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A new type of compact uniform integrability with application to degenerate mean convergence of weighted sums of Banach space valued random elements

Manuel Ordóñez Cabrera[a], Andrew Rosalsky[b,*], Mehmet Ünver[c], Andrei Volodin[d]

[a] Department of Mathematical Analysis, University of Sevilla, 41080 Sevilla, Spain
[b] Department of Statistics, University of Florida, Gainesville, FL 32611, USA
[c] Department of Mathematics, Faculty of Science, Ankara University, Besevler, 06100 Ankara, Turkey
[d] Department of Mathematics and Statistics, University of Regina, S4S 0A2 Saskatchewan, Canada

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A B S T R A C T

In this correspondence, for an array $\{X_{nk} : u_n \le k \le v_n, n \in \mathbb{N}\}$ of integrable random elements in a real separable Banach space and an array $\{a_{nk} : u_n \le k \le v_n, n \in \mathbb{N}\}$ of real numbers, a new type of compact uniform integrability is introduced and it is used to obtain degenerate mean convergence theorems for the weighted sums $\sum_{k=u_n}^{v_n} a_{nk}(X_{nk} - \mathbb{E}X_{nk}), n \in \mathbb{N}$. More specifically, conditions are provided under which $\lim_{n\to\infty} \mathbb{E}\left\| \sum_{k=u_n}^{v_n} a_{nk}(X_{nk} - \mathbb{E}X_{nk})\right\| = 0$.

1. Introduction

Probability limit theorems are crucial for making advances in mathematical statistics and its applications. In the current work, we establish degenerate mean convergence theorems for weighted sums $\sum_{k=u_n}^{v_n} a_{nk}(X_{nk} - \mathbb{E}X_{nk}), n \in \mathbb{N}$ arising from an array $\{X_{nk} : u_n \le k \le v_n, n \in \mathbb{N}\}$ of Banach space valued random elements and an array $\{a_{nk} : u_n \le k \le v_n, n \in \mathbb{N}\}$ of real numbers where $\mathbb{N}$ is the set of positive integers.

The concept of uniform integrability plays an important role in the area of probability limit theorems. For example, it is important for relaxing the condition of identical distribution in the case of weak laws of large numbers. In such area, this condition operates as an additional condition to yield the most important modes of convergence. Thus, it is well known that convergence almost surely (a.s.) (strong convergence) or convergence in probability (weak convergence) do not imply mean convergence; but convergence in proba-

* Corresponding author.
E-mail address: rosalsky@stat.ufl.edu (A. Rosalsky).

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bility with the additional condition of uniform integrability implies mean convergence. Thus, convergence a.s. when combined with uniform integrability implies mean convergence. Moreover, in summability theory there are various applications of uniform integrability [7,14].

A sequence $\{X_k : k \in \mathbb{N}\}$ of random variables is said to be *uniformly integrable* (see, e.g., [4]) if

$$\lim_{c\to\infty} \sup_{k\in\mathbb{N}} \mathbb{E}|X_k| I_{\{|X_k|>c\}} = 0$$

where $\mathbb{E}$ is the expectation operator and I is the indicator function. In [10], this concept was generalized to the concept of $\{a_{nk}\}$-uniform integrability: Let $\{a_{nk} : n, k \in \mathbb{N}\}$ be an array of real constants such that

$$\sup_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} |a_{nk}| < \infty. \tag{1.1}$$

A sequence of random variables $\{X_k : k \in \mathbb{N}\}$ is said to be $\{a_{nk}\}$*-uniformly integrable* if

$$\lim_{c\to\infty} \sup_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} |a_{nk}| \, \mathbb{E}|X_k| I_{\{|X_k|>c\}} = 0.$$

In the particular case of the Cesàro array

$$a_{nk} = \begin{cases} \frac{1}{n} \ , & k \le n \\ 0 \ , & \text{otherwise} \end{cases}$$

$\{a_{nk}\}$-uniform integrability reduces to *Cesàro uniform integrability* [2].

2. Preliminaries

Throughout this paper, all random elements are defined on a fixed but otherwise arbitrary probability space $(\Omega, \mathcal{G}, P)$ and take values in a real separable Banach space $(\mathcal{Y}, \|\cdot\|)$. We consider that $\mathcal{Y}$ is equipped with the Borel sigma algebra $\sigma(\mathcal{Y})$ of the norm topology. A random element X in $\mathcal{Y}$ is a $\mathcal{G}$-measurable function from Ω to the measurable space $(\mathcal{Y}, \sigma(\mathcal{Y}))$. The expected value or mean of a random element X, denoted by $\mathbb{E}X$, is defined to be the Pettis integral provided it exists.

A sequence $\{X_k : k \in \mathbb{N}\}$ of random elements is said to be *compactly uniformly integrable* [5,6] if for any $\varepsilon > 0$ there exists a compact subset K of $\mathcal{Y}$ such that

$$\sup_{k\in\mathbb{N}} \mathbb{E} \, \|X_k\| \, I_{\{X_k \notin K\}} < \varepsilon.$$

Similar to the extension of uniform integrability to $\{a_{nk}\}$-uniform integrability, the concept of compact uniform integrability has been generalized to $\{a_{nk}\}$-compact uniform integrability: Let $\{a_{nk} : n, k \in \mathbb{N}\}$ be an array of real constants such that (1.1) holds. A sequence $\{X_k : k \in \mathbb{N}\}$ of random elements is said to be $\{a_{nk}\}$*-compactly uniformly integrable* [11] if for any $\varepsilon > 0$ there exists a compact subset K of $\mathcal{Y}$ such that

$$\sup_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} |a_{nk}| \, \mathbb{E} \, \|X_k\| \, I_{\{X_k \notin K\}} < \varepsilon.$$

All of the above definitions can be extended to an array of random variables or random elements $\{X_{nk} : u_n \le k \le v_n, n \in \mathbb{N}\}$ where u_n, v_n are integers with $v_n > u_n$ for any n and $v_n - u_n \to \infty$ as $n \to \infty$ (see, [13], [12]). A new type of uniform integrability which generalizes $\{a_{nk}\}$-compact uniform integrability and which is weaker than Cesàro uniform integrability was defined as follows: Let $\{h(n) : n \in \mathbb{N}\}$

be an increasing sequence of positive constants with $\lim_{n\to\infty} h(n) = \infty$. The array of random variables $\{X_{nk} : u_n \le k \le v_n, n \in \mathbb{N}\}$ is said to be *h-integrable* with respect to the array $\{a_{nk}\}$ [13] if

(i) $\sup_{n\in\mathbb{N}} \sum_{k=u_n}^{v_n} |a_{nk}| \, \mathbb{E}\, |X_{nk}| < \infty$

(ii) $\lim_{n\to\infty} \sum_{k=u_n}^{v_n} |a_{nk}| \, \mathbb{E}\, |X_{nk}| \, I_{\{|X_{nk}|>h(n)\}} = 0.$

Note that, in a metric space any compact set is totally bounded. Therefore, if K is a compact subset in a metric space, then for any $\varepsilon > 0$, K can be covered with finite number of open balls with radius of ε. Throughout this study, without loss of generality we assume that the center of each open ball belongs to the compact set itself and we denote by $N(K, \varepsilon)$ the minimal number of open balls with radius of ε with centers from K needed to cover K. In the literature, this number is called the ε-covering number (see, e.g., [8]).

In this paper, we introduce a new type of uniform integrability for arrays of random elements taking values in real separable Banach spaces which is a generalization of the notion of the concept of h-uniform integrability. We give some degenerate mean convergence results via this new type of uniform integrability in real separable Banach spaces. Moreover, we study some degenerate mean convergence results in real separable Hilbert spaces. Let $\mathbb{Z}$ denote the set of integers (not necessarily positive). Throughout this study, we assume $u_n, v_n \in \mathbb{Z}\cup\{-\infty, \infty\}$ with $v_n > u_n$ for any $n \in \mathbb{N}$ and $v_n - u_n \to \infty$ as $n \to \infty$. In Theorems 1-3, if $u_n = -\infty$ or $v_n = \infty$, for any $n \in \mathbb{N}$, we assume that the series $\sum_{k=u_n}^{v_n} a_{nk}(X_{nk} - \mathbb{E}X_{nk})$ converges a.s.

We will point out how our results relate to various uniform integrability results in the literature and how some of them (or special cases of some of them) follow from our results.

Definition 1. Let $\{K_n : n \in \mathbb{N}\}$ be a sequence of compact subsets of $\mathcal{Y}$ and let $\{a_{nk} : u_n \le k \le v_n, n \in \mathbb{N}\}$ be an array of real numbers such that

$$\sup_{n\in\mathbb{N}} \sum_{k=u_n}^{v_n} |a_{nk}| = M < \infty. \tag{2.1}$$

Then an array $\{X_{nk} : u_n \le k \le v_n, n \in \mathbb{N}\}$ of random elements taking values in $\mathcal{Y}$ is said to be $\{K_n\}$-compactly uniformly integrable with respect to $\{a_{nk}\}$ if

(i) $\sup_{n\in\mathbb{N}} \sum_{k=u_n}^{v_n} |a_{nk}| \, \mathbb{E}\, \|X_{nk}\| < \infty$

(ii) $\lim_{n\to\infty} \sum_{k=u_n}^{v_n} |a_{nk}| \, \mathbb{E}\, \|X_{nk}\| \, I_{\{X_{nk}\notin K_n\}} = 0.$

Remark 1. In Definition 1, the sequence of compact sets $\{K_n : n \in \mathbb{N}\}$ can be considered as non-decreasing, because the finite union of compact sets is compact. Hence, in this paper, we assume that $\{K_n : n \in \mathbb{N}\}$ is a non-decreasing sequence of compact subsets of $\mathcal{Y}$.

The following remark shows that h-integrability with respect to $\{a_{nk}\}$ is a particular case of $\{K_n\}$-compact uniform integrability with respect to $\{a_{nk}\}$.

Remark 2. If an array of random variables $\{X_{nk} : u_n \le k \le v_n, n \in \mathbb{N}\}$ is h-integrable with respect to the array $\{a_{nk}\}$, then it is $\{K_n\}$-compactly uniformly integrable with respect to $\{a_{nk}\}$ where $K_n = \{t : |t| \le h(n)\}$ for any $n \in \mathbb{N}$. Therefore, h-integrability is a special case of $\{K_n\}$-compact uniform integrability with respect to $\{a_{nk}\}$.

Note here that, since $h(n) \nearrow \infty$ we have that $\delta(K_n) \nearrow \infty$ as $n \to \infty$ where δ denotes the diameter of a set in the Banach space $(\mathcal{Y}, \|\cdot\|)$. However, we do not need this condition in general in the current study.

The following lemma is used in the proofs of our theorems.

Lemma 1. *Let $\{K_n\}$ be a sequence of compact subsets of $\mathcal{Y}$ and let $\{X_{nk} : u_n \le k \le v_n, n \in \mathbb{N}\}$ be an array of random elements such that X_{nk} takes values in K_n for each $n \in \mathbb{N}$ and for each $u_n \le k \le v_n$. Then for any $\varepsilon > 0$ and $n \in \mathbb{N}$ there exist $\left\{x_j^{(n)} : 1 \le j \le N(K_n, \varepsilon)\right\} \subset K_n$ and a disjoint family $\left\{A_j^{(n)} : 1 \le j \le N(K_n, \varepsilon)\right\}$ of Borel subsets of $\mathcal{Y}$ such that for any $n \in \mathbb{N}$ and any $u_n \le k \le v_n$,*

$$\|X_{nk}(\omega) - Z_{nk}(\omega)\| < \varepsilon, \text{ for any } \omega \in \Omega$$

where for any $n \in \mathbb{N}$ and any $u_n \le k \le v_n$,

$$Z_{nk} = \sum_{j=1}^{N(K_n,\varepsilon)} x_j^{(n)} I_{\left\{X_{nk} \in A_j^{(n)}\right\}}.$$

Proof. Let $\varepsilon > 0$ and $n \in \mathbb{N}$. As K_n is totally bounded there exists $\left\{x_j^{(n)} : 1 \le j \le N(K_n, \varepsilon)\right\} \subset K_n$ such that

$$K_n \subset \bigcup_{j=1}^{N(K_n,\varepsilon)} B(x_j^{(n)}, \varepsilon).$$

If we set

$$A_1^{(n)} := K_n \cap B(x_j^{(n)}, \varepsilon)$$
$$A_j^{(n)} := K_n \cap \left\{ B(x_j^{(n)}, \varepsilon) \setminus \bigcup_{i=1}^{j-1} B(x_i^{(n)}, \varepsilon) \right\}, 2 \le j \le N(K_n, \varepsilon)$$

then it is easy to see for any $n \in \mathbb{N}$ and any $u_n \le k \le v_n$ that

$$\|X_{nk}(\omega) - Z_{nk}(\omega)\| < \varepsilon, \text{ for any } \omega \in \Omega. \quad \square$$

3. Degenerate mean convergence in Banach spaces

In this section, we establish some degenerate mean convergence theorems for weighted sums from arrays of random elements with the help of $\{K_n\}$-compact uniform integrability with respect to $\{a_{nk}\}$. In the following degenerate mean convergence theorem, the array is comprised of rowwise pairwise independent random elements; that is, the random elements from the same row are pairwise independent but there are no independence or dependence conditions imposed on the random elements from different rows.

Theorem 1. *Let $\{a_{nk} : u_n \le k \le v_n, n \in \mathbb{N}\}$ be an array of real numbers such that (2.1) holds and let $\{X_{nk} : u_n \le k \le v_n, n \in \mathbb{N}\}$ be an array of rowwise pairwise independent integrable random elements. If*
(i) *$\{X_{nk}\}$ is $\{K_n\}$-compactly uniformly integrable with respect to $\{a_{nk}\}$,*
(ii) *for any $\varepsilon > 0$*

$$\lim_{n\to\infty} N^2(K_n, \varepsilon) h^2(n) \sum_{k=u_n}^{v_n} a_{nk}^2 = 0,$$

then

$$\lim_{n\to\infty} \mathbb{E}\left\|\sum_{k=u_n}^{v_n} a_{nk}(X_{nk}-\mathbb{E}X_{nk})\right\| = 0$$

and, a fortiori,

$$\sum_{k=u_n}^{v_n} a_{nk}(X_{nk}-\mathbb{E}X_{nk}) \xrightarrow{P} 0$$

where $h(n) := \sup_{x\in K_n} \|x\|$ *for any* $n \in \mathbb{N}$*.*

Proof. Let $\varepsilon > 0$. By (i) there exists $n_1 \in \mathbb{N}$ such that

$$\sum_{k=u_n}^{v_n} |a_{nk}|\, \mathbb{E}\, \|X_{nk}\|\, I_{\{X_{nk}\notin K_n\}} < \varepsilon/6 \tag{3.1}$$

whenever $n \geq n_1$. On the other hand as $X_{nk}I_{\{X_{nk}\in K_n\}}$ takes values in $K_n \cup \{0\}$, by Lemma 1 there exists an array of pairwise independent $\mathcal{Y}$-valued random elements

$$\left\{ Z_{nk} = \sum_{j=1}^{N(K_n,\varepsilon)} x_j^{(n)} I_{\left\{X_{nk}\in A_j^{(n)}\right\}} \right\}$$

such that

$$\sup_{n\in\mathbb{N}} \sup_{u_n\leq k\leq v_n} \left\|X_{nk}I_{\{X_{nk}\in K_n\}} - Z_{nk}\right\| \leq \varepsilon/6M. \tag{3.2}$$

Now, by (3.1) we have for any $n \geq n_1$ that

$$\begin{aligned}
&\mathbb{E}\left\|\sum_{k=u_n}^{v_n} a_{nk}\left(X_{nk}I_{\{X_{nk}\notin K_n\}} - EX_{nk}I_{\{X_{nk}\notin K_n\}}\right)\right\| \\
&\leq \sum_{k=u_n}^{v_n} |a_{nk}|\, \mathbb{E}\left\|X_{nk}I_{\{X_{nk}\notin K_n\}} - EX_{nk}I_{\{X_{nk}\notin K_n\}}\right\| \\
&\qquad \leq 2\sum_{k=u_n}^{v_n} |a_{nk}|\, \mathbb{E}\, \|X_{nk}\|\, I_{\{X_{nk}\notin K_n\}} \\
&\qquad < \varepsilon/3.
\end{aligned} \tag{3.3}$$

By (ii) there exists $n_2 \in \mathbb{N}$ such that

$$N(K_n,\varepsilon)h(n)\left(\sum_{k=u_n}^{v_n} a_{nk}^2\right)^{1/2} < \varepsilon/3 \tag{3.4}$$

whenever $n \geq n_2$. Thus, we have for any $n \geq n_2$ that

$$
\begin{aligned}
&\mathbb{E}\left\|\sum_{k=u_n}^{v_n} a_{nk}(Z_{nk}-\mathbb{E}Z_{nk})\right\| \\
&= \mathbb{E}\left\|\sum_{j=1}^{N(K_n,\varepsilon)}\sum_{k=u_n}^{v_n} a_{nk}x_j^{(n)}\left(I_{\left\{X_{nk}\in A_j^{(n)}\right\}} - \mathbb{E}I_{\left\{X_{nk}\in A_j^{(n)}\right\}}\right)\right\| \\
&\le \sum_{j=1}^{N(K_n,\varepsilon)}\left\|x_j^{(n)}\right\| \mathbb{E}\left\|\sum_{k=u_n}^{v_n} a_{nk}\left(I_{\left\{X_{nk}\in A_j^{(n)}\right\}} - \mathbb{E}I_{\left\{X_{nk}\in A_j^{(n)}\right\}}\right)\right\| \\
&\text{(by the Cauchy-Schwarz inequality)} \\
&\le \sum_{j=1}^{N(K_n,\varepsilon)}\left\|x_j^{(n)}\right\|\left(\mathbb{E}\left(\sum_{k=u_n}^{v_n} a_{nk}\left(I_{\left\{X_{nk}\in A_j^{(n)}\right\}} - \mathbb{E}I_{\left\{X_{nk}\in A_j^{(n)}\right\}}\right)\right)^2\right)^{1/2} \\
&\text{(by pairwise independence)} \\
&\le \sum_{j=1}^{N(K_n,\varepsilon)}\left\|x_j^{(n)}\right\|\left(\sum_{k=u_n}^{v_n} a_{nk}^2\mathbb{E}\left(I_{\left\{X_{nk}\in A_j^{(n)}\right\}} - \mathbb{E}I_{\left\{X_{nk}\in A_j^{(n)}\right\}}\right)^2\right)^{1/2} \\
&\le \sum_{j=1}^{N(K_n,\varepsilon)}\left\|x_j^{(n)}\right\|\left(\sum_{k=u_n}^{v_n} a_{nk}^2\right)^{1/2} \\
&\le N(K_n,\varepsilon)h(n)\left(\sum_{k=u_n}^{v_n} a_{nk}^2\right)^{1/2} \\
&\text{(by 3.4)} \\
&< \varepsilon/3. && (3.5)
\end{aligned}
$$

Finally, using (3.2), (3.3), and (3.5) we obtain for any $n \ge \max\{n_1, n_2\}$ that

$$
\begin{aligned}
\mathbb{E}\left\|\sum_{k=u_n}^{v_n} a_{nk}(X_{nk}-\mathbb{E}X_{nk})\right\| &= \mathbb{E}\Bigg\|\sum_{k=u_n}^{v_n} a_{nk}\left(X_{nk}I_{\{X_{nk}\notin K_n\}} - \mathbb{E}X_{nk}I_{\{X_{nk}\notin K_n\}}\right) \\
&\qquad + \sum_{k=u_n}^{v_n} a_{nk}\left(X_{nk}I_{\{X_{nk}\in K_n\}} - \mathbb{E}X_{nk}I_{\{X_{nk}\in K_n\}}\right) \\
&\quad - \sum_{k=u_n}^{v_n} a_{nk}(Z_{nk}-\mathbb{E}Z_{nk}) + \sum_{k=u_n}^{v_n} a_{nk}(Z_{nk}-\mathbb{E}Z_{nk})\Bigg\| \\
&\le \mathbb{E}\left\|\sum_{k=u_n}^{v_n} a_{nk}\left(X_{nk}I_{\{X_{nk}\notin K_n\}} - \mathbb{E}X_{nk}I_{\{X_{nk}\notin K_n\}}\right)\right\| \\
&\quad + \mathbb{E}\left\|\sum_{k=u_n}^{v_n} a_{nk}\left(X_{nk}I_{\{X_{nk}\in K_n\}} - Z_{nk}\right)\right\| \\
&\quad + \mathbb{E}\left\|\sum_{k=u_n}^{v_n} a_{nk}(\mathbb{E}X_{nk}I_{\{X_{nk}\in K_n\}} - \mathbb{E}Z_{nk})\right\| \\
&\quad + \mathbb{E}\left\|\sum_{k=u_n}^{v_n} a_{nk}(Z_{nk}-\mathbb{E}Z_{nk})\right\|
\end{aligned}
$$

$$< \frac{\varepsilon}{3} + 2M\frac{\varepsilon}{6M} + \frac{\varepsilon}{3}$$
$$= \varepsilon$$

completing the proof. □

Remark 3. Let $\alpha \in (0, \infty)$. A sequence $\{X_k\}$ of random variables is said to be Cesàro α-integrable [3] if

(i) $\sup_{n\in\mathbb{N}} \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}\,|X_k| < \infty$

(ii) $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}\,|X_k|\, I_{\{|X_k|>k^\alpha\}} = 0.$

Now, let $\{X_k\}$ be a sequence of pairwise independent random variables that is Cesàro α-integrable for $\alpha \in (0, 1/4)$. If we define

$$X_{nk} := \begin{cases} X_k\ , & k \le n \\ 0\ , & \text{otherwise} \end{cases} \tag{3.6}$$

and $K_n = [-n^\alpha, n^\alpha]$ for any $n \in \mathbb{N}$, then we have

$$\begin{aligned} \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}\,|X_{nk}|\, I_{\{X_{nk}\notin K_n\}} &= \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}\,|X_k|\, I_{\{|X_k|\ge n^\alpha\}} \\ &\le \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}\,|X_k|\, I_{\{|X_k|\ge k^\alpha\}} \end{aligned}$$

for any $n \in \mathbb{N}$. Thus, $\{X_{nk}\}$ is $\{K_n\}$-compactly uniformly integrable with respect to the Cesàro array. So, (i) of Theorem 1 holds. Moreover, we get for any $\varepsilon > 0$ and sufficiently large $n \in \mathbb{N}$ that

$$0 \le N^2(K_n, \varepsilon)h^2(n)\sum_{k=1}^{n} \frac{1}{n^2} \le \frac{5n^{4\alpha}}{\varepsilon^2 n}$$

which yields

$$\lim_{n\to\infty} N^2(K_n, \varepsilon)h^2(n)\sum_{k=1}^{n} \frac{1}{n^2} = 0.$$

So, (ii) of Theorem 1 holds. Hence, we have

$$\begin{aligned} \lim_{n\to\infty} \mathbb{E}\frac{1}{n}\left|\sum_{k=1}^{n} (X_k - \mathbb{E}X_k)\right| &= \lim_{n\to\infty} \mathbb{E}\frac{1}{n}\left|\sum_{k=1}^{n} (X_{nk} - \mathbb{E}X_{nk})\right| \\ &= 0. \end{aligned}$$

Thus, for $\alpha \in (0, 1/4)$, Theorem 2.2 (a) of [3] is a consequence of Theorem 1. Note here that Theorem 2.2 (a) of [3] holds for $\alpha \in (0, 1/2)$.

The following example, which was inspired by Example 6 of [1], shows that in Theorem 1 the condition

$$\lim_{n\to\infty} \sum_{k=u_n}^{v_n} |a_{nk}|\, \mathbb{E}\,\|X_{nk}\|\, I_{\{X_{nk}\notin K_n\}} = 0$$

cannot be replaced by the condition: for arbitrary $\varepsilon > 0$, there exists a sequence of compact sets $\{K_n : n \in \mathbb{N}\}$ such that

$$\limsup_{n\to\infty} \sum_{k=u_n}^{v_n} |a_{nk}| \, P(X_{nk} \notin K_n) \leq \varepsilon.$$

Example 1. Consider the real separable Banach space l_1 of absolutely summable real sequences $v = \{v_i : i \in \mathbb{N}\}$ with the norm $\|v\| = \sum_{i=1}^{\infty} |v_i|$. Let $v^{(k)}$ denote the member of l_1 having 1 in its kth position and 0 elsewhere, $k \in \mathbb{N}$. Let $u_n = 1$, $v_n = n$, $n \in \mathbb{N}$ and let $a_{nk} = \frac{1}{n}, 1 \leq k \leq n, n \in \mathbb{N}$. Define a sequence $\{X_k : k \in \mathbb{N}\}$ of random elements in l_1 by requiring the $\{X_k : k \in \mathbb{N}\}$ to be independent with $P(X_k = \sqrt{k}v^{(k)}) = P(X_k = -\sqrt{k}v^{(k)}) = \frac{1}{2\sqrt{k}}$, $P(X_k = 0) = 1 - \frac{1}{\sqrt{k}}$, $k \in \mathbb{N}$. Consider the array $\{X_{nk} : 1 \leq k \leq n, n \in \mathbb{N}\}$ of random elements defined by $X_{nk} = X_k, 1 \leq k \leq n, n \in \mathbb{N}$. Note that

$$\sup_{n\in\mathbb{N}} \sum_{k=1}^{n} |a_{nk}| = 1$$

and so (2.1) holds. Next, note that

$$\sup_{n\in\mathbb{N}} \sum_{k=1}^{n} |a_{nk}| \, \mathbb{E} \, \|X_{nk}\| = 1$$

since $\mathbb{E} \, \|X_{nk}\| = 1, 1 \leq k \leq n, n \in \mathbb{N}$.

Let $\varepsilon > 0$ be arbitrary. Let $J_\varepsilon \in \mathbb{N}$ be such that $\dfrac{1}{\sqrt{J_\varepsilon}} \leq \varepsilon$ and set

$$K_\varepsilon = \left\{0, v^{(1)}, -v^{(1)}, \sqrt{2}v^{(2)}, -\sqrt{2}v^{(2)}, ..., \sqrt{J_\varepsilon}v^{(J_\varepsilon)}, -\sqrt{J_\varepsilon}v^{(J_\varepsilon)}\right\}.$$

Let $K_n = K_\varepsilon, n \in \mathbb{N}$. Then $\{K_n : n \in \mathbb{N}\}$ is a sequence of compact subsets of l_1. Since

$$P(\|X_{nk}\| = \sqrt{k}) = \frac{1}{\sqrt{k}}, P(\|X_{nk}\| = 0) = 1 - \frac{1}{\sqrt{k}}, 1 \leq k \leq n, n \in \mathbb{N},$$

it follows that whenever $n \geq k > J_\varepsilon$,

$$\mathbb{E} \, \|X_{nk}\| \, I_{\{X_{nk} \notin K_n\}} = \mathbb{E} \, \|X_{nk}\| \, I_{\{X_{nk} \notin K_\varepsilon\}} = \mathbb{E} \, \|X_{nk}\| = 1.$$

Thus for $n > J_\varepsilon$,

$$\sum_{k=1}^{n} |a_{nk}| \, \mathbb{E} \, \|X_{nk}\| \, I_{\{X_{nk} \notin K_n\}} \geq \sum_{k=J_\varepsilon+1}^{n} \frac{1}{n} = \frac{n - J_\varepsilon}{n} \to 1$$

and so

$$\lim_{n\to\infty} \sum_{k=1}^{n} |a_{nk}| \, \mathbb{E} \, \|X_{nk}\| \, I_{\{X_{nk} \notin K_n\}} = 0$$

fails.

Now for $n > J_\varepsilon$,

$$P(X_{nk} \notin K_\varepsilon) = \begin{cases} 0 < \varepsilon & , 1 \le k \le J_\varepsilon \\ \frac{1}{\sqrt{k}} < \frac{1}{\sqrt{J_\varepsilon}} \le \varepsilon \ , & k > J_\varepsilon \end{cases}$$

and so the sequence of compact sets $\{K_n : n \in \mathbb{N}\}$ satisfies

$$\begin{aligned} \limsup_{n\to\infty} \sum_{k=1}^{n} |a_{nk}|\, P(X_{nk} \notin K_n) &= \limsup_{n\to\infty} \sum_{k=1}^{n} |a_{nk}|\, P(X_{nk} \notin K_\varepsilon) \\ &\le \limsup_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\varepsilon \\ &= \varepsilon. \end{aligned}$$

Next, note that

$$h(n) = \sup_{x \in K_n} \|x\| = \sup_{x \in K_\varepsilon} \|x\| = \sqrt{J_\varepsilon} < \infty$$

and for arbitrary $\varepsilon_0 > 0$,

$$N^2(K_n, \varepsilon_0) = N^2(K_\varepsilon, \varepsilon_0) \le (2J_\varepsilon + 1)^2.$$

Thus

$$\begin{aligned} \lim_{n\to\infty} N^2(K_\varepsilon, \varepsilon_0) h^2(n) \sum_{k=1}^{n} a_{nk}^2 &\le \lim_{n\to\infty} (2J_\varepsilon + 1)^2 J_\varepsilon \sum_{k=1}^{n} \frac{1}{n^2} \\ &= \lim_{n\to\infty} (2J_\varepsilon + 1)^2 J_\varepsilon \frac{1}{n} \\ &= 0. \end{aligned}$$

All of the conditions of Theorem 1 are satisfied except for condition (ii) of Definition 1.

We now verify that the conclusion of Theorem 1 fails. By the structure of the l_1 norm and Kolmogorov's theorem (see [9], p. 250) applied to the sequence of random variables $\left\{\sqrt{k} I_{\left\{\|X_k\| = \sqrt{k}\right\}} : k \in \mathbb{N}\right\}$, it follows that

$$\begin{aligned} \left\| \sum_{k=1}^{n} a_{nk}(X_{nk} - \mathbb{E}X_{nk}) \right\| &= \frac{\left\| \sum_{k=1}^{n} X_{nk} \right\|}{n} \\ &= \frac{\left\| \sum_{k=1}^{n} X_{k} \right\|}{n} \\ &= \frac{\sum_{k=1}^{n} \sqrt{k} I_{\left\{\|X_k\| = \sqrt{k}\right\}}}{n} \to 1 \text{ a.s.} \end{aligned}$$

and so the conclusion of Theorem 1 cannot hold.

The following result establishes degenerate mean convergence for weighted sums from a $\{K_n\}$-compactly uniformly integrable array with respect to $\{a_{nk}\}$ of conditionally mean zero random elements. Note that condition (ii) of Theorem 2 is stronger than condition (ii) of Theorem 1.

Theorem 2. *Let $\{K_n\}$ be a non-decreasing sequence of compact subsets of $\mathcal{Y}$, let $\{a_{nk} : u_n \leq k \leq v_n, n \in \mathbb{N}\}$ be an array of real numbers such that (2.1) holds, let $\{\mathcal{F}_{nk} : u_n \leq k \leq v_n, n \in \mathbb{N}\}$ be an array of non-decreasing subsigma algebras and let $\{X_{nk} : u_n \leq k \leq v_n, n \in \mathbb{N}\}$ be an array of integrable random elements such that $\mathbb{E}\left(X_{nk} \mid \mathcal{F}_{n,k-1}\right) = 0$ for any $n \in \mathbb{N}$ and k. If*
(i) *$\{X_{nk}\}$ is $\{K_n\}$-compactly uniformly integrable with respect to $\{a_{nk}\}$,*
(ii) *for any $\varepsilon > 0$*

$$\lim_{n\to\infty} N(K_n, \varepsilon) h(n) \sum_{k=u_n}^{v_n} |a_{nk}| = 0,$$

then

$$\lim_{n\to\infty} \mathbb{E} \left\| \sum_{k=u_n}^{v_n} a_{nk} X_{nk} \right\| = 0$$

and, a fortiori,

$$\sum_{k=u_n}^{v_n} a_{nk} X_{nk} \xrightarrow{P} 0$$

where $h(n) := \sup_{x\in K_n} \|x\|$ for any $n \in \mathbb{N}$.

Proof. Let $\varepsilon > 0$. By (i) there exists $n_1 \in \mathbb{N}$ such that

$$\sum_{k=u_n}^{v_n} |a_{nk}| \, \mathbb{E} \|X_{nk}\| \, I_{\{X_{nk}\notin K_n\}} < \varepsilon/6 \tag{3.7}$$

whenever $n \geq n_1$. As in the proof of Theorem 1, by Lemma 1, there exists an array of $\mathcal{Y}$-valued random elements

$$\left\{ Z_{nk} = \sum_{j=1}^{N(K_n,\varepsilon)} x_j^{(n)} I_{\left\{X_{nk}\in A_j^{(n)}\right\}} \right\}$$

such that (3.2) holds. Now, from (3.7) and the tower rule we can write for any $n \geq n_1$ that

$$\begin{aligned}
&\mathbb{E} \left\| \sum_{k=u_n}^{v_n} a_{nk} \left(X_{nk} I_{\{X_{nk}\notin K_n\}} - \mathbb{E}\left(X_{nk} I_{\{X_{nk}\notin K_n\}} \mid \mathcal{F}_{n,k-1} \right) \right) \right\| \\
&\leq \sum_{k=u_n}^{v_n} |a_{nk}| \, \mathbb{E} \left\| \left(X_{nk} I_{\{X_{nk}\notin K_n\}} - \mathbb{E}\left(X_{nk} I_{\{X_{nk}\notin K_n\}} \mid \mathcal{F}_{n,k-1} \right) \right) \right\| \\
&\leq \sum_{k=u_n}^{v_n} |a_{nk}| \left(\mathbb{E} \left\| X_{nk} I_{\{X_{nk}\notin K_n\}} \right\| + \mathbb{E}\left(\mathbb{E}\left(\left\| X_{nk} I_{\{X_{nk}\notin K_n\}} \right\| \mid \mathcal{F}_{n,k-1} \right) \right) \right)
\end{aligned}$$

$$= 2 \sum_{k=u_n}^{v_n} |a_{nk}| \, \mathbb{E} \left\| X_{nk} I_{\{X_{nk} \notin K_n\}} \right\| \\ \le \varepsilon/3. \tag{3.8}$$

Moreover, using the tower rule and (3.2) we have that for any $n \in \mathbb{N}$

$$\mathbb{E} \left\| \sum_{k=u_n}^{v_n} a_{nk} \mathbb{E} \left(X_{nk} I_{\{X_{nk} \in K_n\}} - Z_{nk} \mid \mathcal{F}_{n,k-1} \right) \right\| \\ \le \sum_{k=u_n}^{v_n} |a_{nk}| \, \mathbb{E} \left(\mathbb{E} \left(\left\| X_{nk} I_{\{X_{nk} \in K_n\}} - Z_{nk} \right\| \mid \mathcal{F}_{n,k-1} \right) \right) \\ = \sum_{k=u_n}^{v_n} |a_{nk}| \, \mathbb{E} \left\| X_{nk} I_{\{X_{nk} \in K_n\}} - Z_{nk} \right\| \\ \le \varepsilon/6. \tag{3.9}$$

On the other hand, by (ii) there exists $n_2 \in \mathbb{N}$ such that

$$N(K_n, \varepsilon) h(n) \sum_{k=u_n}^{v_n} |a_{nk}| < \varepsilon/6$$

whenever $n \ge n_2$. Thus, using the tower rule we have for $n \ge n_2$ that

$$\mathbb{E} \left\| \sum_{k=u_n}^{v_n} a_{nk} (Z_{nk} - \mathbb{E} (Z_{nk} \mid \mathcal{F}_{n,k-1})) \right\| \\ \le 2 \sum_{k=u_n}^{v_n} |a_{nk}| \, \mathbb{E} \, \|Z_{nk}\| \\ \le 2 \sum_{j=1}^{N(K_n,\varepsilon)} \left\| x_j^{(n)} \right\| \sum_{k=u_n}^{v_n} |a_{nk}| \, \mathbb{E} I_{\left\{X_{nk} \in A_j^{(n)}\right\}} \\ = 2N(K_n, \varepsilon) h(n) \sum_{k=u_n}^{v_n} |a_{nk}| \\ \le \varepsilon/3. \tag{3.10}$$

Finally, using (3.2), (3.8), (3.9), (3.10), and the fact that $\mathbb{E} \left(X_k \mid \mathcal{F}_{n,k-1} \right) = 0$ we have for every positive integer $n \ge \max\{n_1, n_2\}$ that

$$\left\| \sum_{k=u_n}^{v_n} a_{nk} X_{nk} \right\| = \left\| \sum_{k=u_n}^{v_n} a_{nk} \left(X_{nk} - \mathbb{E} \left(X_{nk} \mid \mathcal{F}_{n,k-1} \right) \right) \right\| \\ \le \left\| \sum_{k=u_n}^{v_n} a_{nk} \left(X_{nk} I_{\{X_{nk} \notin K_n\}} - \mathbb{E} \left(X_{nk} I_{\{X_{nk} \notin K_n\}} \mid \mathcal{F}_{n,k-1} \right) \right) \right\| \\ + \left\| \sum_{k=u_n}^{v_n} a_{nk} \left(X_{nk} I_{\{X_{nk} \in K_n\}} - \mathbb{E} \left(X_{nk} I_{\{X_{nk} \in K_n\}} \mid \mathcal{F}_{n,k-1} \right) \right) \right.$$

$$
\begin{aligned}
&- \sum_{k=u_n}^{v_n} a_{nk}(Z_{nk} - \mathbb{E}(Z_{nk} \mid \mathcal{F}_{n,k-1})) \\
&+ \sum_{k=u_n}^{v_n} a_{nk}(Z_{nk} - \mathbb{E}(Z_{nk} \mid \mathcal{F}_{n,k-1}))\Bigg\|
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\left\| \sum_{k=u_n}^{v_n} a_{nk} X_{nk} \right\| \\
&\leq \left\| \sum_{k=u_n}^{v_n} a_{nk} \left(X_{nk} I_{\{X_{nk} \notin K_n\}} - \mathbb{E}\left(X_{nk} I_{\{X_{nk} \notin K_n\}} \mid \mathcal{F}_{n,k-1} \right) \right) \right\| \\
&+ \left\| \sum_{k=u_n}^{v_n} a_{nk} \left(X_{nk} I_{\{X_{nk} \in K_n\}} - Z_{nk} \right) \right\| \\
&+ \left\| \sum_{k=u_n}^{v_n} a_{nk} \mathbb{E}\left(X_{nk} I_{\{X_{nk} \in K_n\}} - Z_{nk} \mid \mathcal{F}_{n,k-1} \right) \right\| \\
&+ \left\| \sum_{k=u_n}^{v_n} a_{nk}(Z_{nk} - \mathbb{E}(Z_{nk} \mid \mathcal{F}_{n,k-1})) \right\| \\
&= \frac{\varepsilon}{3} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{3} \\
&= \varepsilon
\end{aligned}
$$

whenever $n \geq \max\{n_1, n_2\}$. Hence the proof is completed. □

Remark 4. Let $\{X_k\}$ be a sequence of pairwise independent random variables which is Cesàro α-integrable for $\alpha \in (0, 1/2)$. If we consider the array $\{X_{nk}\}$ in (3.6), then by taking $K_n = [-n^\alpha, n^\alpha]$ for any $n \in \mathbb{N}$ we have $\{X_{nk}\}$ is $\{K_n\}$-compactly uniformly integrable with respect to the Cesàro array. So, (i) of Theorem 2 holds. Furthermore, we get for any $\varepsilon > 0$ and sufficiently large $n \in \mathbb{N}$ that

$$
0 \leq N(K_n, \varepsilon) h(n) \sum_{k=1}^{n} \frac{1}{n^2} \leq \frac{3n^{2\alpha}}{\varepsilon n}
$$

which implies

$$
\lim_{n \to \infty} N(K_n, \varepsilon) h(n) \sum_{k=1}^{n} \frac{1}{n^2} = 0.
$$

So, (ii) of Theorem 2 holds. Hence, Theorem 3.1 of [3] is a consequence of Theorem 2. Note here that in Theorem 3.1 of [3], $\alpha \in (0, 1/2)$ as well.

4. Degenerate mean convergence in Hilbert spaces

In this section, we provide a result on degenerate mean convergence in real separable Hilbert spaces. Throughout this section $\mathcal{H}$ denotes a real separable Hilbert space.

By considering the Riesz representation theorem, we obtain the following equivalent definition of the expected value or mean in Hilbert spaces:

Definition 2. Let X be a random element taking values in $\mathcal{H}$. If there exists $z \in \mathcal{H}$ such that

$$\langle z, h \rangle = \mathbb{E} \langle X, h \rangle$$

for each $h \in \mathcal{H}$ then X is said to be integrable and the vector z is said to be the expected value or mean of X. In this case we write $\mathbb{E}X := z$.

Definition 3. Let $\{K_n\}$ be a sequence of compact subsets of $\mathcal{H}$. An array $\{X_{nj} : u_n \leq j \leq v_n, n \in \mathbb{N}\}$ of integrable random elements in $\mathcal{H}$ is said to be $\{K_n\}$-non-negative if for any j, k, n

$$\mathbb{E} \langle X_{nj}, X_{nk} \rangle \geq \mathbb{E} \left\langle X_{nj} I_{\{X_{nj} \in K_n\}}, X_{nk} I_{\{X_{nk} \in K_n\}} \right\rangle .$$

Remark 5. Note that in the case of where $\mathcal{H}$ is the space of all real numbers, we have that if $X_{nj} \geq 0$ for all $j, n \in \mathbb{N}$, then

$$\langle X_{nj}, X_{nk} \rangle \geq \left\langle X_{nj} I_{\{X_{nj} \in K_n\}}, X_{nk} I_{\{X_{nk} \in K_n\}} \right\rangle .$$

Thus, the condition of Definition 3 holds. This explains the terminology "non-negative".

Definition 4. An array $\{X_{nj} : u_n \leq j \leq v_n, n \in \mathbb{N}\}$ of integrable random elements in $\mathcal{H}$ is said to be negatively correlated if for any $j \neq k$, and $n \in \mathbb{N}$,

$$\mathbb{E} \langle X_{nj}, X_{nk} \rangle - \langle \mathbb{E}X_{nj}, \mathbb{E}X_{nk} \rangle \leq 0.$$

Theorem 3. *Let $\{a_{nk} : u_n \leq k \leq v_n, n \in \mathbb{N}\}$ be an array of non-negative real numbers such that (2.1) holds and let $\{X_{nk} : u_n \leq k \leq v_n, n \in \mathbb{N}\}$ be an array of $\{K_n\}$-non-negative and negatively correlated integrable random elements in $\mathcal{H}$. If*
(i) *$\{X_{nk}\}$ is $\{K_n\}$-compactly uniformly integrable with respect to $\{a_{nk}\}$,*
(ii) $\lim_{n\to\infty} h(n) \left(\sup_{u_n \leq k \leq v_n} a_{nk} \right) = 0$,
then

$$\lim_{n\to\infty} \mathbb{E} \left\| \sum_{k=u_n}^{v_n} a_{nk}(X_{nk} - \mathbb{E}X_{nk}) \right\| = 0$$

and, a fortiori,

$$\sum_{k=u_n}^{v_n} a_{nk}(X_{nk} - \mathbb{E}X_{nk}) \xrightarrow{P} 0$$

where $h(n) := \sup_{x \in K_n} \|x\|$ for any $n \in \mathbb{N}$.

Proof. We have for any $n \in \mathbb{N}$ that

$$\begin{aligned} \mathbb{E} \left\| \sum_{k=u_n}^{v_n} a_{nk}(X_{nk} - \mathbb{E}X_{nk}) \right\| = \mathbb{E} \Bigg\| & \sum_{k=u_n}^{v_n} a_{nk}(X_{nk} - X_{nk} I_{\{X_{nk} \in K_n\}} \\ & + \sum_{k=u_n}^{v_n} a_{nk} \left(X_{nk} I_{\{X_{nk} \in K_n\}} - \mathbb{E}X_{nk} I_{\{X_{nk} \in K_n\}} \right) \end{aligned}$$

$$+\sum_{k=u_n}^{v_n} a_{nk}\left(\mathbb{E}X_{nk}I_{\{X_{nk}\in K_n\}}-\mathbb{E}X_{nk}\right)\Bigg\|$$
$$\leq 2\sum_{k=u_n}^{v_n} a_{nk}\mathbb{E}\left\|X_{nk}\right\| I_{\{X_{nk}\notin K_n\}}$$
$$+\mathbb{E}\left\|\sum_{k=u_n}^{v_n} a_{nk}\left(X_{nk}I_{\{X_{nk}\in K_n\}}-\mathbb{E}X_{nk}I_{\{X_{nk}\in K_n\}}\right)\right\|. \tag{4.1}$$

Denote $Y_{nk}:=X_{nk}I_{\{X_{nk}\in K_n\}}$. For any $n\in\mathbb{N}$ we obtain

$$\begin{aligned}
0 &\leq \mathbb{E}\left\|\sum_{k=u_n}^{v_n} a_{nk}\left(Y_{nk}-\mathbb{E}Y_{nk}\right)\right\|^2\\
&= \mathbb{E}\Bigg[\sum_{k=u_n}^{v_n}\Big\{a_{nk}^2\left\langle Y_{nk}-\mathbb{E}Y_{nk},Y_{nk}-\mathbb{E}Y_{nk}\right\rangle\\
&\quad +\sum_{j\neq k}\left\langle a_{nj}\left(Y_{nj}-\mathbb{E}Y_{nj}\right),a_{nk}\left(Y_{nk}-\mathbb{E}Y_{nk}\right)\right\rangle\Big\}\Bigg]\\
&= \sum_{k=u_n}^{v_n}\{a_{nk}^2\left(\mathbb{E}\left\|Y_{nk}\right\|^2-2\mathbb{E}\left\langle Y_{nk},\mathbb{E}Y_{nk}\right\rangle+\left\|\mathbb{E}Y_{nk}\right\|^2\right)\\
&\quad +\sum_{j\neq k}a_{nj}a_{nk}\left(\mathbb{E}\left\langle Y_{nj},Y_{nk}\right\rangle-\mathbb{E}\left\langle Y_{nj},\mathbb{E}Y_{nk}\right\rangle\right.\\
&\quad \left.-\mathbb{E}\left\langle \mathbb{E}Y_{nj},Y_{nk}\right\rangle+\mathbb{E}\left\langle \mathbb{E}Y_{nj},\mathbb{E}Y_{nk}\right\rangle\right)\}\\
&= \sum_{k=u_n}^{v_n}\{a_{nk}^2\left(\mathbb{E}\left\|Y_{nk}\right\|^2-\left\|\mathbb{E}Y_{nk}\right\|^2\right)\\
&\quad +\sum_{j\neq k}a_{nj}a_{nk}\left(\mathbb{E}\left\langle Y_{nj},Y_{nk}\right\rangle-\left\langle \mathbb{E}Y_{nj},\mathbb{E}Y_{nk}\right\rangle\right)\}\\
&\leq \sum_{k=u_n}^{v_n}\{a_{nk}^2\mathbb{E}\left\|Y_{nk}\right\|^2\\
&\quad +\sum_{j\neq k}a_{nj}a_{nk}\left(\mathbb{E}\left\langle Y_{nj},Y_{nk}\right\rangle-\left\langle \mathbb{E}Y_{nj},\mathbb{E}Y_{nk}\right\rangle\right)\}.
\end{aligned} \tag{4.2}$$

We have from (i) and (ii) that

$$\begin{aligned}
0 &\leq \sum_{k=u_n}^{v_n} a_{nk}^2\mathbb{E}\left\|Y_{nk}\right\|^2\\
&\leq h(n)\left(\sup_{u_n\leq k\leq v_n} a_{nk}\right)\sum_{k=u_n}^{v_n} a_{nk}\mathbb{E}\left\|Y_{nk}\right\|\\
&\leq h(n)\left(\sup_{u_n\leq k\leq v_n} a_{nk}\right)\sum_{k=u_n}^{v_n} a_{nk}\mathbb{E}\left\|X_{nk}\right\|\\
&\leq h(n)\left(\sup_{u_n\leq k\leq v_n} a_{nk}\right)\left(\sup_{n\in\mathbb{N}}\sum_{k=u_n}^{v_n} a_{nk}\mathbb{E}\left\|X_{nk}\right\|\right)\to 0.
\end{aligned} \tag{4.3}$$

Now, we have for any $n \in \mathbb{N}$ that

$$
\begin{aligned}
&\sum_{k=u_n}^{v_n} \sum_{j \neq k} a_{nj} a_{nk} \left(\mathbb{E} \langle Y_{nj}, Y_{nk} \rangle - \langle \mathbb{E} Y_{nj}, \mathbb{E} Y_{nk} \rangle \right) \\
&\text{(by } \{K_n\}\text{-non-negativity)} \\
&\leq \sum_{k=u_n}^{v_n} \sum_{j \neq k} a_{nj} a_{nk} \left(\mathbb{E} \langle X_{nj}, X_{nk} \rangle - \langle \mathbb{E} Y_{nj}, \mathbb{E} Y_{nk} \rangle \right) \\
&\text{(by negative correlation)} \\
&\leq \sum_{k=u_n}^{v_n} \sum_{j \neq k} a_{nj} a_{nk} \left(\langle \mathbb{E} X_{nj}, \mathbb{E} X_{nk} \rangle - \langle \mathbb{E} Y_{nj}, \mathbb{E} Y_{nk} \rangle \right) \\
&\leq \sum_{j,k=u_n}^{v_n} a_{nj} a_{nk} \left(\langle \mathbb{E} X_{nj}, \mathbb{E} X_{nk} \rangle - \langle \mathbb{E} Y_{nj}, \mathbb{E} Y_{nk} \rangle \right) \\
&= \sum_{j,k=u_n}^{v_n} a_{nj} a_{nk} \left(\langle \mathbb{E} X_{nj} - \mathbb{E} Y_{nj}, \mathbb{E} X_{nk} \rangle + \langle \mathbb{E} X_{nk} - \mathbb{E} Y_{nk}, \mathbb{E} Y_{nj} \rangle \right) \\
&= \sum_{j,k=u_n}^{v_n} a_{nj} a_{nk} \left(\left\langle \mathbb{E} X_{nj} I_{\{X_{nj} \notin K_n\}}, \mathbb{E} X_{nk} \right\rangle + \left\langle \mathbb{E} X_{nk} I_{\{X_{nk} \notin K_n\}}, \mathbb{E} Y_{nj} \right\rangle \right) \\
&\leq \sum_{j,k=u_n}^{v_n} a_{nj} a_{nk} \left\| \mathbb{E} X_{nj} I_{\{X_{nj} \notin K_n\}} \right\| \left\| \mathbb{E} X_{nk} \right\| \\
&+ \sum_{j,k=u_n}^{v_n} a_{nj} a_{nk} \left\| \mathbb{E} X_{nk} I_{\{X_{nk} \notin K_n\}} \right\| \left\| \mathbb{E} Y_{nj} \right\| \\
&= \left(\sum_{j=u_n}^{v_n} a_{nj} \left\| \mathbb{E} X_{nj} I_{\{X_{nj} \notin K_n\}} \right\| \right) \left(\sum_{k=u_n}^{v_n} a_{nk} \left\| \mathbb{E} X_{nk} \right\| \right) \\
&+ \left(\sum_{k=u_n}^{v_n} a_{nk} \left\| \mathbb{E} X_{nk} I_{\{X_{nk} \notin K_n\}} \right\| \right) \left(\sum_{j=u_n}^{v_n} a_{nj} \left\| \mathbb{E} Y_{nj} \right\| \right) \\
&\leq 2 \left(\sum_{j=u_n}^{v_n} a_{nj} \mathbb{E} \left\| X_{nj} \right\| \right) \left(\sum_{j=u_n}^{v_n} a_{nj} \mathbb{E} \left\| X_{nj} \right\| I_{\{X_{nj} \notin K_n\}} \right) \\
&\text{(by (i))} \\
&= o(1).
\end{aligned}
\tag{4.4}
$$

Hence, from (4.2), (4.3), and (4.4) we obtain

$$
\lim_{n \to \infty} \mathbb{E} \left\| \sum_{k=u_n}^{v_n} a_{nk} \left(Y_{nk} - \mathbb{E} Y_{nk} \right) \right\|^2 = 0
$$

which yields

$$
\lim_{n \to \infty} \mathbb{E} \left\| \sum_{k=u_n}^{v_n} a_{nk} \left(Y_{nk} - \mathbb{E} Y_{nk} \right) \right\| = 0, \tag{4.5}
$$

since convergence in L_2 implies convergence in L_1. Finally, considering (i), (4.1), and (4.5) we get

$$\lim_{n\to\infty} \mathbb{E}\left\|\sum_{k=u_n}^{v_n} a_{nk}(X_{nk} - \mathbb{E}X_{nk})\right\| = 0$$

thereby completing the proof. □

Remark 6. Assume that conditions of Theorem 2 of [13] hold for the Cesàro array. Then, $\{X_{nk}\}$ is h-integrable with respect to the Cesàro array and

$$\lim_{n\to\infty} h^2(n)\sum_{k=1}^{n}\frac{1}{n^2} = \lim_{n\to\infty}\frac{h^2(n)}{n} = 0. \tag{4.6}$$

Now, let $K_n = [-h(n), h(n)]$ for any $n \in \mathbb{N}$. Then $\{X_{nk}\}$ is $\{K_n\}$-compactly uniformly integrable with respect to the Cesàro array. Therefore, (i) of Theorem 3 holds. Moreover, as $h(n) \to \infty$ for $n \to \infty$ (4.6) yields that

$$\lim_{n\to\infty} h(n) \sup_{1\le k\le n}\frac{1}{n} = \lim_{n\to\infty}\frac{h(n)}{n} = 0.$$

Thus, (ii) of Theorem 3 holds. Hence, for the Cesàro array, Theorem 2 of [13] is a consequence of Theorem 3.

Moreover, assume that $\{X_k\}$ is a sequence of pairwise independent random variables which is Cesàro α-integrable for $\alpha \in (0,1)$. If we consider the array $\{X_{nk}\}$ in (3.6), then by taking $K_n = [n^{-\alpha}, n^{\alpha}]$ for any $n \in \mathbb{N}$ we have that $\{X_{nk}\}$ is $\{K_n\}$-compactly uniformly integrable with respect to the Cesàro array. So, (i) of Theorem 3 holds. Moreover, from Remark 5, $\{K_n\}$-non-negativity holds. Finally, we have

$$\lim_{n\to\infty} n^{\alpha}\frac{1}{n} = 0$$

which yields (ii) of Theorem 3. Thus, for $\alpha \in (0,1)$, Theorem 2.1 (a) of [3] is a consequence of Theorem 3.

Declaration of competing interest

None.

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