Lobachevskii Journal of Mathematics
http://ljm.ksu.ru
ISSN 1818-9962
Vol. 26, 2007, 17–25

Kamon Budsaba, Pingyan Chen, Andrei Volodin

LIMITING BEHAVIOUR OF MOVING AVERAGE PROCESSES BASED ON A SEQUENCE OF ρ^- MIXING AND NEGATIVELY ASSOCIATED RANDOM VARIABLES

(submitted by D. Kh. Mushtari)

Abstract. Let $\{Y_i, -\infty < i < \infty\}$ be a doubly infinite sequence of identically distributed ρ^--mixing or negatively associated random variables, $\{a_i, -\infty < i < \infty\}$ a sequence of real numbers. In this paper, we prove the rate of convergence and strong law of large numbers for the partial sums of moving average processes $\{\sum_{i=-\infty}^{\infty} a_i Y_{i+n}, n \geq 1\}$ under some moment conditions.

1. Preliminaries

Let $\{Y_i, -\infty < i < +\infty\}$ be a doubly infinite sequence of identically distributed random variables and $\{a_i, -\infty < i < +\infty\}$ be an absolutely summable sequence of real numbers. Next, let

$$X_n = \sum_{i=-\infty}^{\infty} a_i Y_{i+n}, n \geq 1$$

2000 Mathematical Subject Classification. 60F15.
Key words and phrases. Moving average process, Kolmogorov and Marcinkiewicz-Zygmund strong law of large numbers, Rate of complete convergence, ρ^--mixing, ρ^*-mixing, Negative association.
The work of A. Volodin is supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

be the *moving average process based on the sequence* $\{Y_i, -\infty < i < +\infty\}$. As usual, we denote $S_n = \sum_{k=1}^{n} X_k,\ n \geq 1$, the sequence of partial sums.

Under the assumption that $\{Y_i, -\infty < i < +\infty\}$ is a sequence of independent identically distributed random variables, many limiting results have been obtained for the moving average process $\{X_n, n \geq 1\}$. For example, Ibragimov (1962) established the central limit theorem, Burton and Dehling (1990) obtained a large deviation principle, and Li, Rao, and Wang (1992) obtained the complete convergence result for $\{X_n, n \geq 1\}$.

Certainly, even if $\{Y_i, -\infty < i < +\infty\}$ is the sequence of independent identically distributed random variables, the moving average random variables $\{X_n, n \geq 1\}$ are dependent. This kind of dependence is called *weak dependence*. The partial sums of weakly dependent random variables $\{X_n, n \geq 1\}$ have similar limiting behaviour properties in comparison with the limiting properties of independent identically distributed random variables.

Very few results for a moving average process based on a dependent sequence are known. In this paper, we provide two results on the limiting behaviour of a moving average process based on a ρ^--mixing and negatively associated sequences.

Let $\{Y_i, -\infty < i < \infty\}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. For a set of integer numbers T denote σ-algebra $\mathcal{F}(T) = \sigma(Y_i, i \in T)$ and as usual, for a σ-algebra $\mathcal{F}$ we denote by $\mathcal{L}^2(\mathcal{F})$ the class of all $\mathcal{F}$-measurable random variables with the finite second moment.

For two sets S and T of real numbers we denote

$$\operatorname{dist}(S, T) = \inf\{|s - t|, s \in S, t \in T\}.$$

The following definition was introduced in Zang and Wang (1999). A sequence of random variables $\{Y_i, -\infty < i < \infty\}$ is called ρ^--mixing if

$$\rho^-(s) = \sup\{\rho(S, T); S, T \text{ are sets of integers }, \operatorname{dist}(S, T) \geq s\} \to 0$$

as $s \to \infty$, where

$$\rho^-(S, T) = \max\left\{0, \sup\left(\operatorname{Corr}\left[f(Y_i, i \in S), g(Y_j, j \in T)\right]\right)\right\},$$

where supremum is taken over all coordinatewise increasing real functions f on R^S and g on R^T and by $\operatorname{Corr}(\cdot, \cdot)$ we denote the classical correlation coefficient.

Next, a sequence of random variables $\{Y_i, -\infty < i < \infty\}$ is called ρ^*-mixing if for some integer $s \geq 1$

$$\rho(s)^* = \sup \sup\{\mathrm{Corr}(X, Y) : X \in \mathcal{L}^2(\mathcal{F}_S), Y \in \mathcal{L}^2(\mathcal{F}_T)\} < 1,$$

where the first sup is taken over all pairs of nonempty finite sets S, T of integers, such that $\mathrm{dist}(S, T) \geq s$. The notion of ρ^*-mixing seems to be similar to the notion of ρ-mixing, but Bryc and Smolenski (1993) showed that they are quite different from each other.

Recall that a finite family of random variables $\{Y_i, 1 \leq i \leq n\}$ is said to be *negatively associated*, if for any disjoint finite subsets S and T of integers and any real coordinatewise nondecreasing functions f on R^S and g on R^T,

$$\mathrm{Cov}\left(f(Y_i, i \in S), g(Y_j, j \in T)\right) \leq 0$$

whenever the covariance exists. This concept was studied in Joag-Dev and Proschan (1983).

It is easy to see that $\{Y_i, -\infty < i < \infty\}$ is negatively associated if and only if $\rho^-(s) = 0$ for all $s \geq 1$ and $\rho^-(s) \leq \rho^*(s)$. Hence the notion of ρ^--mixing is weaker than both notions of negative association and ρ^*-mixing.

We also need the following simple statement (see Property P2 in Wang and Lu (2006))

Property of ρ^--mixing random variables. *Let $\{Y_n, n \geq 1\}$ be a sequence of ρ^--mixing random variables. If $\{f_n, n \geq 1\}$ is a sequence of real functions all of which are monotone nondecreasing (or all monotone nonincreasing), then $\{f_n(Y_n), n \geq 1\}$ is a sequence of ρ^--mixing random variables.*

Note that Property P2 in Wang and Lu (2006) is stated only for increasing functions. It is simple to see that this property remains true for nondecreasing functions, too. The statement for nonincreasing functions follows for the observation that if a function f_n is nonincreasing, then the function $-f_n$ is nondecreasing.

Recall that a measurable function l is said to be *slowly varying* if for each $\lambda > 0$

$$\lim_{x\to\infty} \frac{l(\lambda x)}{l(x)} = 1.$$

We refer to Seneta (1976) for other equivalent definitions and for detailed and comprehensive study of properties of such functions.

In the following, C will represent a positive constants although its value may change from one appearance to the next.

The following result was proved in Budsaba, Chen, and Volodin (2007).

Theorem BCV. *Let $l(x)$ be a positive slowly varying function and $1 \le p < 2, r \ge 1, pr \ne 1$. Suppose $\{Y_i, -\infty < i < \infty\}$ is a sequence of identically distributed and ρ^--mixing random variables and $\{X_n, n \ge 1\}$ is the moving average process based on the sequence $\{Y_i, -\infty < i < \infty\}$. Then $EY_1 = 0$ and $E|Y_1|^{rp} l(|Y_1|^p) < \infty$ imply that for all $\varepsilon > 0$*

$$\sum_{n=1}^{\infty} n^{r-2} l(n) P\left(\max_{k \le n} \left|\sum_{j=1}^{k} X_j\right| \ge \varepsilon n^{1/p}\right) < \infty.$$

In particular, the assumptions $EY_1 = 0$ and $E|Y_1|^p < \infty, 1 < p < 2$ imply Marcinkiewicz-Zygmund strong law of large numbers

$$n^{-1/p} \sum_{k=1}^{n} X_k \to 0 \text{ a.s. as } n \to \infty.$$

Next, in Concluding Remark 4, Budsaba, Chen, and Volodin (2007) mentioned that the case $p = r = 1$ is not treated in Theorem BCV and that the authors believe that the result can be proved under the additional assumption $\sum_{i=-\infty}^{\infty} |a_i|^s < \infty$ for some $0 < s < 1$.

In this paper it is shown that this suggestion is true. Namely, we prove the following result.

Theorem 1. *Let $l(x)$ be a positive slowly varying function and suppose that $\{Y_i, -\infty < i < \infty\}$ is a sequence of identically distributed ρ^--mixing random variables. Let $\{X_n, n \ge 1\}$ be the moving average process based on the sequence $\{Y_i, -\infty < i < \infty\}$. Let moreover $\{a_i, -\infty < i < \infty\}$ be a sequence of real numbers with $\sum_{i=-\infty}^{\infty} |a_i|^s < \infty$ for some $0 < s < 1$. Then $EY_1 = 0$ and $E|Y_1| l(|Y_1|) < \infty$ imply that for all $\varepsilon > 0$*

$$\sum_{n=1}^{\infty} n^{-1} l(n) P\{\max_{1 \le k \le n} |\sum_{j=1}^{k} X_j| > \varepsilon n\} < \infty.$$

In particular, $EY_1 = 0$ implies the following Kolmogorov strong law of large numbers

$$n^{-1} \sum_{k=1}^{n} X_k \to 0 \text{ a.s. as } n \to \infty.$$

Proof. Let $Y_{nj}^{(1)} = -nI(Y_j < -n) + Y_i I(|Y_j| \le n) + nI(Y_j > n)$, and $Y_{nj}^{(2)} = Y_j - Y_{nj}^{(1)}$. Then by Property of ρ^--mixing random variables, for any $n \geq 1$, $\{Y_{nj}^{(1)}, -\infty < j < \infty\}$ and $\{Y_{nj}^{(2)}, -\infty < j < \infty\}$ are sequences of ρ^--mixing random variables. Note that

$$\sum_{k=1}^{n} X_k = \sum_{k=1}^{n} \sum_{i=-\infty}^{\infty} a_i Y_{i+k} = \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_j$$

and

$$\begin{aligned} & n^{-1} \max_{1\leq k\leq n} |E \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_{nj}^{(1)}| \leq n^{-1} \sum_{i=-\infty}^{\infty} |a_i| \max_{1\leq k\leq n} |E \sum_{j=i+1}^{i+k} Y_{nj}^{(1)}| \\ & \leq Cn^{-1} n(|EY_1 I(|Y_1| \leq n)| + nP\{|Y_1| > n\}) \\ & = CE|Y_1|I(|Y_1| > n) + CnP\{|Y_1| > n\} \to 0, \end{aligned}$$

as $n \to \infty$. Hence for n large enough

$$n^{-1} \max_{1\leq k\leq n} |\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_{nj}^{(1)}| < \varepsilon/4.$$

Therefore it is enough to prove that

$$I_1 := \sum_{n=1}^{\infty} n^{-1} l(n) P\{\max_{1\leq k\leq n} |\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} (Y_{nj}^{(1)} - EY_{nj}^{(1)})| > \varepsilon n/4\} < \infty$$

and

$$I_2 := \sum_{n=1}^{\infty} n^{-1} l(n) P(\max_{1\leq k\leq n} |\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_{nj}^{(2)}| > \varepsilon n/2) < \infty.$$

For I_2, by Markov inequality, we have

$$\begin{aligned} I_2 \quad & \leq C \sum_{n=1}^{\infty} n^{-1-s} l(n) E \max_{1\leq k\leq n} |\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_{nj}^{(2)}|^s \\ & \leq C \sum_{n=1}^{\infty} n^{-s} l(n) (E|Y_1|^s I(|Y_1| > n) + n^s P\{|Y_1| > n\}) \\ & \leq C \sum_{n=1}^{\infty} n^{-s} l(n) E|Y_1|^s I(|Y_1| > n) \\ & = \sum_{n=1}^{\infty} n^{-s} l(n) \sum_{m=n}^{\infty} E|Y_1|^s P(m < |Y_1| \leq m+1) \end{aligned}$$

$$= \sum_{m=1}^{\infty} E|Y_1|^s P(m < |Y_1| \le m+1) \sum_{n=1}^{m} n^{-s} l(n)$$
$$\le C \sum_{m=1}^{\infty} m^{1-s} l(m) E|Y_1|^s P(m < |Y_1| \le m+1)$$
$$\le CE|Y_1| l(|Y_1|) < \infty.$$

For I_1, by Markov and Hölder inequalities, we have

$$I_1 \quad \le C \sum_{n=1}^{\infty} n^{-1} l(n) n^{-2} E \max_{1\le k\le n} |\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} (Y_{nj}^{(1)} - EY_{nj}^{(1)})|^2$$
$$\le C \sum_{n=1}^{\infty} n^{-1} l(n) n^{-2} E \left(\sum_{i=-\infty}^{\infty} |a_i| \max_{1\le k\le n} | \sum_{j=i+1}^{i+k} (Y_{nj}^{(1)} - EY_{nj}^{(1)})| \right)^2$$

$$\le C \sum_{n=1}^{\infty} n^{-3} l(n) (\sum_{i=-\infty}^{\infty} |a_i|) \sum_{i=-\infty}^{\infty} |a_i| E \max_{1\le k\le n} | \sum_{j=i+1}^{i+k} (Y_{nj}^{(1)} - EY_{nj}^{(1)})|^2$$
$$\le C \sum_{n=1}^{\infty} n^{-2} l(n) (E|Y_1|^2 I(|Y_1| \le n) + n^2 P\{|Y_1| > n\})$$
$$\le C \sum_{n=1}^{\infty} n^{-2} l(n) E|Y_1|^2 I(|Y_1| \le n) + CE|Y_1| l(|Y_1|)$$
$$= C \sum_{n=1}^{\infty} n^{-2} l(n) \sum_{m=1}^{n} E|Y_1|^2 I(m-1 < |Y_1| \le m) + CE|Y_1| l(|Y_1|)$$
$$= C \sum_{m=1}^{\infty} E|Y_1|^2 I(m-1 < |Y_1| \le m) \sum_{n=m}^{\infty} n^{-2} l(n) + CE|Y_1| l(|Y_1|)$$
$$\le C \sum_{m=1}^{\infty} m^{-1} l(m) E|Y_1|^2 I(m-1 < |Y_1| \le m) + CE|Y_1| l(|Y_1|)$$
$$\le C \sum_{m=1}^{\infty} E|Y_1| l(|Y_1|) I(m-1 < |Y_1| \le m) + CE|Y_1| l(|Y_1|)$$
$$\le CE|Y_1| l(|Y_1|) < \infty.$$

Now we show the almost sure convergence. By the first part of Theorem, $EY_1 = 0$ (and hence $E|Y_1| < \infty$) implies

$$\sum_{n=1}^{\infty} n^{-1} P \left\{ \max_{1\le k\le n} |S_n| \ge \varepsilon n^{1/p} \right\} < \infty, \text{ for all } \varepsilon > 0.$$

Therefore

$$\infty > \sum_{n=1}^{\infty} n^{-1} P\{\max_{1\le m\le n} |S_m| > \varepsilon n\}$$

$$= \sum_{k=1}^{\infty} \sum_{n=2^{k-1}}^{2^k} n^{-1} P\{\max_{1\le m\le n} |S_m| > \varepsilon n\}$$

$$\ge 1/2 \sum_{k=1}^{\infty} P\{\max_{1\le m\le 2^{k-1}} |S_m| > \varepsilon 2^k\}.$$

By Borel-Cantelli lemma

$$2^{-k} \max_{1\le m\le 2^k} |S_m| \to 0 \text{ almost surely}$$

which implies that $S_n/n \to 0$ almost surely.□

The second theorem treats the case when the sequence $\{a_i, -\infty < i < +\infty\}$ is not absolutely summable.

Theorem 2. *Let $1 < q \le 2$. Assume that there exists s, $1 < s < q$, such that*

$$\sum_{i=-\infty}^{+\infty} |a_i|^s < \infty.$$

Suppose $\{Y_i, -\infty < i < \infty\}$ is a sequence of identically distributed negatively associated random variables and $\{X_n, n \ge 1\}$ is the moving average process based on the sequence $\{Y_i, -\infty < i < \infty\}$. Then for all p, $0 < p < \frac{sq}{sq-q+s}$, the assumptions $EY_1 = 0$ and $E|Y_1|^q < \infty$ imply Marcinkiewicz-Zygmund strong law of large numbers

$$n^{-1/p} \sum_{k=1}^{n} X_k \to 0 \text{ a.s. as } n \to \infty.$$

Remark. Note that the assumption $1 < s < q \le 2$ implies that $p < 2$.

Proof. Without loss of generality, we assume that $a_i \ge 0$ for all i. Let $\{b_n, n \ge 1\}$ be sequence of positive numbers that will be specified later. By Theorem 2 (1.6) of Shao (2000), we have

$$E\left|\sum_{k=1}^{n} b_k X_k\right|^q = E\left|\sum_{i=-\infty}^{+\infty}\left(\sum_{k=1}^{n} b_k a_{i-k}\right) Y_i\right|^q \le C \sum_{i=-\infty}^{+\infty}\left(\sum_{k=1}^{n} b_k a_{i-k}\right)^q.$$

An application of Hölder and then Jensen inequalities yields

$$\left|\sum_{k=1}^{n} b_k a_{i-k}\right|^q \leq n^{t(1-1/s)} \left(\sum_{k=1}^{n} b_k^s a_{i-k}^s\right)^{q/s}$$

$$\leq n^{q(1-1/s)} \sum_{k=1}^{n} a_{i-k}^s b_k^q \left(\sum_{i=-\infty}^{+\infty} a_i^s\right)^{q/s-1}.$$

Hence

$$E\left|\sum_{k=1}^{n} b_k X_k\right|^q \leq C \left(\sum_{i=-\infty}^{+\infty} a_i^s\right)^{q/s} n^{q(1-1/s)} \sum_{k=1}^{n} b_k^q = C n^{q(1-1/s)} \sum_{k=1}^{n} b_k^q.$$

By Theorem 3.3 of Móricz, Serfling, and Stout (1982), we have the following maximal inequality

$$E\left(\max_{1\leq m\leq n}\left|\sum_{k=1}^{m} b_k X_k\right|\right)^q \leq C \log_2^q(2n) n^{q(1-1/s)} \sum_{k=1}^{n} b_k^q.$$

This maximal inequality implies the almost sure convergence of the series $\sum_{k=1}^{\infty} b_k X_k$ as soon as $\sum_{k=1}^{\infty} b_k^q k^{q(1-1/s)} \log_2^q(2k) < \infty$ (see for instance, Loève (1978) Section 36.1). The application of Kronecker lemma with $b_k = k^{-1/p}$ concludes the proof of Theorem 2. □

References

[1] Bryc, W. and Smolenski, W. Moment conditions for almost sure convergence of weakly correlated random variables. Proc. Amer. Math. Soc. 119 (1993), 629-635.

[2] Budsaba, K., Chen, P., and Volodin, A. Limiting behaviour of moving average processes based on a sequence of ρ^- mixing random variables. To appear in Thai Statistician: Journal of Thailand Statistical Association, (2007).

[3] Burton, R.M. and Dehling, H. Large deviations for some weakly dependent random processes. Statist. Probab. Lett. 9 (1990), 397-401.

[4] Ibragimov, I.A. Some limit theorem for stationary processes. Theory Probab. Appl. 7 (1962), 349-382.

[5] Joag-Dev, K. and Proschan, F. Negative association of random variables with applications. Ann. Statist. 11 (1983), 286-295.

[6] Li, D., Rao, M.B., and Wang, X.C. Complete convergence of moving average processes. Statist. Probab. Lett. 14 (1992), 111-114.

[7] Loève M. Probability Theory II, 4 edition, Spring-Verlag, New York, 1978.

[8] Móricz, F. A., Serfling, R. J., Stout, W. F. Moment and probability bounds with quasisuperadditive structure for the maximum partial sum. Ann. Probab. 10 (1982), 1032-1040.

[9] Seneta. E. Regularly varying functions, Lecture Notes in Math. 508 (1976) Springer-Berlin.

[10] Shao, Q.M. A comparison theorem on inequalities between negatively associated and independent random variables, J. Theor. Probab. 13 (2000), 343-56.

[11] Wang, J.F. and Lu, F.B. Inequalities of maximum of partial sums and weak convergence for a class of weak dependent random variables. Acta Math. Sin. (Engl. Ser.) 22 (2006), 693-700

Department of Mathematics and Statistics, Thammasat University, Rangsit Center, Pathumthani, 12121, Thailand

E-mail address: kamon@mathstat.sci.tu.ac.th

Department of Mathematics, Jinan University, Guangzhou, 510630, China

E-mail address: chenpingyan@yahoo.com.cn

Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan, Canada, S4S 0A2

E-mail address: andrei@math.uregina.ca

Received March 18, 2007