^{u^{-1}} f\left(\frac{1}{t}\right) \left(\frac{1}{u^{1-t}} \right) \frac{dt}{t^2 (u^{-1}-t)^2}.$$

Finally, we achieve the required expressions which complete the proof. However, the pdf of τ has no explicit form. This may be difficult to directly find important functions such as characteristic and moment generating functions. Figures 1-2 show variety of the probability density functions for TS-LBIG. It is indicated that the TS-LBIG is the positively skewed distribution which is a useful choice in this framework.

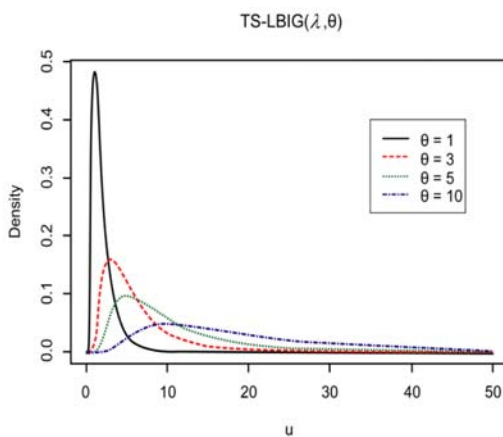


Figure 1. The two-sided length biased inverse Gaussian density function for $\lambda = 2$.

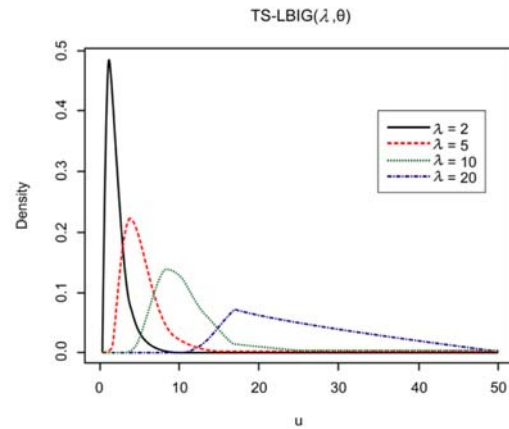


Figure 2. The two-sided length biased inverse Gaussian density function for $\theta = 1$.

4. THEORETICAL RESULTS

4.1 Reciprocal Properties

Proposition 4.1.1 If random variable $\tau > 0$ has the density function $f_{\tau}(x)$, then the reciprocal random variable $1/\tau$ has the pdf, $f_{1/\tau}(x) = x^{-2} f_{\tau}(1/x)$.

Proof For the reciprocal random variable $1/\tau$, applying a definition of a cumulative distribution function, then its distribution function is defined by $F_{1/\tau}(x) = P\left(\frac{1}{\tau} \leq x\right) = P\left(\tau \geq \frac{1}{x}\right) = 1 - F_{\tau}\left(\frac{1}{x}\right)$, and applying the chain rule, the density function is $f_{1/\tau}(x) = F'_{1/\tau}(x) = x^{-2} f_{\tau}(1/x)$.

Proposition 4.1.2 If random variable $\tau = 0$ has LBIG (λ, θ) distribution, then the reciprocal random variable $1/\tau$ is IG $[\lambda, 1/(\lambda^2 \theta)]$ distributed.

Proof By proposition 4.1.1,

$$\begin{aligned}
 f_{1/\tau}(x) &= x^{-2} f_{LBIG}\left(\frac{1}{x}; \lambda, \theta\right) \\
 &= \frac{x^{-3/2} \theta^{-1/2}}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(\lambda(\theta x)^{1/2} - (\theta x)^{-1/2})^2\right\}
 \end{aligned}$$

$$= \frac{\lambda(1/\lambda^2\theta)^{1/2}x^{-3/2}}{\sqrt{2\pi}} \times \exp\left\{-\frac{1}{2}\left(\lambda\sqrt{\frac{1/(\lambda^2\theta)}{x}} - \sqrt{\frac{x}{1/(\lambda^2\theta)}}\right)^2\right\}$$

$$= IG[x; \lambda, 1/(\lambda^2\theta)].$$

4.2 The First Four Moments

Finding the first four moments of the TS-LBIG distribution, we deal with the following way. The reciprocal property is necessarily needed. The characteristic function of the IG distribution is required. Because of the difficulty of direct derivation, Maclaurin expansion is applied. The first four terms are considered to obtain the first four cumulants. Then, they are modified to gain the first four moments. The required theorems are presented as follows.

Theorem 4.2.1 If a continuous random variable X is inverse Gaussian distributed with parameters λ and θ denoted as $IG(\lambda, \theta)$, then its characteristic function is

$$\varphi_{IG}(x; \lambda, \theta) = \exp\{\lambda[1 - (1 - 2i\theta t)^{1/2}]\}.$$

Proof By definition, the characteristic function of the $IG(\lambda, \theta)$ is defined as

$$\varphi_{IG}(x; \lambda, \theta) = \int_{-\infty}^{\infty} \exp(itx)f_{IG}(x; \lambda, \theta)dx$$

$$= \lambda\sqrt{\frac{\theta}{2\pi}} \int_0^{\infty} \exp(itx)x^{-3/2} \times \exp\left\{-\frac{\lambda^2\theta}{2x} + \lambda - \frac{x}{2\theta}\right\} dx$$

$$= \lambda e^\lambda \sqrt{\frac{\theta}{2\pi}} \int_0^{\infty} x^{-3/2} \times \exp\left\{-\left(\frac{1 - 2i\theta t}{2\theta}\right)x - \frac{\lambda^2\theta}{2x}\right\} dx.$$

By applying the formula 3.472.5 from Gradshteyn and Ryzhik [6], (p. 369), we have $p = \frac{1 - 2i\theta t}{2\theta}$ and $q = \frac{\lambda^2\theta}{2x}$, then we get

$$\varphi_{IG}(x; \lambda, \theta) = \lambda e^\lambda \sqrt{\frac{\theta}{2\pi}} \times \left[\sqrt{\frac{2\pi}{\lambda^2\theta}} \exp\left\{-2\sqrt{\left(\frac{1 - 2i\theta t}{2\theta}\right)\left(\frac{\lambda^2\theta}{2x}\right)}\right\} \right]$$

$$= e^\lambda \exp\{-\lambda\sqrt{1 - 2i\theta t}\}$$

$$= \exp\{\lambda[1 - (1 - 2i\theta t)^{1/2}]\}.$$

Theorem 4.2.2 If a random variable X is two-sided length biased inverse Gaussian distributed with parameters λ and θ denoted as $TS-LBIG(\lambda, \theta)$, then the first four moments are

$$\mu(X) = k_1(X) = \frac{2}{\lambda\theta}, \sigma^2(X) = k_2(X) = \frac{2}{\lambda^3\theta^2},$$

$$\mu_3(X) = k_3(X) = \frac{6}{\lambda^5\theta^3},$$

$$\mu_4(X) = k_4(X) + 3\sigma^4(X) = \frac{30 + 12\lambda}{\lambda^7\theta^4},$$

Proof If $\varphi(t)$ is a characteristic function, then its cumulants $k_j; j = 1, 2, \dots, m$, are defined from the Maclaurin expansion of the logarithm of the characteristic function:

$$\ln\varphi(t) = \sum_{j=1}^m \frac{k_j}{j!} (it)^j + o(|t|^m).$$

Hence, the logarithm of the characteristic function of $IG(\lambda, \theta)$ distribution is

$$\ln\varphi(t) = \lambda - \lambda(1 - 2i\theta t)^{1/2} \tag{5}$$

For the Maclaurin expansion, we use $(1-x)^{1/2}$

$$= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{1}{128}x^4 - O(x^5), \tag{6}$$

where $|x| < 1$.

In this case, $x = 2i\theta t$ and we assume that $0 < t < 1/2\theta$. Applying the equations (5) and (6), we obtain

$$\begin{aligned} \ln\varphi(t) &= \lambda - \lambda \left[1 - \frac{2(i\theta t)}{2} - \frac{4(i\theta t)^2}{8} - \frac{8(i\theta t)^3}{16} \right] \\ &= \lambda(i\theta t) + \frac{\lambda}{2} (i\theta t)^2 + \frac{\lambda}{2} (i\theta t)^3 + \frac{5\lambda}{8} (i\theta t)^4 \\ &+ O(t^5) \\ &= \frac{(i)}{1!} \lambda\theta + \frac{(i)^2}{2!} \lambda\theta^2 + \frac{(i)^3}{3!} 3\lambda\theta^3 + \frac{(i)^4}{4!} 15\lambda\theta^4 \\ &+ O(t^5) \end{aligned}$$

This expression gives us the cumulants of the IG distribution:

$$\begin{aligned} k_1 &= \mu = \lambda\theta, \quad k_2 = \sigma^2 = \lambda\theta^2, \\ k_3 &= \mu_3 = 3\lambda\theta^3 \text{ and } k_4 = \mu_4 - 3\sigma^4 = 15\lambda\theta^4. \end{aligned}$$

By proposition 4.1.2, if $\tau \sim \text{LBIG}(\lambda, \theta)$, then the reciprocal $1/\tau \sim \text{IG}[\lambda, 1/(\lambda^2\theta)]$. Substituting θ by $1/(\lambda^2\theta)$, $1/\tau$ has the following cumulants:

$$\begin{aligned} k_1(1/\tau) &= \frac{1}{\lambda\theta}, \quad k_2(1/\tau) = \frac{1}{\lambda^3\theta^2}, \\ k_3(1/\tau) &= \frac{3}{\lambda^5\theta^3} \text{ and } k_4(1/\tau) = \frac{15}{\lambda^7\theta^4}. \end{aligned}$$

Because of the i.i.d. property, the moments for the random variable $X = \tau_1^{-1} + \tau_2^{-1}$ will be obtained by combining the cumulants for $1/\tau$. Finally, the first four moments for the two-sided length biased inverse Gaussian distribution are:

$$\begin{aligned} \mu(X) &= k_1(X) = \frac{2}{\lambda\theta}, \quad \sigma^2(X) = k_2(X) = \frac{2}{\lambda^3\theta^2}, \\ \mu_3(X) &= k_3(X) = \frac{6}{\lambda^5\theta^3}, \\ \mu_4(X) &= k_4(X) + 3\sigma^4(X) = \frac{30 + 12\lambda}{\lambda^7\theta^4}. \end{aligned}$$

4.3 Method of Moment Estimation

Estimation of the parameters λ and θ by the method of moment for the TS-LBIG distribution can be derived in the following way. In the previous section, the first four moments are investigated, and thus we obtain formulas for the expectation and variance;

$$E(X) = \mu(X) = \frac{2}{\lambda\theta} \text{ and } Var(X) = \sigma^2(X) = \frac{2}{\lambda^3\theta^2}.$$

Hence, the method of moment estimation is

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i = \frac{2}{\lambda\theta}; \tag{7}$$

$$S^2 = \frac{1}{n} \sum_{i=1}^n X_i^2 - \bar{X}^2 = \frac{2}{\lambda^3\theta^2} \tag{8}$$

Solving the equations (7) and (8) for λ and θ , we obtain the desired estimates. Let $T = S^2/\bar{X}^2$. Then, . Therefore, the method of moment estimators are $\hat{\lambda} = \frac{1}{2T}$ and $\hat{\theta} = \frac{2}{\lambda\bar{X}}$.

The formulas of the asymptotic variances and covariance are presented in the theorem below.

Theorem 4.3.1 As $n \rightarrow \infty$, the $Var(\hat{\lambda})$, $Var(\hat{\theta})$ and $Cov(\hat{\lambda}, \hat{\theta})$ are given by

$$\begin{aligned} Var(\hat{\lambda}) &= \frac{4\lambda^2 n^2 + 7\lambda n^2 + 5\lambda n - 12\lambda}{2n^2(n-1)}, \\ Var(\hat{\theta}) &= \frac{(2\lambda + 3)\theta^2 n^2 + 6\theta^2 n - 9\theta^2}{\lambda n^2(n-1)}, \\ Cov(\hat{\lambda}, \hat{\theta}) &= \frac{(4\lambda\theta + 18\theta)n^2 - 15\theta n - 3\theta}{2n^2(n-1)} \end{aligned}$$

Proof The asymptotic distribution of an estimate which smoothly depends on sample moments is commonly obtained by their decomposition into the Taylor series expansion. For our case, $\lambda = \lambda(X, S^2)$, $\theta = \theta(X, S^2)$, $\mu(X)$ is the true value of X , and $\sigma^2(X)$ is that of S^2 . Let a_1 and a_2 be the values of partial derivatives of λ by X and S^2 at the point $(\mu(X), \sigma^2(X))$, correspondingly.

The b_1 and b_2 are denoted as the values of analogous derivatives of θ . Therefore, the Taylor expansion can be written as

$$\lambda = \lambda + a_1(\bar{X} - \mu(X)) + a_2(S^2 - \sigma^2(X)) + O_p\left(\frac{1}{n}\right),$$

$$\theta = \theta + b_1(\bar{X} - \mu(X)) + b_2(S^2 - \sigma^2(X)) + O_p\left(\frac{1}{n}\right).$$

The $X_n = O_p\left(\frac{1}{n}\right)$, where $\{X_n, n \geq 1\}$ is a

sequence of random variables, means that there exists a constant $C > 0$ such that $\limsup_{n \rightarrow \infty} P|X_n - C/n| > t = 0$. These

expressions are understood in terms of convergence in probability. Thus, the vector $(\sqrt{n}(\lambda - \lambda), \sqrt{n}(\theta - \theta))$ has limit in distribution, the two-dimensional normal distribution with zero means, variances

$$Var(\lambda) = a_1^2 Var(X) + a_2^2 Var(S^2) + 2a_1 a_2 Cov(X, S^2) \quad (9)$$

$$Var(\lambda) = b_1^2 Var(X) + b_2^2 Var(S^2) + 2b_1 b_2 Cov(X, S^2) \quad (10)$$

and

$$Cov(\lambda, \theta) = a_1 b_1 Var(X) + a_2 b_2 Var(S^2) + 2(a_1 b_2 + a_2 b_1) Cov(X, S^2). \quad (11)$$

As presented results in section 4.2, we apply the formulas outlined in Cho and Cho [5] and Cramer [6], (p. 352-358), then we obtain:

$$Var(X) = \frac{2}{\lambda^3 \theta^2 n},$$

$$Var(S^2) = \frac{8\lambda n + 30n - 30}{n(n-1)\lambda^7 \theta^4}, \quad (12)$$

$$Cov(X, S^2) = \frac{6(n-1)}{n^2 \lambda^5 \theta^2}.$$

We firstly consider the derivatives of the estimate λ and we have

$$\frac{\partial \lambda}{\partial X} = \frac{d\lambda}{dT} \cdot \frac{\partial T}{\partial X} = \frac{1}{2T^2} \cdot \frac{\partial T}{\partial X}.$$

Exchanging X and S^2 on μ and σ^2 , we get

$$a_1 = \frac{\partial \lambda}{\partial X} = \left(\frac{4\lambda^2}{2}\right) \cdot \left(\frac{\theta}{2}\right) = \lambda^2 \theta.$$

Similarly, based on the appropriate substitution, we obtain

$$a_2 = \frac{\partial \lambda}{\partial S^2} = \left(-\frac{4\lambda^2}{2}\right) \cdot \left(\frac{\lambda^2 \theta^2}{4}\right) = -\frac{\lambda^2 \theta^2}{2}.$$

We secondly provide the analogous derivations of the derivatives of the θ and we have

$$\frac{\partial \theta}{\partial X} = \frac{d\theta}{d\lambda} \cdot \frac{\partial \lambda}{\partial X}$$

$$= \left(d\left(\frac{2}{\lambda}\right)/d\lambda\right) \cdot \frac{\partial \lambda}{\partial X} \cdot X^{-1} + \left(\frac{2}{\lambda}\right)(-1)X^{-2}$$

$$= -\left[\frac{2}{\lambda^2 X} \cdot \frac{\partial \lambda}{\partial X} + \frac{2}{\lambda X^2}\right].$$

Note that $\frac{\partial \lambda}{\partial X}(\mu, \sigma^2) = a_1$, and hence

$$b_1 = \frac{\partial \theta}{\partial X}(\mu, \sigma^2) = -\frac{3}{2} \lambda^2 \theta.$$

Equivalently,

$$b_2 = \frac{\partial \theta}{\partial S^2}(\mu, \sigma^2) = \frac{\lambda^3 \theta^3}{2}.$$

Then, we substitute a_1, a_2, b_1, b_2 and (12) into the equations (9), (10) and (11) to obtain $Var(\lambda)$, and $Cov(\lambda, \theta)$.

Hence,

$$Var(\lambda) = (\lambda^2 \theta)^2 \left(\frac{2}{\lambda^3 \theta^2 n}\right) + \left(\frac{\lambda^4 \theta^2}{2}\right) \times$$

$$\left(\frac{8\lambda n + 30n - 30}{n(n-1)\lambda^7 \theta^4}\right) + 2(\lambda^2 \theta) \left(-\frac{\lambda^2 \theta^2}{2}\right) \left(\frac{n-1}{2}\right) \left(\frac{6}{\lambda^5 \theta^3}\right)$$

$$= \frac{2\lambda}{n} + \frac{4\lambda^2 n + 15\lambda n - 15\lambda}{2n(n-1)} - \frac{(n-1)6\lambda}{n^2}$$

$$= \frac{4\lambda^2 n^2 + 7\lambda n^2 + 5\lambda n - 12\lambda}{2n^2(n-1)}.$$

Thus,

$$\begin{aligned} Var(\theta) &= \left(-\frac{3\lambda\theta^2}{2}\right)^2 \left(\frac{2}{\lambda^3\theta^{2n}}\right) + \left(\frac{\lambda^3\theta^3}{2}\right)^2 \times \\ &\left(\frac{8\lambda n + 30n - 30}{n(n-1)\lambda^7\theta^4}\right) + 2\left(-\frac{3\lambda\theta^2}{2}\right)\left(\frac{\lambda^3\theta^3}{2}\right)\left(\frac{6(n-1)}{n^2\lambda^5\theta^3}\right) \\ &= \frac{9\theta^2}{2\lambda n} + \frac{4\lambda\theta^{2n} + 15\theta^{2n} - 15\theta^2}{2\lambda n(n-1)} + \frac{18\theta^{2(n-1)}}{2\lambda n^2} \\ &= \frac{(2\lambda + 3)\theta^{2n} + 6\theta^{2n} - 9\theta^2}{\lambda n^2(n-1)}. \end{aligned}$$

Therefore,

$$\begin{aligned} Cov(\lambda, \theta) &= (\lambda^2\theta)\left(-\frac{3\lambda\theta^2}{2}\right)\left(\frac{2}{\lambda^3\theta^{2n}}\right) \\ &+ \left(-\frac{\lambda^4\theta^2}{2}\right)\left(\frac{\lambda^3\theta^3}{2}\right)\left(\frac{8\lambda n + 30n - 30}{n(n-1)\lambda^7\theta^4}\right) \\ &+ 2\left\{(\lambda^2\theta)\left(\frac{\lambda^3\theta^3}{2}\right) + \left(-\frac{\lambda^4\theta^2}{2}\right)\left(\frac{3\lambda\theta^2}{2}\right)\right\} \times \\ &\left(\frac{6(n-1)}{n^2\lambda^5\theta^3}\right) \\ &= -\frac{3\theta}{n} - \frac{4\lambda\theta n^2 + 15\theta n - 15\theta}{2n(n-1)} + \frac{15\theta(n-1)}{n^2} \\ &= -\frac{(4\lambda\theta + 18\theta)n^2 - 15\theta n - 3\theta}{2n^2(n-1)}. \end{aligned}$$

Finally, the proof is complete.

5. NUMERICAL RESULTS

5.1 Simulaion Study

We studied the properties of the presented estimators by using the numerical method. The results were reported to investigate the behavior of the estimators via calculating the estimated bias and mean square error. Monte Carlo simulations were performed for different sample sizes. The R program version 3.1.2 was used to generate and analyze the data. The number of iterations was fixed at 5,000 for each combination of λ , θ , and sample sizes n . We considered all

combinations of the following values of λ , θ , and n as follows: $\lambda = 2, 5, 10$ and 50 , $\theta = 1, 5, 10$ and 50 , $n = 10, 50$ and 100 . Tables 1-3 show the estimated bias and MSE of λ and θ for $n = 10, 50$ and 100 , respectively.

As seen in tables 1-3, the method of moment estimators has generally a systematic positive bias (underestimate). In some situations, the estimators have a negative bias (overestimate) when single or both parameters are large. Furthermore, the bigger parameters, the larger are the bias. For instance, at $n = 100$, $\lambda = 2$ and $\theta = 5$, the bias of λ is 0.0892, while for $\lambda = 50$ and $\theta = 50$, the bias of λ is -0.9187. However, the magnitude of the bias may be assumed to be relatively small. Tables 1-3 reveal that the more increasing values of the parameters, the more growing are the mean square error.

As the results from the numerical study, it is observed that the λ and θ are consistent. Thus, they are asymptotically unbiased estimators. The simulated bias corresponds to the theoretical background as it is a decreasing function of sample sizes n . That is, when sample sizes increase, the amount of the bias decreases and tends to zero as $n \rightarrow \infty$. Similarly, the MSE is a decreasing function of n . The larger sample size, the smaller is the MSE, and it approaches to zero as $n \rightarrow \infty$.

5.2 Illustrative Examples

The practical applications of the suggested estimators are illustrated in this section. Two real data sets are considered as the followings.

Example 1. The following data provided by Lieblein and Zelen [12] on the fatigue life of the 23 deep groove ball bearings:
 17.88 28.92 33.00 41.52 42.12 45.60 48.48
 51.84 51.96 54.12 55.56 67.80 68.64 68.64
 68.88 84.12 93.12 98.64 105.12 105.84
 127.92 128.04 173.40.

The above data had been analyzed by Gupta and Akman [8] and the result indicated that the data set comes from the length biased inverse Gaussian distribution. For two-sided case, we divide all data in pairs, we have only 11 pairs of observations, and the last one is dropped. We obtain 11 observations: $(y_1, y_2), (y_3, y_4), \dots, (y_{21}, y_{22})$. Let $u_1 = (1/y_1) + (1/y_2)$, $u_2 = (1/y_3) + (1/y_4)$, ..., $u_{11} = (1/y_{21}) + (1/y_{22})$ be the observations drawn from the TS-LBIG distribution. In this example, the point estimates are reported in Table 4. Moreover, using the relationship given in section 2.1, the point estimates for parameters μ and β are also computed.

Example 2. This example was taken from Nichols and Padgett [14] consisting of 100 observations on breaking stress of carbon fibers (in GPa). These data had been analyzed by assuming the Weibull distribution. In our case, the data was ascendingly ordered. Dealing with the analogous manner of the example 1, finally, we obtain 50 observations drawn from the TS-LBIG distribution. The point estimates for the proposed and original parameters are shown in Table 5.

Most importantly, the estimators λ and θ represent the thickness of the object under study and the nominal treatment pressure on the object, correspondingly. On the other hand, the estimators μ and β lack this physical explanation.

6. CONCLUSIONS AND DISCUSSION

The new lifetime distribution based on re-parametrization model called the two-sided length biased inverse Gaussian distribution is proposed. The reciprocal properties and the first four moments of the distribution are investigated. The conventional point estimation, method of moment, is developed for estimating the parameters of the distribution and the asymptotic variances and

covariance of the suggested estimators are also provided.

In this article, we discussed on some statistical properties of the TS-LBIG distribution. According to previous results, because the density function of the distribution has no closed form, the characteristic and moment generating functions cannot be directly solved. We may use the reciprocal property to accomplish this problem. The first four cumulants are defined from the Maclaurin expansion of the logarithm of the characteristic function. There are two important points we should concern: the remainder of the Maclaurin expansion and the existing condition of the Maclaurin extended terms. In the first point, our goal is investigation of the first four cumulants since the first four terms involve important characteristics, i.e., mean, variance, skewness, and kurtosis of the distribution, then the fifth and higher exponents are considered to be trivial. In the second point, the first four cumulants for the TS-LBIG(λ, θ) distribution exist when the condition $0 < t < 1/2\theta$ is satisfied. In parameter estimation scheme, we only focus on a point estimation. The results ensure us that the method of moment estimators provides consistent and asymptotically unbiased statistics. Importantly, we estimate the parameters which reflect the physical nature of an object analyzed by a statistical point of view of the distribution. Consequently, one could continue investigations for other point estimation schemes such as maximum likelihood estimators, although we expect to encounter difficulties of mathematical derivation. The density function of the TS-LBIG distribution involves an integral, and finding a maximum of their products even numerically is not an easy task. Furthermore, we may consider new other estimators, i.e., a regression-quantile (least square). This

method is dealt with the regression analysis of sample quantiles. However, the method of moment estimation can be examined as a preliminary topic of studying for the TS-LBIG distribution since it has a satisfying property, e.g. simple computation. Interval estimation and hypothesis testing issues remain to be interesting for further

investigation. As presented results of the asymptotic analysis, tests and confident interval estimation procedures will be explored regarding on a power of the test and coverage probabilities. Nevertheless, they are above the scope of this article and the investigation will be separately communicated.

Table 1. The estimated bias and mean square error of $\hat{\lambda}$ and $\hat{\theta}$ for $n = 10$.

λ	θ	$\hat{\lambda}$	$\hat{\theta}$	$\lambda - \hat{\lambda}$	$\theta - \hat{\theta}$	$MSE(\hat{\lambda})$	$MSE(\hat{\theta})$
2	1	1.3703	0.9329	0.6297	0.0671	5.8276	0.2760
	5	1.6259	4.6382	0.3741	0.3618	5.0295	6.9039
	10	1.4576	9.3924	0.5424	0.6076	4.6658	26.6801
	50	1.3064	44.5593	0.6936	5.4407	4.6971	675.4794
5	1	4.1880	1.0852	0.8120	-0.0852	20.8110	0.2032
	5	4.0990	5.3294	0.9010	-0.3294	20.5830	4.9728
	10	3.6782	10.4845	1.3218	-0.4845	20.4278	19.3153
	50	4.0014	50.9950	0.9986	-0.9950	25.5249	522.8751
10	1	9.5862	0.9202	0.4138	0.0798	79.4018	0.1950
	5	11.2015	4.0934	-1.2015	0.9066	80.4220	4.8915
	10	9.7968	8.5040	0.2030	1.4960	73.0999	19.4146
	50	10.7888	48.7552	-0.7888	1.2448	85.8734	522.2085
50	1	51.7070	1.0124	-1.7070	-0.0124	1924.1440	0.2113
	5	52.8344	4.6693	-2.8344	0.3307	1867.7820	5.4653
	10	46.4884	10.9861	3.5116	-0.9861	1815.5120	21.7748
	50	47.2446	51.9597	2.7554	-1.9597	1895.5340	528.9137

Table 2. The estimated bias and mean square error of $\hat{\lambda}$ and $\hat{\theta}$ for $n = 50$.

λ	θ	$\hat{\lambda}$	$\hat{\theta}$	$\lambda - \hat{\lambda}$	$\theta - \hat{\theta}$	$MSE(\hat{\lambda})$	$MSE(\hat{\theta})$
2	1	1.9160	0.7330	0.0840	0.2670	0.5081	0.1938
	5	1.7219	3.6200	0.2781	1.3800	0.4855	4.7132
	10	1.2650	9.5448	0.7350	0.4552	0.4783	18.7898
	50	1.6161	41.5526	0.3839	8.4474	0.5048	477.7856
5	1	4.9301	0.8442	0.0699	0.1558	1.7735	0.0708
	5	4.1089	4.9859	0.8911	0.0141	1.7389	1.7413
	10	4.7881	8.7293	0.2119	1.2707	1.7864	7.0728
	50	4.1589	48.1984	0.8411	1.8016	1.7332	176.8778
10	1	9.0434	1.0543	0.9566	-0.0543	5.8227	0.0473
	5	10.1666	4.2394	-0.1666	0.7606	5.7515	1.1636
	10	9.8077	9.5888	0.1923	0.4112	5.7532	4.7292
	50	9.9115	46.7098	0.0885	3.2902	5.9288	118.8181
50	1	51.3669	0.9592	-1.3669	0.0480	129.5475	0.0403
	5	51.6333	4.7817	-1.6333	0.2183	130.6555	1.0128
	10	49.8187	10.0550	0.1813	-0.0550	128.0641	3.9821
	50	48.9986	50.031	1.0014	-0.0301	132.8144	103.7690

Table 3. The estimated bias and mean square error of $\hat{\lambda}$ and $\hat{\theta}$ for $n = 100$.

λ	θ	$\hat{\lambda}$	$\hat{\theta}$	$\lambda - \hat{\lambda}$	$\theta - \hat{\theta}$	$MSE(\hat{\lambda})$	$MSE(\hat{\theta})$
2	1	1.9383	0.6956	0.0617	0.3044	0.2707	0.1788
	5	1.9108	3.6939	0.0892	1.3061	0.2597	4.4037
	10	1.6465	7.9884	0.3535	2.0116	0.2600	17.8233
	50	1.5441	43.6269	0.4559	6.3731	0.2605	444.7151
5	1	4.5678	0.9309	0.4322	0.0691	0.8041	0.0536
	5	4.4455	4.7929	0.5545	0.2071	0.7772	1.3023
	10	4.1601	9.8065	0.8399	0.1935	0.8175	5.3521
	50	4.4546	47.9102	0.5454	2.0898	0.7933	132.2765
10	1	9.5431	0.9185	0.4569	0.0815	2.5924	0.0284
	5	10.6294	4.2904	-0.6294	0.7096	2.6800	0.7216
	10	9.7182	9.2669	0.2818	0.7331	2.5553	2.8027
	50	9.6252	47.8684	0.3748	2.1315	2.7423	73.0557
50	1	48.9022	0.9880	1.0978	0.0120	57.9392	0.0202
	5	48.9931	5.1267	1.0069	-0.1267	57.4006	0.5032
	10	48.6309	10.0512	1.3691	-0.0512	57.9988	2.0488
	50	50.9187	48.8310	-0.9187	1.1690	55.9675	49.1647

Table 4. Point estimates for Example 1.

Ahmed's Parameter		Usual Parameter	
λ	θ	μ	β
1.56484	34.12668	53.40291	83.56719

Table 5. Point estimates for Example 2.

Ahmed's Parameter		Usual Parameter	
λ	θ	μ	β
1.37361	1.55789	2.13994	2.93945

ACKNOWLEDGEMENTS

The authors gratefully acknowledge the referees for providing valuable recommendations to complete the manuscript.

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