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ABSTRACT
In this correspondence, we investigate mean convergence of order p for the weighted sums of Banach space valued random elements under a suitable (compactly) uniformly integrable condition with or without a geometric condition placed on the Banach space.

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1. Introduction and main results

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathbb{B}$ be a real separable Banach space with norm $\|\cdot\|$. A $\mathbb{B}$-valued *random element* X is defined as an $\mathcal{F}$-measurable function from $(\Omega, \mathcal{F})$ into $\mathbb{B}$ equipped with its *Borel* σ*-algebra* $\mathcal{B}$ which is the σ-algebra generated by the topology of open subsets of $\mathbb{B}$ determined by $\|\cdot\|$.

For a $\mathbb{B}$-valued random element sequence $\{X_n,\ n \ge 1\}$ and an array of real constants $\{a_{nk},\ n \ge 1,\ k \ge 1\}$, the limiting behaviour of the weighted sums $\sum_{k=1}^{\infty} a_{nk}X_k$ enjoys a large literature of investigation (see, e.g. [6,8,15,23–25]). The impetus for the study of limiting behaviour of weighted sums comes from the fact that such sums can play an important role in various applied and theoretical problems. For example, in a quality control study, if we consider that each random variable X_k (random element taking values in the real line) is a statistic computed from a sample of size n, then the sum $\sum_{k=1}^{\infty} a_{nk}X_k$ is used to determine the variation in the quality of the output, summing over the entire past.

In this paper we study mean convergence for weighted sums of random elements. Our results will extend and generalize some well-known results.

We first review some concepts.

The *expected value* or *mean* of a random element X, denoted $\mathbb{E}X$, is defined to be the *Pettis integral* provided it exists; that is, X has expected value $\mathbb{E}X \in \mathbb{B}$ if $f(\mathbb{E}X) = \mathbb{E}(f(x))$

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for every $f \in \mathbb{B}^*$, where $\mathbb{B}^*$ is the (*dual*) space of all continuous linear functionals on $\mathbb{B}$. A sufficient condition for X to have an expected value is that $\mathbb{E}\|X\| < \infty$ (see, e.g. [22]).

Let $\mathbb{B}^\infty = \mathbb{B} \times \mathbb{B} \times \mathbb{B} \times \cdots$ and let $\{Y_n,\ n \geq 1\}$ be a symmetric *Bernoulli sequence*; that is, $\{Y_n,\ n \geq 1\}$ is a sequence of independent and identically distributed (i.i.d.) random variables where $\mathbb{P}\{Y_1 = 1\} = \mathbb{P}\{Y_1 = -1\} = 1/2$. Let

$$\mathcal{C}(\mathbb{B}) = \left\{ (x_1, x_2, \ldots) \in \mathbb{B}^\infty : \sum_{n=1}^{\infty} Y_n x_n \text{ converges in probability} \right\}.$$

Let $p \in [1, 2]$. Then $\mathbb{B}$ is said to be of *Rademacher type p* if there exists a constant $C_p \in (0, \infty)$ such that

$$\mathbb{E} \left\| \sum_{n=1}^{\infty} Y_n x_n \right\|^p \leq C_p \sum_{n=1}^{\infty} \|x_n\|^p \quad \text{for all } (x_1, x_2, \ldots) \in \mathcal{C}(\mathbb{B}).$$

The condition of $\mathbb{B}$ being of Rademacher type p is actually equivalent to the structurally simpler condition that there exists a constant $C_p \in (0, \infty)$ such that

$$\mathbb{E} \left\| \sum_{n=1}^{N} Y_n x_n \right\|^p \leq C_p \sum_{n=1}^{N} \|x_n\|^p \quad \text{for all } N \geq 1 \text{ and all } x_n \in \mathbb{B},\ 1 \leq n \leq N.$$

This equivalence is established in [18].

It is proved in [7] for $p \in [1, 2]$ that a real separable Banach space is of Rademacher type p if and only if there exists a constant $C_p \in (0, \infty)$ such that

$$\mathbb{E} \left\| \sum_{k=1}^{n} X_k \right\|^p \leq C_p \sum_{k=1}^{n} \mathbb{E}\|X_k\|^p$$

for every finite collection $\{X_1, \ldots, X_n\}$ of independent mean 0 random elements.

If a real separable Banach space is of Rademacher type p for some $p \in (1, 2]$, then it is of Rademacher type q for all $q \in [1, p]$.

For a random element X and a sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$, the *conditional expectation* $\mathbb{E}(X \mid \mathcal{G})$ was introduced in [20] and is defined in a manner which is analogous to that in the real-valued random variable case and enjoys similar properties. A complete development of the notion of conditional expectation may be found in [20] including Banach space valued martingales (which will be defined below) and martingale convergence theorems.

A sequence $\{X_n,\ n \geq 1\}$ of $\mathbb{B}$-valued random elements is said to be a *martingale* with respect to a non-decreasing sequence of sub-sigma-algebras $\{\mathcal{F}_n,\ n \geq 1\}$ of $\mathcal{F}$ (a *filtration*) if $\mathbb{E}X_n$ exists for all $n \geq 1$, X_n is $\mathcal{F}_n$-measurable for all $n \geq 1$, and

$$\mathbb{E}(X_{n+1} \mid \mathcal{F}_n) = X_n, \quad n \geq 1.$$

In this case, the sequence $\{Y_n, \mathcal{F}_n,\ n \geq 1\}$ where $Y_n = X_n - X_{n-1},\ n \geq 1, X_0 = 0$ is said to be a *martingale difference sequence*.

Let $p \in [1, 2]$. Then $\mathbb{B}$ is said to be of *martingale type p* if there exists a constant $C_p \in (0, \infty)$ such that for any pth order integrable martingale difference sequence $\{X_n, \mathcal{F}_n, n \geq 1\}$,

$$\sup_{n\geq 1} \mathbb{E} \left\| \sum_{k=1}^{n} X_k \right\|^p \leq C_p \sum_{k=1}^{\infty} \mathbb{E}\|X_k\|^p.$$

It can be shown (see [16,17]) that $\mathbb{B}$ being of martingale type p is indeed equivalent to the apparently stronger condition that for all $q \in [1, \infty)$, there exists a constant $C_{p,q} \in (0, \infty)$ such that for any pth order integrable martingale sequence $\{X_n, \mathcal{F}_n, \ n \geq 1\}$,

$$\mathbb{E} \left(\sup_{n\geq 1} \left\| \sum_{k=1}^{n} X_k \right\| \right)^q \leq C_{p,q} \mathbb{E} \left(\sum_{k=1}^{\infty} \|X_k\|^p \right)^{q/p}. \tag{1}$$

It follows from (1) that if $\mathbb{B}$ is of martingale type p for some $p \in (1, 2]$, then $\mathbb{B}$ is of martingale type q for all $q \in [1, p]$.

Clearly every real separable Banach space is of martingale type (at least) 1. For $p \in [1, \infty)$, the L_p-spaces and ℓ_p-spaces are of martingale type $p \wedge 2$. We refer the reader to [16,17,21,26,27] for detailed discussion of martingale type p Banach spaces including many interesting examples.

It follows from the Hoffmann-Jørgensen and Pisier [7] characterization of Rademacher type p Banach spaces discussed above that if a Banach space is of martingale type p, then it is of Rademacher type p. The real line $\mathbb{R}$ is of martingale type 2 and hence $\mathbb{R}$ is of Rademacher type 2. The notion of martingale type p spaces is only superficially similar to that of Rademacher type p spaces. Indeed, a Banach space can be of Rademacher type 2 (and hence be of Rademacher type p for all $p \in [1, 2]$) yet be of martingale type p only for $p = 1$; for details see [9,17].

We note that the real line, as well as any Hilbert space, has much more geometry than only being of Rademacher type $p \leq 2$. It is also of cotype $q \geq 2$. Hence it is not surprising that our assumptions are stronger than the assumptions of the same results that pertain to real-valued random variables (see, for example, Theorem 1.1 in [5]).

The following two concepts were introduced by [12,13], respectively.

Definition 1.1: Let $p > 0$ and let $\{a_{nk}, \ n \geq 1, \ k \geq 1\}$ be an array of real constants with

$$\sup_{n\geq 1} \sum_{k=1}^{\infty} |a_{nk}| < \infty.$$

A sequence $\{X_n, \ n \geq 1\}$ of random elements is said to be *$\{a_{nk}\}$-uniformly pth order integrable* if

$$\lim_{x\to\infty} \sup_{n\geq 1} \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}\|X_k\|^p I(\|X_k\| > x) = 0.$$

If $p = 1$, then $\{X_n, \ n \geq 1\}$ is said to be *$\{a_{nk}\}$-uniformly integrable.*

Definition 1.2: Let $p > 0$ and let $\{a_{nk},\ n \geq 1,\ k \geq 1\}$ be an array of real constants with

$$\sup_{n\geq 1} \sum_{k=1}^{\infty} |a_{nk}| < \infty.$$

A sequence $\{X_n,\ n \geq 1\}$ of random elements is said to be $\{a_{nk}\}$*-compactly uniformly pth order integrable* if for any $\varepsilon > 0$, there exists a compact subset K in $\mathbb{B}$ such that

$$\sup_{n\geq 1} \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}\|X_k\|^p I(X_k \notin K) < \varepsilon.$$

If $p = 1$, $\{X_n,\ n \geq 1\}$ is said to be $\{a_{nk}\}$*-compactly uniformly integrable*.

Ordóñez Cabrera and Volodin [14] showed that a sequence of random elements $\{X_n,\ n \geq 1\}$ is $\{a_{nk}\}$-compactly uniformly pth order integrable if and only if it is $\{a_{nk}\}$-uniformly pth order integrable and $\{a_{nk}\}$-tight; that is, for every $\varepsilon > 0$, there exists a compact subset K of $\mathbb{B}$ such that

$$\sup_{n\geq 1} \sum_{k=1}^{\infty} |a_{nk}| \mathbb{P}\{X_k \notin K\} < \varepsilon.$$

It is worthwhile to point out that the notions of $\{a_{nk}\}$-uniformly pth order integrable and $\{a_{nk}\}$-compactly uniformly pth order integrable are equivalent in the case when $\{X_n,\ n \geq 1\}$ is a sequence of real-valued random variables. To see this, let the sequence of real-valued random variables $\{X_n,\ n \geq 1\}$ be $\{a_{nk}\}$-uniformly pth order integrable; that is,

$$\lim_{x\to\infty} \sup_{n\geq 1} \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}|X_k|^p I(|X_k| > x) = 0.$$

This expression can be rewritten in the equivalent form: For any $\varepsilon > 0$ there exists $A > 0$ such that

$$\sup_{n\geq 1} \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}|X_k|^p I(|X_k| > A) < \varepsilon.$$

Let $K = [-A, A]$. Then K is a compact set and the expression above can be rewritten as

$$\sup_{n\geq 1} \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}|X_k|^p I(|X_k| \notin K) < \varepsilon.$$

Hence $\{X_n,\ n \geq 1\}$ is $\{a_{nk}\}$-compactly uniformly pth order integrable.

With the preliminaries accounted for, we can now state the main results. Their proofs are given in the next section. We assume that the random series in Theorems 1.1–1.6 and Lemma 2.4 are almost surely convergent for each $n \geq 1$. Of course, almost sure convergence is automatic for any $n \geq 1$ in which $a_{nk} = 0$ for all large k.

Theorem 1.1: *Let* $\{a_{nk},\ n \geq 1,\ k \geq 1\}$ *be an array of real constants with*

$$\sup_{n\geq 1}\sum_{k=1}^{\infty}|a_{nk}| < \infty,$$

let $\{X_n,\ n \geq 1\}$ *be a sequence of random elements, and let* $p \geq 1$. *Suppose that* $\{X_n,\ n \geq 1\}$ *is* $\{a_{nk}\}$*-uniformly pth order integrable. Then* $\sum_{k=1}^{\infty} a_{nk}X_k \to 0$ *in probability if and only if*

$$\mathbb{E}\left\|\sum_{k=1}^{\infty} a_{nk}X_k\right\|^p \to 0. \tag{2}$$

Remark 1.1: Theorem 1.1 extends the equivalence of (v) and (vi) of Theorem 2.3 of [23]. Also, under the conditions of Theorem 1.1, (i)–(iv) of Theorem 2.3 of [23] are equivalent. Moreover, replacing the assumption that $\{X_n,\ n \geq 1\}$ is $\{a_{nk}\}$-uniformly pth order integrable by the assumption that $\{X_n,\ n \geq 1\}$ is $\{a_{nk}\}$-compactly uniformly pth order integrable, it is easy to show that (i)–(vi) of Theorem 2.3 of [23] are all equivalent.

Theorem 1.2: *Let* $\{a_{nk},\ n \geq 1,\ k \geq 1\}$ *be an array of real constants with*

$$\sup_{n\geq 1}\sum_{k=1}^{\infty}|a_{nk}| < \infty \quad \textit{and} \quad \sup_{k\geq 1}|a_{nk}| \to 0,$$

let $\{X_n,\ n \geq 1\}$ *be a sequence of pairwise independent random elements, and let* $p \geq 1$. *Suppose that* $\{X_n,\ n \geq 1\}$ *is* $\{a_{nk}\}$*-compactly uniformly pth order integrable. Then*

$$\mathbb{E}\left\|\sum_{k=1}^{\infty} a_{nk}(X_k - \mathbb{E}X_k)\right\|^p \to 0. \tag{3}$$

Remark 1.2: Theorem 1.2 extends Theorem 4.2 of [13] and Theorem 2.4 of [23].

Theorem 1.3: *Let* $\{a_{nk},\ n \geq 1,\ k \geq 1\}$ *be an array of real constants with*

$$\sup_{n\geq 1}\sum_{k=1}^{\infty}|a_{nk}| < \infty \quad \textit{and} \quad \sup_{k\geq 1}|a_{nk}| \to 0,$$

let $\{X_n, \mathcal{F}_n,\ n \geq 1\}$ *be a* $\mathbb{B}$*-valued martingale difference sequence, and let* $p \geq 1$. *Suppose that* $\{X_n,\ n \geq 1\}$ *is* $\{a_{nk}\}$*-compactly uniformly pth order integrable. Then* (2) *holds.*

Theorem 1.4: *Let* $\{a_{nk},\ n \geq 1,\ k \geq 1\}$ *be an array of real constants with*

$$\sup_{n\geq 1}\sum_{k=1}^{\infty}|a_{nk}|^p < \infty \quad \textit{and} \quad \sup_{k\geq 1}|a_{nk}| \to 0,$$

let $\{X_n,\ n \geq 1\}$ *a sequence of independent random elements, and let* $1 \leq p < 2$. *Suppose that* $\{X_n,\ n \geq 1\}$ *is* $\{|a_{nk}|^p\}$*-uniformly pth order integrable. Then* $\sum_{k=1}^{\infty} a_{nk}X_k \to 0$ *in probability if and only if* (2) *holds.*

Theorem 1.5: *Let $1 < p < 2$. Then the following two statements are equivalent:*

(1) *$\mathbb{B}$ is of Rademacher type p;*
(2) *(3) holds for any array of real constants $\{a_{nk},\ n \geq 1,\ k \geq 1\}$ with*

$$\sup_{n\geq 1}\sum_{k=1}^{\infty}|a_{nk}|^p < \infty \quad \textit{and} \quad \sup_{k\geq 1}|a_{nk}| \to 0$$

and for any sequence of independent $\{|a_{nk}|^p\}$-compactly uniformly pth order integrable random elements $\{X_n,\ n \geq 1\}$.

Remark 1.3: (1) If $\mathbb{B}$ is of Rademacher type p, $1 < p < 2$ and $\{X_n,\ n \geq 1\}$ is a sequence of i.i.d. random elements, then it is shown in [3] that the following two statements are equivalent:

$$\mathbb{E}X_1 = 0, \quad \mathbb{E}\|X_1\|^p < \infty;$$

$$\mathbb{E}\left\|n^{-1/p}\sum_{k=1}^{n}X_k\right\|^p \to 0.$$

It is clear that Theorem 1.5 extends the above mentioned result to the non-identically distributed case and weighted sums. Note that [2,6,8,15,19,25] also discussed the mean convergence for random elements, but their results do not generalize the result of [3] mentioned above.

(2) Theorem 1.5 gives a characterization of a Rademacher type p Banach space by mean convergence of weighted sums when $1 < p < 2$. However, the case when $p = 2$ (the real line, or more generally, any Hilbert space falls into this situation) is not covered by this theorem. The following example shows that Theorem 1.5 is not true for $p = 2$. Let $\{X, X_n,\ n \geq 1\}$ be a sequence of i.i.d. real-valued random variables with $\mathbb{E}X = 0$ and $\mathbb{E}X^2 = 1$. Then $\{X_n,\ n \geq 1\}$ is uniformly 2-nd integrable, and hence compactly uniformly 2-nd integrable, but $\mathbb{E}|\sum_{k=1}^{n}X_k/n^{1/2}|^2 = 1 \not\to 0$.

Theorem 1.6: *Let $\{a_{nk},\ n \geq 1,\ k \geq 1\}$ be an array of real constants with*

$$\sup_{n\geq 1}\sum_{k=1}^{\infty}|a_{nk}|^p < \infty \quad \textit{and} \quad \sup_{k\geq 1}|a_{nk}| \to 0,$$

let $\{X_n, \mathcal{F}_n,\ n \geq 1\}$ be a martingale difference sequence, and let $1 < p < 2$. Suppose that $\mathbb{B}$ is of martingale type p and $\{X_n,\ n \geq 1\}$ is $\{|a_{nk}|^p\}$-compactly uniformly pth order integrable. Then (2) *holds.*

In the rest of this paper, we denote by c a generic positive number which may be different at different places.

2. Proofs of the main results

The following lemmas play a very important role in the proofs of the main results.

Lemma 2.1: *Let K be a compact subset of $\mathbb{B}$ and let $\{X_n,\ n \geq 1\}$ be a sequence of $\mathbb{B}$-valued random elements taking values in K. Then for any $\varepsilon > 0$, there exist $m \geq 1, \{x_j, 1 \leq j \leq m\} \subset \mathbb{B}$, and $\{A_j, 1 \leq j \leq m\} \subset \mathcal{B}$ such that for any $n \geq 1$,*

$$\|X_n - Y_n\| < \varepsilon,$$

where $Y_n = \sum_{j=1}^m x_j I(X_n \in A_j)$ is X_n-measurable.

Proof: Since K is a compact subset of $\mathbb{B}$, we can get for any $\varepsilon > 0$ that there exists $\{x_j, 1 \leq j \leq m\} \subset \mathbb{B}$, such that $K \subset \bigcup_{j=1}^m B(x_j, \varepsilon)$, where $B(x_j, \varepsilon) = \{x : \|x - x_j\| < \varepsilon\}, 1 \leq j \leq m$. Set

$$A_1 = K \cap B(x_1, \varepsilon), \quad A_j = K \cap \left(B(x_j, \varepsilon) \Big\backslash \bigcup_{i=1}^{j-1} B(x_i, \varepsilon) \right), \quad 2 \leq j \leq m.$$

It is easy to verify that for any $n \geq 1$, $\|X_n - Y_n\| < \varepsilon$ and Y_n is X_n-measurable. ■

In the proofs of Lemmas 2.2 and 2.3 and Theorems 1.2 and 1.5, we apply the following well-known inequality which is referred to as the c_p-inequality (see, e.g. Loève [11], p. 157): For any $p > 0$ and any two random variables X and Y,

$$\mathbb{E}|X + Y|^p \leq c_p \left(\mathbb{E}|X|^p + \mathbb{E}|Y|^p \right)$$

where

$$c_p = \begin{cases} 1 & \text{if } p \leq 1 \\ 2^{p-1} & \text{if } p > 1. \end{cases}$$

Lemma 2.2: *Assume that $r > 0$, $\{Z_n,\ n \geq 1\}$ is a sequence of $\mathbb{B}$-valued random elements, $\{Z'_n,\ n \geq 1\}$ is an independent copy of $\{Z_n,\ n \geq 1\}$, and $Z_n \to 0$ in probability. Then $\mathbb{E}\|Z_n\|^r \to 0$ if and only if $\mathbb{E}\|Z_n - Z'_n\|^r \to 0$.*

Proof: It is obvious that $\mathbb{E}\|Z_n\|^r \to 0$ implies that $\mathbb{E}\|Z_n - Z'_n\|^r \to 0$ by the c_p-inequality. Now suppose that $\mathbb{E}\|Z_n - Z'_n\|^r \to 0$. For any $\varepsilon > 0$

$$\mathbb{E}\|Z_n\|^r = \int_0^\infty \mathbb{P}\{\|Z_n\| > t^{1/r}\}\,\mathrm{d}t \leq \varepsilon + \int_\varepsilon^\infty \mathbb{P}\{\|Z_n\| > t^{1/r}\}\,\mathrm{d}t.$$

Note that $\sup_{t \geq \varepsilon} \mathbb{P}\{\|Z_n\| > t^{1/r}/2\} = \mathbb{P}\{\|Z_n\| > \varepsilon^{1/r}/2\} \to 0$. Hence, by (6.1) of [10] we get that for n large enough,

$$\mathbb{E}\|Z_n\|^r \leq \varepsilon + 2\int_\varepsilon^\infty \mathbb{P}\{\|Z_n - Z'_n\| > t^{1/r}/2\}\,\mathrm{d}t \leq \varepsilon + 2^{r+1}\mathbb{E}\|Z_n - Z'_n\|^r,$$

which leads to the desired result by the arbitrariness of $\varepsilon > 0$. ■

Lemma 2.3: *Let $p > 0$, let $\{a_{nk},\ n \geq 1,\ k \geq 1\}$ be an array of real constants with $\sup_{n \geq 1} \sum_{k=1}^\infty |a_{nk}|^p < \infty$, and let $\{X_n,\ n \geq 1\}$ be a sequence of random elements. Suppose* *that $\{X'_n,\ n \geq 1\}$ is an independent copy of $\{X_n,\ n \geq 1\}$. If $\{X_n,\ n \geq 1\}$ is $\{|a_{nk}|^p\}$-uniformly pth order integrable, then $\{X_n - X'_n,\ n \geq 1\}$ is also $\{|a_{nk}|^p\}$-uniformly pth order integrable.*

Proof: By the c_p-inequality and Markov's inequality, for any $k \geq 1$ and $x > 0$

$$\begin{aligned}
\mathbb{E}\|X_k - X'_k\|^p I(\|X_k - X'_k\| > x) &\leq c\mathbb{E}\|X_k\|^p I(\|X_k - X'_k\| > x) \\
&= c\mathbb{E}\{\|X_k\|^p I(\|X_k - X'_k\| > x, \|X_k\| > x) \\
&\quad + \|X_k\|^p I(\|X_k - X'_k\| > x, \|X_k\| \leq x)\} \\
&\leq c\{\mathbb{E}\|X_k\|^p I(\|X_k\| > x) + x^p \mathbb{P}\{\|X_k - X'_k\| > x\}\} \\
&\leq c\mathbb{E}\|X_k\|^p I(\|X_k\| > x) + cx^p \mathbb{P}\{\|X_k\| > x/2\} \\
&\leq c\mathbb{E}\|X_k\|^p I(\|X_k\| > x) + c\mathbb{E}\|X_k\|^p I(\|X_k\| > x/2) \\
&\leq c\mathbb{E}\|X_k\|^p I(\|X_k\| > x/2).
\end{aligned}$$

Hence we complete the proof. ■

Lemma 2.4: *Let $\{X_n,\ n \geq 1\}$ be a sequence of independent and symmetric random elements and $p > 0$, $r > 0$. If $\sum_{k=1}^{\infty} a_{nk}X_k \to 0$ in probability, then*

$$\lim_{n\to\infty} \sup_{t\geq 1} \mathbb{E} \left\| t^{-1/r} \sum_{k=1}^{\infty} a_{nk}X_k I\left(\|a_{nk}X_k\| \leq t^{1/r}\right) \right\|^p = 0.$$

Proof: Using the contraction principle in the form of Lemma 6.5 of [10] we can derive that for any $\varepsilon > 0$, there exists a positive integer $n_0 = n_0(\varepsilon)$ such that for all $n \geq n_0$,

$$\begin{aligned}
\sup_{t\geq 1} \mathbb{P}\left\{\left\|\sum_{k=1}^{\infty} a_{nk}X_k I(\|a_{nk}X_k\| \leq t^{1/r})\right\| > t^{1/r}\varepsilon\right\} &\leq \sup_{t\geq 1} \mathbb{P}\left\{\left\|\sum_{k=1}^{\infty} a_{nk}X_k\right\| > t^{1/r}\varepsilon\right\} \\
&\leq \mathbb{P}\left\{\left\|\sum_{k=1}^{\infty} a_{nk}X_k\right\| > \varepsilon\right\} \leq \frac{1}{8\cdot 3^p}. \quad (4)
\end{aligned}$$

Clearly $\sup_{k\geq 1} \|a_{nk}X_k I(\|a_{nk}X_k\| \leq t^{1/r})\| \leq t^{1/r}$. When $n \geq n_0$, for all $t \geq 1$, by (6.7) of [10], (4), Lévy's inequality, and the contraction principle (see Proposition 2.3 and Lemma 6.5 of [10], respectively) we have for $A > 0$ that

$$\begin{aligned}
&\int_0^A \mathbb{P}\left\{\left\|\sum_{k=1}^{\infty} a_{nk}X_k I(\|a_{nk}X_k\| \leq t^{1/r})\right\| > t^{1/r}y^{1/p}\right\} \mathrm{d}y \\
&= 3^p \int_0^{A/3^p} \mathbb{P}\left\{\left\|\sum_{k=1}^{\infty} a_{nk}X_k I(\|a_{nk}X_k\| \leq t^{1/r})\right\| > 3t^{1/r}y^{1/p}\right\} \mathrm{d}y \\
&\leq 3^p \left(4\int_0^{A/3^p} \left[\mathbb{P}\left\{\left\|\sum_{k=1}^{\infty} a_{nk}X_k I(\|a_{nk}X_k\| \leq t^{1/r})\right\| > t^{1/r}y^{1/p}\right\}\right]^2 \mathrm{d}y\right.
\end{aligned}$$

$$+\int_0^{A/3^p} \mathbb{P}\left\{\sup_{k\geq 1} \|a_{nk}X_kI(\|a_{nk}X_k\| \leq t^{1/r})\| > t^{1/r}y^{1/p}\right\} \mathrm{d}y\Bigg)$$

$$\equiv 3^p(4J_1 + J_2), \quad \text{say.}$$

Note that

$$\begin{aligned}
J_1 &= \int_0^{A/3^p} \left[\mathbb{P}\left\{\left\|\sum_{k=1}^{\infty} a_{nk}X_kI(\|a_{nk}X_k\| \leq t^{1/r})\right\| > t^{1/r}y^{1/p}\right\}\right]^2 \mathrm{d}y \\
&= \left\{\int_0^{\varepsilon} + \int_{\varepsilon}^{A/3^p}\right\} \left[\mathbb{P}\left\{\left\|\sum_{k=1}^{\infty} a_{nk}X_kI(\|a_{nk}X_k\| \leq t^{1/r})\right\| > t^{1/r}y^{1/p}\right\}\right]^2 \mathrm{d}y \\
&\leq \varepsilon + \frac{1}{8\cdot 3^p}\int_{\varepsilon}^{A/3^p} \mathbb{P}\left\{\left\|\sum_{k=1}^{\infty} a_{nk}X_kI(\|a_{nk}X_k\| \leq t^{1/r})\right\| > t^{1/r}y^{1/p}\right\} \mathrm{d}y \quad \text{(by (4))} \\
&\leq \varepsilon + \frac{1}{8\cdot 3^p}\int_0^{A/3^p} \mathbb{P}\left\{\left\|\sum_{k=1}^{\infty} a_{nk}X_kI(\|a_{nk}X_k\| \leq t^{1/r})\right\| > t^{1/r}y^{1/p}\right\} \mathrm{d}y.
\end{aligned}$$

Also, since $\sup_{k\geq 1} \|a_{nk}X_kI(\|a_{nk}X_k\| \leq t^{1/r})\| \leq t^{1/r}$

$$\begin{aligned}
J_2 &= \int_0^{A/3^p} \mathbb{P}\left\{\sup_{k\geq 1} \|a_{nk}X_kI(\|a_{nk}X_k\| \leq t^{1/r})\| > t^{1/r}y^{1/p}\right\} \mathrm{d}y \\
&= \left\{\int_0^1 + \int_1^{A/3^p}\right\} \mathbb{P}\left\{\sup_{k\geq 1} \|a_{nk}X_kI(\|a_{nk}X_k\| \leq t^{1/r})\| > t^{1/r}y^{1/p}\right\} \mathrm{d}y \\
&= \int_0^1 \mathbb{P}\left\{\sup_{k\geq 1} \|a_{nk}X_kI(\|a_{nk}X_k\| \leq t^{1/r})\| > t^{1/r}y^{1/p}\right\} \mathrm{d}y.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_0^A \mathbb{P}\left\{\left\|\sum_{k=1}^{\infty} a_{nk}X_kI(\|a_{nk}X_k\| \leq t^{1/r})\right\| > t^{1/r}y^{1/p}\right\} \mathrm{d}y \\
&\quad\leq 3^p\Bigg(4\varepsilon + \frac{4}{8\cdot 3^p}\int_0^A \mathbb{P}\left\{\left\|\sum_{k=1}^{\infty} a_{nk}X_kI(\|a_{nk}X_k\| \leq t^{1/r})\right\| > t^{1/r}y^{1/p}\right\} \mathrm{d}y \\
&\qquad + \int_0^1 \mathbb{P}\left\{\sup_{k\geq 1} \|a_{nk}X_kI(\|a_{nk}X_k\| \leq t^{1/r})\| > t^{1/r}y^{1/p}\right\} \mathrm{d}y\Bigg) \\
&\quad\leq 3^p\Bigg(4\varepsilon + \frac{4}{8\cdot 3^p}\int_0^A \mathbb{P}\left\{\left\|\sum_{k=1}^{\infty} a_{nk}X_kI(\|a_{nk}X_k\| \leq t^{1/r})\right\| > t^{1/r}y^{1/p}\right\} \mathrm{d}y \\
&\qquad +2\int_0^1 \mathbb{P}\left\{\left\|\sum_{k=1}^{\infty} a_{nk}X_k\right\| > y^{1/p}\right\} \mathrm{d}y\Bigg).
\end{aligned}$$

Hence

$$\sup_{t\geq 1}\int_0^A \mathbb{P}\left\{\left\|\sum_{k=1}^{\infty} a_{nk}X_k I(\|a_{nk}X_k\|\leq t^{1/r})\right\| > t^{1/r}y^{1/p}\right\} \mathrm{d}y$$
$$\leq 8\cdot 3^p\varepsilon + 4\cdot 3^p\int_0^1 \mathbb{P}\left\{\left\|\sum_{k=1}^{\infty} a_{nk}X_k\right\| > y^{1/p}\right\} \mathrm{d}y.$$

Letting first $A\to\infty$ and then $n\to\infty$ we obtain the desired result by the Lebesgue dominated convergence theorem and the arbitrariness of $\varepsilon>0$. ∎

Proof of Theorem 1.1.: The sufficiency half is obvious. We only prove the necessity half. By the Mean Convergence Criterion (see, e.g. Theorem 3(i), Section 4.2 of [4]), it suffices to show that $\{\|\sum_{k=1}^{\infty} a_{nk}X_k\|^p,\ n\geq 1\}$ is uniformly integrable. According to the definition of uniform integrability given in [4] on pages 93–94, we need to prove that

(i) $\sup_{n\geq 1}\mathbb{E}\|\sum_{k=1}^{\infty} a_{nk}X_k\|^p<\infty$

and

(ii) For any $\varepsilon>0$ there exists $\delta>0$ such that for any $A\in\mathcal{F}$ such that $\mathbb{P}(A)<\delta$, we have that

$$\sup_{n\geq 1}\int_A \left\|\sum_{k=1}^{\infty} a_{nk}X_k\right\|^p \mathrm{d}\mathbb{P}<\varepsilon.$$

First suppose that $p>1$. Note that

$$\begin{aligned}\left\|\sum_{k=1}^{\infty} a_{nk}X_k\right\| &\leq \sum_{k=1}^{\infty}|a_{nk}|\|X_k\|\\ &=\sum_{k=1}^{\infty}|a_{nk}|^{1/p}\|X_k\||a_{nk}|^{\frac{p-1}{p}}\\ &\leq\left(\sum_{k=1}^{\infty}|a_{nk}|\|X_k\|^p\right)^{1/p}\left(\sum_{k=1}^{\infty}|a_{nk}|\right)^{\frac{p-1}{p}} \qquad \text{(by Hölder's inequality).}\end{aligned}$$

Thus

$$\left\|\sum_{k=1}^{\infty} a_{nk}X_k\right\|^p\leq\left(\sum_{k=1}^{\infty}|a_{nk}|\right)^{p-1}\left(\sum_{k=1}^{\infty}|a_{nk}|\|X_k\|^p\right)$$

and so for any $A\in\mathcal{F}$,

$$\sup_{n\geq 1}\int_A\left\|\sum_{k=1}^{\infty} a_{nk}X_k\right\|^p\mathrm{d}\mathbb{P}\leq\sup_{n\geq 1}\left(\sum_{k=1}^{\infty}|a_{nk}|\right)^{p-1}\sum_{k=1}^{\infty}|a_{nk}|\int_A\|X_k\|^p\,\mathrm{d}\mathbb{P}. \tag{5}$$

Note that (5) holds trivially when $p=1$. Let

$$M=\sup_{n\geq 1}\sum_{k=1}^{\infty}|a_{nk}|.$$

Thus for all $p \geq 1$,

$$\sup_{n\geq 1} \int_A \left\| \sum_{k=1}^{\infty} a_{nk} X_k \right\|^p \mathrm{d}\mathbb{P} \leq M^{p-1} \sup_{n\geq 1} \sum_{k=1}^{\infty} |a_{nk}| \int_A \|X_k\|^p \,\mathrm{d}\mathbb{P}. \tag{6}$$

We first prove (i). Since $\{X_n\}$ is $\{a_{nk}\}$-uniformly pth order integrable, there exists $x_0 > 0$ such that

$$\sup_{n\geq 1} \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}\|X_k\|^p I(\|X_k\| > x_0) \leq 1.$$

Then taking $A = \Omega$ in (6),

$$\begin{aligned} \sup_{n\geq 1} \mathbb{E} \left\| \sum_{k=1}^{\infty} a_{nk} X_k \right\|^p &\leq M^{p-1} \sup_{n\geq 1} \left(\sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}\|X_k\|^p I(\|X_k\| \leq x_0) \right. \\ &\quad \left. + \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}\|X_k\|^p I(\|X_k\| > x_0) \right) \\ &\leq M^p x_0^p + 1 < \infty \end{aligned}$$

thereby proving (i).

We now prove (ii). According to Theorem 2 of [12], the assumption that $\{X_n,\ n \geq 1\}$ is $\{a_{nk}\}$-uniformly pth order integrable is equivalent to the pair of conditions

(a) $\sup_{n\geq 1} \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}\|X_k\|^p < \infty$

and

(b) For any $\varepsilon > 0$, there exists $\delta_0 > 0$ such that if $\{A_k,\ k \geq 1\}$ is any sequence in $\mathcal{F}$ satisfying

$$\sup_{n\geq 1} \sum_{k=1}^{\infty} |a_{nk}| \mathbb{P}(A_k) < \delta_0,$$

then

$$\sup_{n\geq 1} \sum_{k=1}^{\infty} |a_{nk}| \int_{A_k} \|X_k\|^p \,\mathrm{d}\mathbb{P} < \frac{\varepsilon}{M^{p-1}}.$$

Let $\varepsilon > 0$ and let $\delta_0 > 0$ be as in (b). Let $\delta = \frac{\delta_0}{M}$, let $A \in \mathcal{F}$ be such that $\mathbb{P}(A) < \delta$, and let $A_k = A,\ k \geq 1$. Then

$$\sup_{n\geq 1} \sum_{k=1}^{\infty} |a_{nk}| \mathbb{P}(A) < M \frac{\delta_0}{M} = \delta_0$$

and so by (6) and (b),

$$\sup_{n\geq 1} \int_A \left\| \sum_{k=1}^{\infty} a_{nk} X_k \right\|^p \mathrm{d}\mathbb{P} < M^{p-1} \frac{\varepsilon}{M^{p-1}} = \varepsilon$$

thereby proving (ii). Hence $\{\| \sum_{k=1}^{\infty} a_{nk} X_k \|^p,\ n \geq 1\}$ is uniformly integrable and the proof is completed. ■

Remark 2.1: Part (ii) of Theorem 1.1 can also be proved without referring to Theorem 2 of [12]. In fact, since $\{X_n,\ n \geq 1\}$ is $\{a_{nk}\}$-uniformly pth order integrable, there exists $B > 0$ such that

$$\sup_{n\geq 1}\sum_{k=1}^{\infty}|a_{nk}|\mathbb{E}\|X_k\|^p I(\|X_k\| > B) < \frac{\varepsilon}{2M^{p-1}}.$$

Let $\delta = \varepsilon/(2M^pB^p)$. Then for any $A \in \mathcal{F}$ with $\mathbb{P}(A) < \delta$, it follows from (6) that

$$\begin{aligned}
&\sup_{n\geq 1}\int_A \left\|\sum_{k=1}^{\infty}a_{nk}X_k\right\|^p \mathrm{d}\mathbb{P} \\
&\leq M^{p-1}\sup_{n\geq 1}\left(\sum_{k=1}^{\infty}|a_{nk}|\int_{A[\|X_k\|\leq B]}\|X_k\|^p\,\mathrm{d}\mathbb{P} + \sum_{k=1}^{\infty}|a_{nk}|\int_{A[\|X_k\|>B]}\|X_k\|^p\,\mathrm{d}\mathbb{P}\right) \\
&\leq M^{p-1}\sup_{n\geq 1}\sum_{k=1}^{\infty}|a_{nk}|\int_{A[\|X_k\|\leq B]}\|X_k\|^p\,\mathrm{d}\mathbb{P} \\
&\quad + M^{p-1}\sup_{n\geq 1}\sum_{k=1}^{\infty}|a_{nk}|\int_{A[\|X_k\|>B]}\|X_k\|^p\,\mathrm{d}\mathbb{P} \\
&\leq M^pB^p\mathbb{P}(A) + M^{p-1}\sup_{n\geq 1}\sum_{k=1}^{\infty}|a_{nk}|\mathbb{E}\|X_k\|^pI(\|X_k\| > B) \\
&< M^pB^p\frac{\varepsilon}{2M^pB^p} + M^{p-1}\frac{\varepsilon}{2M^{p-1}} = \varepsilon
\end{aligned}$$

thereby proving (ii).

Proof of Theorem 1.2.: For any $\varepsilon > 0$, since $\{X_n,\ n \geq 1\}$ is $\{a_{nk}\}$-compactly uniformly pth order integrable, there exists a compact subset K of $\mathbb{B}$ such that

$$\sup_{n\geq 1}\sum_{k=1}^{\infty}|a_{nk}|\mathbb{E}\|X_k\|^pI(X_k \notin K) < \varepsilon.$$

It is obvious that $X_nI(X_n \in K)$ takes values in $K \cup \{0\}$ for all $n \geq 1$. Hence by Lemma 2.1 there exists a sequence of pairwise independent $\mathbb{B}$-valued random elements $\{Y_n = \sum_{j=1}^{m} x_jI(X_n \in A_j),\ n \geq 1\}$ such that

$$\sup_{n\geq 1}\|X_nI(X_n \in K) - Y_n\| \leq \varepsilon,$$

where $\{x_j, 1 \leq j \leq m\}$ is a finite subset of $\mathbb{B}$.

Set $\Gamma = \sup_{n\geq 1}\sum_{k=1}^{\infty}|a_{nk}|$, $\Gamma_1 = \max_{1\leq j\leq m}\|x_j\|$. Note that

$$\begin{aligned}
\left\|\sum_{k=1}^{\infty}a_{nk}(X_k - \mathbb{E}X_k)\right\| = \Bigg\|&\sum_{k=1}^{\infty}a_{nk}(X_kI(X_k \notin K) - \mathbb{E}X_kI(X_k \notin K)) \\
&+ \sum_{k=1}^{\infty}a_{nk}(X_kI(X_k \in K) - \mathbb{E}X_kI(X_k \in K))
\end{aligned}$$

$$
\begin{aligned}
&\quad - \sum_{k=1}^{\infty} a_{nk}(Y_k - \mathbb{E}Y_k)] + \sum_{k=1}^{\infty} a_{nk}(Y_k - \mathbb{E}Y_k)\Bigg\| \\
&\leq \left\| \sum_{k=1}^{\infty} a_{nk}(X_k I(X_k \notin K) - \mathbb{E}X_k I(X_k \notin K)) \right\| \\
&\quad + 2\Gamma\varepsilon + \left\| \sum_{k=1}^{\infty} a_{nk}(Y_k - \mathbb{E}Y_k) \right\|.
\end{aligned}
$$

Hence, by Hölder's inequality, Jensen's inequality, and the c_p-inequality, for all $n \geq 1$

$$
\begin{aligned}
&\mathbb{E}\left\| \sum_{k=1}^{\infty} a_{nk}(X_k I(X_k \notin K) - \mathbb{E}X_k I(X_k \notin K)) \right\|^p \\
&\leq \Gamma^{p-1} \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}\|X_k I(X_k \notin K) - \mathbb{E}X_k I(X_k \notin K)\|^p \\
&\leq 2^{p-1}\Gamma^{p-1} \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}\|X_k\|^p I(X_k \notin K) \\
&< 2^{p-1}\Gamma^{p-1}\varepsilon.
\end{aligned}
$$

By employing Jensen's inequality and the c_p-inequality again we obtain

$$
\begin{aligned}
&\mathbb{E}\left\| \sum_{k=1}^{\infty} a_{nk}(Y_k - \mathbb{E}Y_k) \right\|^p \\
&= \mathbb{E}\left\| \sum_{j=1}^{m} \sum_{k=1}^{\infty} a_{nk} x_j (I(X_k \in A_j) - \mathbb{E}I(X_k \in A_j)) \right\|^p \\
&\leq (2m\Gamma\Gamma_1)^{p-1} \sum_{j=1}^{m} \|x_j\| \mathbb{E}\left| \sum_{k=1}^{\infty} a_{nk}(I(X_k \in A_j) - \mathbb{E}I(X_k \in A_j)) \right| \\
&\leq (2m\Gamma\Gamma_1)^{p-1} \sum_{j=1}^{m} \|x_j\| \left(\mathbb{E}\left| \sum_{k=1}^{\infty} a_{nk}(I(X_k \in A_j) - \mathbb{E}I(X_k \in A_j)) \right|^2 \right)^{1/2}.
\end{aligned}
$$

By the pairwise independence of the random elements $\{X_k,\ k \geq 1\}$, we obtain that for any $1 \leq j \leq m$, the random variables $\{I(X_k \in A_j) - \mathbb{E}I(X_k \in A_j),\ k \geq 1\}$ are also pairwise independent and hence

$$
\begin{aligned}
&(2m\Gamma\Gamma_1)^{p-1} \sum_{j=1}^{m} \|x_j\| \left(\mathbb{E}\left| \sum_{k=1}^{\infty} a_{nk}(I(X_k \in A_j) - \mathbb{E}I(X_k \in A_j)) \right|^2 \right)^{1/2} \\
&\quad \leq (2m\Gamma\Gamma_1)^{p-1} \sum_{j=1}^{m} \|x_j\| \sum_{k=1}^{\infty} |a_{nk}|^2 \left(\mathbb{E}\left| (I(X_k \in A_j) - \mathbb{E}I(X_k \in A_j)) \right|^2 \right)^{1/2}
\end{aligned}
$$

$$\leq (2m\Gamma\Gamma_1)^{p-1} \sum_{j=1}^{m} \|x_j\| \left(\sum_{k=1}^{\infty} |a_{nk}|^2 \right)^{1/2}$$

$$\leq 2^{p-1} (m\Gamma_1)^p \Gamma^{p-1/2} \left(\sup_{k\geq 1} |a_{nk}| \right)^{1/2} \to 0 \quad \text{as } n \to \infty.$$

By the arbitrariness of $\varepsilon > 0$, we get that (3) holds. ■

Remark 2.2: Note that we cannot remove the assumption of the pairwise independence from the formulation of Theorem 1.2. In order to see this, consider the following simple example. Take $p = 1$, $a_{nk} = \frac{1}{n}$, if $1 \leq k \leq n$ and $a_{nk} = 0$, if $k > n$. Let $X_1 = X_2 = \ldots$ where X_1 has the uniform distribution on $[0, 1]$. Then

$$\mathbb{E} \left| \frac{1}{n} \sum_{k=1}^{n} (X_k - \mathbb{E}X_k) \right| = \mathbb{E}|X_1 - \mathbb{E}X_1| = \frac{1}{4} \nrightarrow 0.$$

Proof of Theorem 1.3.: For any $\varepsilon > 0$, since $\{X_n,\ n \geq 1\}$ is $\{a_{nk}\}$-compactly uniformly pth order integrable, there exists a compact subset K of $\mathbb{B}$ such that

$$\sup_{n\geq 1} \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}\|X_k\|^p I(X_k \notin K) < \varepsilon.$$

It is obvious that $X_n I(X_n \in K)$ takes values in $K \cup \{0\}$ for all $n \geq 1$. Hence we get from Lemma 2.1 that there exists a sequence of $\mathbb{B}$-valued random elements $\{Y_n = \sum_{j=1}^{m} x_j I(X_n \in A_j),\ n \geq 1\}$ such that

$$\sup_{n\geq 1} \|X_n I(X_n \in K) - Y_n\| \leq \varepsilon,$$

where $\{x_j, 1 \leq j \leq m\}$ is finite subset of $\mathbb{B}$.

Set $\Gamma = \sup_{n\geq 1} \sum_{k=1}^{\infty} |a_{nk}|$. We split $\sum_{k=1}^{\infty} a_{nk} X_k$ by

$$\begin{aligned}
\left\| \sum_{k=1}^{\infty} a_{nk} X_k \right\| &= \left\| \sum_{k=1}^{\infty} a_{nk} (X_k I(X_k \notin K) - \mathbb{E}(X_k I(X_k \notin K) \mid \mathcal{F}_{n-1})) \right. \\
&\quad + \sum_{k=1}^{\infty} a_{nk} (X_k I(X_k \in K) - \mathbb{E}(X_k I(X_k \in K) \mid \mathcal{F}_{n-1})) \\
&\quad \left. - \sum_{k=1}^{\infty} a_{nk} (Y_k - \mathbb{E}(Y_k \mid \mathcal{F}_{n-1})) + \sum_{k=1}^{\infty} a_{nk} (Y_k - \mathbb{E}(Y_k \mid \mathcal{F}_{n-1})) \right\| \\
&\leq \left\| \sum_{k=1}^{\infty} a_{nk} (X_k I(X_k \notin K) - \mathbb{E}(X_k I(X_k \notin K) \mid \mathcal{F}_{n-1})) \right\| \\
&\quad + 2\Gamma\varepsilon + \left\| \sum_{k=1}^{\infty} a_{nk} (Y_k - \mathbb{E}(Y_k \mid \mathcal{F}_{n-1})) \right\|,
\end{aligned}$$

where $\mathcal{F}_0 = \{\Omega, \emptyset\}$. The rest of the proof is similar to that of Theorem 1.2 and hence we omit the details. ■

Proof of Theorem 1.4.: We only need to show that $\sum_{k=1}^{\infty} a_{nk}X_k \to 0$ in probability implies (2). By Lemmas 2.2 and 2.3, without loss of generality, we can assume that $\{X_n,\ n \geq 1\}$ is a sequence of symmetric random elements. Note that

$$\begin{aligned}
\mathbb{E}\left\|\sum_{k=1}^{\infty} a_{nk}X_k\right\|^p &= \int_0^{\infty} \mathbb{P}\left\{\left\|\sum_{k=1}^{\infty} a_{nk}X_k\right\| > t^{1/p}\right\} \mathrm{d}t \\
&\leq \int_0^{1} \mathbb{P}\left\{\left\|\sum_{k=1}^{\infty} a_{nk}X_k\right\| > t^{1/p}\right\} \mathrm{d}t + \int_1^{\infty} \mathbb{P}\left\{\left\|\sum_{k=1}^{\infty} a_{nk}X_k\right\| > t^{1/p}\right\} \mathrm{d}t \\
&= I_1 + I_2, \quad \text{say.}
\end{aligned}$$

By the Lebesgue dominated convergence theorem, $I_1 \to 0$. Hence it is enough to prove that $I_2 \to 0$. We have

$$\begin{aligned}
I_2 &\leq \int_1^{\infty} \sum_{k=1}^{\infty} \mathbb{P}\{\|a_{nk}X_k\| > t^{1/p}\} \mathrm{d}t + \int_1^{\infty} \mathbb{P}\left\{\left\|\sum_{k=1}^{\infty} a_{nk}X_k I\left(\|a_{nk}X_k\| \leq t^{1/p}\right)\right\| > t^{1/p}\right\} \mathrm{d}t \\
&= I_3 + I_4, \quad \text{say.}
\end{aligned}$$

For I_3,

$$\begin{aligned}
I_3 &= \sum_{k=1}^{\infty} \int_1^{\infty} \mathbb{P}\{\|a_{nk}X_k\| > t^{1/p}\} \mathrm{d}t \\
&= \sum_{k=1}^{\infty} \int_1^{\infty} \mathbb{P}\{\|a_{nk}X_k\| I(\|a_{nk}X_k\| > 1) > t^{1/p}\} \mathrm{d}t \\
&\leq \sum_{k=1}^{\infty} \int_0^{\infty} \mathbb{P}\{\|a_{nk}X_k\| I(\|a_{nk}X_k\| > 1) > t^{1/p}\} \mathrm{d}t \\
&= \sum_{k=1}^{\infty} |a_{nk}|^p \mathbb{E}\|X_k\|^p I(\|a_{nk}X_k\| > 1) \\
&\leq \sup_{m\geq 1} \sum_{k=1}^{\infty} |a_{mk}|^p \mathbb{E}\|X_k\|^p I\left(\|X_k\| > \left(\sup_{k\geq 1} |a_{nk}|\right)^{-1}\right) \\
&\to 0.
\end{aligned}$$

For I_4, by Lemma 2.4, there exists $N \geq 1$ such that for all $n \geq N$ for all $t \geq 1$ we have that $\mathbb{E}\|t^{-1/p} \sum_{k=1}^{\infty} a_{nk}X_k I(\|a_{nk}X_k\| \leq t^{1/p})\|^p \leq 1/2$. Hence,

$$I_4 = \int_1^{\infty} \mathbb{P}\left\{\left\|\sum_{k=1}^{\infty} a_{nk}X_k I\left(\|a_{nk}X_k\| \leq t^{1/p}\right)\right\| > t^{1/p}\right\} \mathrm{d}t$$

$$\leq \int_1^\infty \mathbb{P}\left\{\left\|\sum_{k=1}^\infty a_{nk}X_kI(\|a_{nk}X_k\| \leq t^{1/p})\right\|\right.$$

$$\left. -\mathbb{E}\left\|\sum_{k=1}^\infty a_{nk}X_kI(\|a_{nk}X_k\| \leq t^{1/p})\right\| > t^{1/p}/2\right\} \mathrm{d}t$$

$$\leq c\sum_{k=1}^\infty \int_1^\infty t^{-2/p}\int_0^{t^{2/p}} \mathbb{P}\{\|a_{nk}X_k\| > s^{1/2}\}\,\mathrm{d}s\,\mathrm{d}t \quad \text{(by Theorem 2.1 of [1])}$$

$$= c\sum_{k=1}^\infty \int_1^\infty \mathbb{P}\{\|a_{nk}X_k\| > s^{1/2}\}\int_{s^{p/2}}^\infty t^{-2/p}\,\mathrm{d}t\,\mathrm{d}s$$

$$= c\int_1^\infty s^{(1-2/p)p/2}\sum_{k=1}^\infty \mathbb{P}\{\|a_{nk}X_k\| > s^{1/2}\}\,\mathrm{d}s$$

$$= c\int_1^\infty s^{p/2-1}\sum_{k=1}^\infty \mathbb{P}\{\|a_{nk}X_k\| > s^{1/2}\}\,\mathrm{d}s \quad (s = t^{2/p})$$

$$= c\int_1^\infty \sum_{k=1}^\infty \mathbb{P}\{\|a_{nk}X_k\| > t^{1/p}\}\,\mathrm{d}t$$

$$\to 0.$$

Hence we complete the proof. ∎

Proof of Theorem 1.5: We first prove that (1) implies (2). For any $\varepsilon > 0$, since $\{X_n,\ n \geq 1\}$ is $\{|a_{nk}|^p\}$-compactly uniformly pth order integrable, we get that there exists a compact subset K of $\mathbb{B}$ such that

$$\sup_{n\geq 1}\sum_{k=1}^\infty |a_{nk}|^p\mathbb{E}\|X_k\|^pI(X_k \notin K) < \varepsilon.$$

It is obvious that $X_nI(X_n \in K)$ takes values in $K \cup \{0\}$ for all $n \geq 1$. For any $\varepsilon > 0$, we get from Lemma 2.1 that there exists a sequence of independent $\mathbb{B}$-valued random elements $\{Y_n = \sum_{j=1}^m x_jI(X_n \in A_j),\ n \geq 1\}$ such that

$$\sup_{n\geq 1}\|X_nI(X_n \in K) - Y_n\| \leq \varepsilon,$$

where $\{x_j, 1 \leq j \leq m\}$ is a finite subset of $\mathbb{B}$. Note that by the c_p-inequality and the Hoffmann-Jørgensen and Pisier [7] characterization of Rademacher type p Banach spaces discussed above,

$$\mathbb{E}\left\|\sum_{k=1}^\infty a_{nk}(X_k - \mathbb{E}X_k)\right\|^p$$

$$= \mathbb{E}\left\|\sum_{k=1}^\infty a_{nk}(X_kI(X_k \notin K) - \mathbb{E}X_kI(X_k \notin K))\right.$$

$$
\begin{aligned}
&+\left[\sum_{k=1}^{\infty} a_{nk}(X_k I(X_k \in K) - \mathbb{E}X_k I(X_k \in K)) - \sum_{k=1}^{\infty} a_{nk}(Y_k - \mathbb{E}Y_k)\right] \\
&+\sum_{k=1}^{\infty} a_{nk}(Y_k - \mathbb{E}Y_k)\Bigg\|^p \\
\leq\ & c\sum_{k=1}^{\infty} |a_{nk}|^p \mathbb{E}\|X_k\|^p I(X_n \notin K) + c\varepsilon^p \sum_{k=1}^{\infty} |a_{nk}|^p + c\mathbb{E}\left\|\sum_{k=1}^{\infty} a_{nk}(Y_k - \mathbb{E}Y_k)\right\|^p.
\end{aligned}
$$

Hence, by Jensen's inequality and the c_p-inequality we get

$$
\begin{aligned}
\mathbb{E}\left\|\sum_{k=1}^{\infty} a_{nk}(Y_k - \mathbb{E}Y_k)\right\|^p &= \mathbb{E}\left\|\sum_{j=1}^{m} x_j \sum_{k=1}^{\infty} a_{nk}(I(X_i \in A_j) - \mathbb{E}I(X_i \in A_j))\right\|^p \\
&\leq c\sum_{j=1}^{m} \|x_j\|^p \left(\mathbb{E}\left|\sum_{k=1}^{\infty} a_{nk}(I(X_i \in A_j) - \mathbb{E}I(X_i \in A_j))\right|^2\right)^{p/2} \\
&\leq c\left(\sup_{k\geq 1}|a_{nk}|\right)^{p-p^2/2} \to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary, we have that (2) holds.

Next we show that (2) implies (1). Set $a_{nk} = n^{-1/p}$ if $1 \leq k \leq n$ and $a_{nk} = 0$ if $k > n$ and $\{X, X_n,\ n \geq 1\}$ is a sequence of i.i.d. random elements with $\mathbb{E}X = 0$ and $\mathbb{E}\|X\|^p < \infty$. Note that this choice of $\{a_{nk}\}$ and $\{X_n,\ n \geq 1\}$ satisfies the assumptions of (2) and hence (3) holds. Then $\mathbb{E}\|n^{-1/p}\sum_{k=1}^{n} X_k\| \to 0$ and thus $n^{-1/p}\sum_{k=1}^{n} X_k \to 0$ in probability. Hence by Theorem 3.1 of [1], $n^{-1/p}\sum_{k=1}^{n} X_k \to 0$ almost surely and then by Theorem 4.1 of [1], we get that (2) implies (1). This completes the proof of the theorem. ■

Proof Theorem 1.6.: It is similar to the proofs of Theorems 1.3 and 1.5 and hence we omit it. ■

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