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On the Weak Law with Random Indices for Arrays of Banach Space Valued Random Elements

Andrew Rosalsky
University of Florida, USA
Andrei Volodin
University of Regina, Canada

Abstract

For a sequence of constants $\{a_n,\ n \geq 1\}$, an array of rowwise independent and stochastically dominated random elements $\{V_{nj},\ j \geq 1, n \geq 1\}$ in a real separable Rademacher type p Banach space for some $p \in [1, 2]$, and a sequence of positive integer-valued random variables $\{T_n,\ n \geq 1\}$, a general weak law of large numbers of the form $\sum_{j=1}^{T_n} a_j(V_{nj} - c_{nj})/b_{\lfloor \alpha_n \rfloor} \xrightarrow{P} 0$ is established, where $\{c_{nj},\ j \geq 1, n \geq 1\}$ is an array of truncated expectations, and $\alpha_n \to \infty, b_n \to \infty$ are suitable sequences. No assumption is made concerning the existence of expected values or absolute moments of the random elements $\{V_{nj},\ j \geq 1, n \geq 1\}$. The current work is a new version of a result of Adler, Rosalsky, and Volodin (*J. Theoret. Probab.* vol. 10, 1997, 605–623).

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1 Introduction

In this paper, for an array $\{V_{nj},\ j \geq 1, n \geq 1\}$ of rowwise independent Banach space valued random elements, a general weak law of large numbers (WLLN) will be established for the weighted sums $\sum_{j=1}^{T_n} a_j V_{nj}$, where $\{T_n,\ n \geq 1\}$ is a sequence of positive integer-valued random variables.

The general setting will now be described. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{X}$ be a real separable Banach space with norm $||\cdot||$. Let $\{V_{nj},\ j \geq 1, n \geq 1\}$ be an array of rowwise independent $\mathcal{X}$-valued random elements defined on $(\Omega, \mathcal{F}, P)$, and let $\{a_n \neq 0,\ n \geq 1\}, \{b_n,\ n \geq 1\}$, and $\{\alpha_n,\ n \geq 1\}$ be sequences of constants with $0 < b_n \to \infty, 1 \leq \alpha_n \to \infty$. Let $\{T_n,\ n \geq 1\}$ be a sequence of positive integer-valued random variables, and let $\{c_{nj},\ j \geq 1, n \geq 1\}$ be a "centering" array consisting of (suitably selected) elements in $\mathcal{X}$. In this paper, the main result, Theorem 3.1, establishes a general WLLN of the form

$$\frac{\sum_{j=1}^{T_n} a_j(V_{nj} - c_{nj})}{b_{\lfloor \alpha_n \rfloor}} \xrightarrow{P} 0 \tag{1.1}$$

where for $x > 0$, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. The number of terms in the sum in (1.1) is random, and the $\{T_n,\ n \geq 1\}$ are referred to as *random indices*.

In the current work, the Banach space $\mathcal{X}$ is assumed to satisfy the geometric condition of being of Rademacher type p for some $p \in [1, 2]$. (Technical definitions such as this will be reviewed in Section 2.) Conditions are placed on the growth behaviour of the constants $\{a_n,\ n \geq 1\}$ and $\{b_n,\ n \geq 1\}$. The random elements $\{V_{nj},\ j \geq 1, n \geq 1\}$ are assumed to be stochastically dominated by a random element V in the sense that (2.1) holds. The tail $P\{||V|| > t\}$ of the distribution of $||V||$ as $t \to \infty$ is controlled by (3.4), as given in Section 3. No conditions are imposed on the joint distributions of the random indices $\{T_n,\ n \geq 1\}$, whose marginal distributions are constrained solely by (3.2), as described in Section 3, and no independence conditions are imposed between $\{T_n,\ n \geq 1\}$ and $\{V_{nj},\ j \geq 1, n \geq 1\}$.

Theorem 3.1 is a new version of Theorem 1 of Adler et al. (1997). Theorem 3.1 was obtained by Adler et al. (1997) with the assumption

$$T_n = \mathcal{O}_P(\alpha_n) \tag{1.2}$$

(that is, $\lim_{\lambda\to\infty} \sup_{n\geq 1} P\{T_n/\alpha_n > \lambda\} = 0$) replaced by the stronger condition

$$P\left\{\frac{T_n}{\alpha_n} > \lambda\right\} = o(1) \text{ as } n \to \infty \text{ for some constant } 0 < \lambda < \infty. \tag{1.3}$$

In Proposition 1.1 below, it will be shown that (1.3) indeed implies (1.2). An example will then be provided wherein (1.3) fails but (1.2) holds. However, the condition (3.3) of Theorem 3.1 is slightly stronger than its counterpart in Theorem 1 of Adler et al. (1997).

PROPOSITION 1.1. *If (1.3) holds, then so does (1.2).*

PROOF. Let λ_0 be a value of λ satisfying (1.3), and let $\varepsilon > 0$ be arbitrary. By (1.3), there exists an integer $N \geq 2$ such that

$$P\left\{\frac{T_n}{\alpha_n} > \lambda_0\right\} \leq \varepsilon \text{ for all } n \geq N.$$

Choose $\lambda_n,\ 1 \leq n \leq N-1$ such that

$$P\left\{\frac{T_n}{\alpha_n} > \lambda_n\right\} \leq \varepsilon \text{ for all } 1 \leq n \leq N-1,$$

and let $\lambda^* = \max\{\lambda_n,\ 0 \leq n \leq N-1\}$. Then, for all $\lambda \geq \lambda^*$,

$$\sup_{n\geq 1} P\left\{\frac{T_n}{\alpha_n} > \lambda\right\} \leq \sup_{n\geq 1} P\left\{\frac{T_n}{\alpha_n} > \lambda^*\right\} \leq \varepsilon.$$

Thus, since $\varepsilon > 0$ is arbitrary,

$$\lim_{\lambda\to\infty} \sup_{n\geq 1} P\left\{\frac{T_n}{\alpha_n} > \lambda\right\} = 0,$$

which establishes (1.2). □

For $x > 0$, let $\lceil x \rceil$ denote the smallest integer greater than or equal to x.

EXAMPLE 1.1. Let $\{\tau_n,\ n \geq 1\}$ be a sequence of identically distributed random variables, where

$$P\{\tau_1 = j\} = \frac{1}{j(j+1)} = \frac{1}{j} - \frac{1}{j+1},\ j \geq 1.$$

Then

$$P\{\tau_1 \geq j\} = \frac{1}{j},\ j \geq 1.$$

Let $T_n = n\tau_n$ and $\alpha_n = n,\ n \geq 1$. Now, for all $0 < \lambda < \infty$ and all $n \geq 1$,

$$P\left\{\frac{T_n}{\alpha_n} > \lambda\right\} = P\{\tau_1 > \lambda\} = \frac{1}{\lceil\lambda\rceil}$$

and so (1.3) fails. On the other hand,

$$\lim_{\lambda\to\infty} \sup_{n\geq 1} P\left\{\frac{T_n}{\alpha_n} > \lambda\right\} = \lim_{\lambda\to\infty} P\{\tau_1 > \lambda\} = 0$$

and so (1.2) holds.

The plan of the paper is as follows. For convenience, technical definitions will be consolidated into Section 2. The main result and three corollaries of it will be established in Section 3, and some final remarks concerning the main result are provided in Section 4.

2 Preliminaries

Throughout this paper, the symbol C denotes a generic constant ($0 < C < \infty$), which is not necessarily the same one in each appearance. Technical definitions relevant to the current work will be discussed in this section.

The *expected value* or *mean* of a random element V, denoted by EV or by $E(V)$, is defined to be the *Pettis integral* provided it exists; that is, V has expected value $EV \in \mathcal{X}$ if $f(EV) = E(f(V))$ for every $f \in \mathcal{X}^*$ where $\mathcal{X}^*$ is the (*dual*) space of all continuous linear functionals on $\mathcal{X}$. If $E\|V\| < \infty$, then (see, e.g., Taylor, 1978, p. 40) V has an expected value.

Let $\{Y_n, n \geq 1\}$ be a symmetric *Bernoulli sequence;* that is, $\{Y_n, n \geq 1\}$ is a sequence of independent and identically distributed (i.i.d.) random variables with $P\{Y_1 = 1\} = P\{Y_1 = -1\} = 1/2$. Let $1 \leqslant p \leqslant 2$. Then $\mathcal{X}$ is said to be of *Rademacher type* p if there exists a constant $0 < C < \infty$ such that

$$E\left\|\sum_{n=1}^{N} Y_n v_n\right\|^p \leqslant C\sum_{n=1}^{N} \|v_n\|^p \text{ for all } N \geq 1 \text{ and } v_n \in \mathcal{X}, 1 \leq n \leq N.$$

Let $\mathcal{X}^\infty = \mathcal{X} \times \mathcal{X} \times \mathcal{X} \times \cdots$ and define

$$\mathcal{C}(\mathcal{X}) = \left\{(v_1, v_2, \dots) \in \mathcal{X}^\infty : \sum_{n=1}^{\infty} Y_n v_n \text{ converges in probability}\right\}.$$

Now the condition that $\mathcal{X}$ is of Rademacher type p is equivalent to the condition that there exists a constant $0 < C < \infty$ such that

$$E\left\|\sum_{n=1}^{\infty} Y_n v_n\right\|^p \leqslant C\sum_{n=1}^{\infty} \|v_n\|^p \text{ for all } (v_1, v_2, \dots) \in \mathcal{C}(\mathcal{X}).$$

This equivalence follows immediately from a famous theorem of Itô and Nisio (1968) (which asserts that convergence in probability and almost sure convergence are equivalent for series of independent random elements) and Fatou's lemma. Moreover, Hoffmann-Jørgensen and Pisier (1976) proved for

$1 \leqslant p \leqslant 2$ that a real separable Banach space is of Rademacher type p if and only if there exists a constant $0 < C < \infty$ such that

$$E\left|\left|\sum_{j=1}^{n} V_j\right|\right|^p \leq C \sum_{j=1}^{n} E||V_j||^p$$

for every finite collection $\{V_1, \ldots, V_n\}$ of independent mean 0 random elements.

If a real separable Banach space is of Rademacher type p for some $1 < p \leq 2$, then it is of Rademacher type q for all $1 \leq q < p$. Every real separable Banach space is of Rademacher type (at least) 1, while the $\mathcal{L}_p$-spaces and l_p-spaces are of Rademacher type $2 \wedge p$ for $p \geq 1$. Every real separable Hilbert space and real separable finite-dimensional Banach space is of Rademacher type 2. In particular, the real line is of Rademacher type 2.

An array of random elements $\{V_{nj}, j \geq 1, n \geq 1\}$ is said to be *stochastically dominated* by a random element V if for some constant $D < \infty$,

$$P\{||V_{nj}|| > t\} \leq DP\{||DV|| > t\},\ t \geq 0, j \geq 1, n \geq 1. \tag{2.1}$$

This condition is, of course, automatic with $V = V_{11}$ and $D = 1$ if the random elements $\{V_{nj}, j \geq 1, n \geq 1\}$ are identically distributed. It follows from Lemma 5.2.2 of Taylor (1978), p. 123 (or Lemma 3 of Wei and Taylor, 1978) that stochastic dominance can be accomplished by the array of random elements having a bounded absolute r^{th} moment ($r > 0$). Specifically, if $\sup_{n\geq 1, j\geq 1} E||V_{nj}||^r < \infty$ for some $r > 0$, then there exists a random element V with $E||V||^p < \infty$ for all $0 < p < r$ such that (2.1) holds with $D = 1$. (The proviso that $r > 1$ in Lemma 5.2.2 of Taylor, 1978, p. 123 (or Lemma 3 of Wei and Taylor, 1978) is not needed, as was pointed out by Adler et al., 1992.)

3 The Main Result

With the preliminaries accounted for, the main result of this paper, Theorem 3.1, may be established. Theorem 3.1 is apparently a new result even when the Banach space is the real line. It should be noted that the first condition of (3.1) ensures that $b_n \to \infty$. However, it is not assumed that $\{b_n, n \geq 1\}$ is monotone. Moreover, the condition (3.4) is in the spirit of the condition $nP\{|X_1| > n\} = o(1)$ of the classical WLLN with random indices for a sequence of i.i.d. random variables $\{X_n,\ n \geq 1\}$ (see, e.g., Chow and

Teicher, 1997, p. 133). Also, note that for all $1 < \lambda < \infty$, if either

$$\sum_{j=1}^{\lfloor \lambda \alpha_n \rfloor} |a_j|^p = \mathcal{O}\left(\sum_{j=1}^{\lfloor \alpha_n \rfloor} |a_j|^p\right) \text{ or } b_{\lfloor \lambda \alpha_n \rfloor} = \mathcal{O}(b_{\lfloor \alpha_n \rfloor}),$$

then (3.3) holds with $\kappa_n = \lfloor \alpha_n \rfloor,\ n \geq 1$ or $\kappa_n = \lfloor \lambda \alpha_n \rfloor,\ n \geq 1$, respectively.

Of course in Theorem 3.1, the larger is the Rademacher type $p \in [1, 2]$, the stronger is the condition on the Banach space. However, there is a trade-off between the Rademacher type and the condition (3.1) when $|a_n| \uparrow$. Specifically, when $|a_n| \uparrow$, then (3.1) is weaker for larger p. To see this, let $1 \leq p_0 < p \leq 2$ and suppose that p_0 satisfies (3.1). It follows from $|a_n| \uparrow$ and the assumption $b_n/|a_n| \uparrow$ that $b_n \uparrow$. Then

$$\begin{aligned}
\frac{\sum_{j=1}^n |a_j|^p}{b_n^p} &= \frac{\sum_{j=1}^n |a_j|^{p_0} |a_j|^{p-p_0}}{b_n^{p_0} b_n^{p-p_0}} \\
&\leq \frac{\sum_{j=1}^n |a_j|^{p_0} \left(\frac{|a_j|}{b_j}\right)^{p-p_0}}{b_n^{p-p_0}} \text{ (since } b_n \uparrow) \\
&\leq \left(\frac{|a_1|}{b_1}\right)^{p-p_0} \frac{\sum_{j=1}^n |a_j|^{p_0}}{b_n^{p_0}} \text{ (since } b_n/|a_n| \uparrow) \\
&= o(1) \text{ (since } p_0 \text{ satisfies (3.1))},
\end{aligned}$$

and so p satisfies the first condition of (3.1). Similarly, when $|a_n| \uparrow$, the third condition of (3.1) is weaker for larger p whereas the second condition of (3.1) is automatic irrespective of the value of p.

THEOREM 3.1. *Let $\{V_{nj},\ j \geq 1, n \geq 1\}$ be an array of rowwise independent random elements in a real separable Rademacher type p Banach space for some $p \in [1, 2]$, and suppose that $\{V_{nj},\ j \geq 1, n \geq 1\}$ is stochastically dominated by a random element V. Let $\{a_n,\ n \geq 1\}$ and $\{b_n,\ n \geq 1\}$ be sequences of constants with $a_n \neq 0, b_n > 0,\ n \geq 1$ and suppose that $b_n/|a_n| \uparrow$ and*

$$\sum_{j=1}^n |a_j|^p = o(b_n^p), \sum_{j=1}^n |a_j|^p = \mathcal{O}(n|a_n|^p), \text{ and } \sum_{j=1}^n \frac{b_j^p}{j^2 |a_j|^p} = \mathcal{O}\left(\frac{b_n^p}{\sum_{j=1}^n |a_j|^p}\right). \tag{3.1}$$

Let $\{T_n, n \geq 1\}$ be a sequence of positive integer-valued random variables and let $1 \leq \alpha_n \to \infty$ be constants such that

$$T_n = \mathcal{O}_P(\alpha_n). \tag{3.2}$$

Suppose that for all constants $0 < \lambda < \infty$, there exists a sequence of integers $\{\kappa_n,\ n \geq 1\}$ such that

$$\kappa_n \geq \lfloor \alpha_n \rfloor,\ n \geq 1 \text{ and } b_{\lfloor \alpha_n \rfloor}^{-p} \sum_{j=1}^{\lfloor \lambda \alpha_n \rfloor} |a_j|^p = \mathcal{O}\left(b_{\kappa_n}^{-p} \sum_{j=1}^{\kappa_n} |a_j|^p \right). \tag{3.3}$$

Then if

$$nP\left\{ ||DV|| > \frac{b_n}{|a_n|} \right\} = o(1), \tag{3.4}$$

where D is as in (2.1), then the WLLN

$$\frac{\sum_{j=1}^{T_n} a_j \left(V_{nj} - E(V_{nj} I(||V_{nj}|| \leq b_{\lfloor \alpha_n \rfloor}/|a_{\lfloor \alpha_n \rfloor}|)) \right)}{b_{\lfloor \alpha_n \rfloor}} \xrightarrow{P} 0 \tag{3.5}$$

holds.

Proof. Set

$$c_0 = 0,\ \ c_n = \frac{b_n}{|a_n|},\ \ U_{nj} = V_{nj} I(||V_{nj}|| \leq c_{\lfloor \alpha_n \rfloor}),\ \ j \geq 1,\ \ n \geq 1.$$

First, it will be verified that

$$\frac{\sum_{j=1}^{T_n} a_j (V_{nj} - U_{nj})}{b_{\lfloor \alpha_n \rfloor}} \xrightarrow{P} 0. \tag{3.6}$$

Let $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ be arbitrary. By (3.2), we can choose $\lambda_0 > 0$ such that

$$\sup_{k \geq 1} P\left\{ \frac{T_k}{\alpha_k} > \lambda_0 \right\} \leq \varepsilon_2. \tag{3.7}$$

Then, for $n \geq 1$,

$$
\begin{aligned}
&P\left\{\frac{\|\sum_{j=1}^{T_n} a_j(V_{nj}-U_{nj})\|}{b_{\lfloor \alpha_n \rfloor}} > \varepsilon_1\right\} \\
&\quad \leq P\left\{\sum_{j=1}^{T_n} a_j V_{nj} \neq \sum_{j=1}^{T_n} a_j U_{nj}\right\} \\
&\quad \leq P\left\{\left[\sum_{j=1}^{T_n} a_j V_{nj} \neq \sum_{j=1}^{T_n} a_j U_{nj}\right] \bigcap [T_n \leq \lambda_0 \alpha_n]\right\} + P\{T_n > \lambda_0 \alpha_n\} \\
&\quad \leq P\left\{\bigcup_{j=1}^{\lfloor \lambda_0 \alpha_n \rfloor} [\|V_{nj}\| > c_{\lfloor \alpha_n \rfloor}]\right\} + \varepsilon_2 \qquad \text{(by (3.7))} \\
&\quad \leq \sum_{j=1}^{\lfloor \lambda_0 \alpha_n \rfloor} P\{\|V_{nj}\| > c_{\lfloor \alpha_n \rfloor}\} + \varepsilon_2 \\
&\quad \leq D \lfloor \lambda_0 \alpha_n \rfloor P\{\|DV\| > c_{\lfloor \alpha_n \rfloor}\} + \varepsilon_2 \qquad \text{(by (2.1))} \\
&\quad = (1+o(1)) D \lambda_0 \lfloor \alpha_n \rfloor P\{\|DV\| > c_{\lfloor \alpha_n \rfloor}\} + \varepsilon_2 \\
&\quad = o(1) + \varepsilon_2 \qquad \text{(by (3.4))}
\end{aligned}
$$

thereby establishing (3.6), since $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ are arbitrary.

The proof will thus be completed if it can be demonstrated that

$$\frac{\sum_{j=1}^{T_n} a_j(U_{nj} - EU_{nj})}{b_{\lfloor \alpha_n \rfloor}} \xrightarrow{P} 0. \tag{3.8}$$

To this end, again let $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ be arbitrary and let λ_0 be as in (3.7). Let $\{\kappa_n,\ n \geq 1\}$ be a sequence of integers corresponding to λ_0 and satisfying (3.3). Then for $n \geq 1$, arguing as in the proof of Theorem 1 of Adler et al. (1997) wherein the Rademacher type p hypothesis and (2.1) are utilized,

$$P\left\{\frac{||\sum_{j=1}^{T_n} a_j(U_{nj}-EU_{nj})||}{b_{\lfloor\alpha_n\rfloor}} > \varepsilon_1\right\}$$

$$\leq P\left\{\left[\frac{||\sum_{j=1}^{T_n} a_j(U_{nj}-EU_{nj})||}{b_{\lfloor\alpha_n\rfloor}} > \varepsilon_1\right]\bigcap [T_n\leq\lambda_0\alpha_n]\right\} + P\{T_n>\lambda_0\alpha_n\}$$

$$\leq P\left\{\bigcup_{k=1}^{\lfloor\lambda_0\alpha_n\rfloor}\left[\left|\left|\sum_{j=1}^{k} a_j(U_{nj}-EU_{nj})\right|\right| > \varepsilon_1 b_{\lfloor\alpha_n\rfloor}\right]\right\} + \varepsilon_2 \qquad \text{(by (3.7))}$$

$$\leq \left(\frac{C}{b^p_{\kappa_n}}\sum_{j=1}^{\kappa_n}|a_j|^p\right)\sum_{k=1}^{\kappa_n}\frac{c_k^p-c_{k-1}^p}{k}kP\{\|DV\|>c_{k-1}\}+\varepsilon_2$$

$$= o(1)+\varepsilon_2,$$

since $\kappa_n \geq \lfloor\alpha_n\rfloor \to \infty$, and it was shown in Adler et al. (1997) that

$$\left(\frac{1}{b_n^p}\sum_{j=1}^{n}|a_j|^p\right)\sum_{k=1}^{n}\frac{c_k^p-c_{k-1}^p}{k}kP\{\|DV\|>c_{k-1}\}=o(1).$$

Since $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ are arbitrary, (3.8) holds, thereby completing the proof of the theorem. □

The first corollary is also apparently a new result even when the Banach space is the real line. The condition (3.10) is of course weaker than $E\|V_{11}\| < \infty$.

COROLLARY 3.1. *Let $\{V_{nj},\ j \geq 1, n \geq 1\}$ be an array of identically distributed and rowwise independent random elements in a real separable Rademacher type p Banach space for some $p \in (1,2]$. Let $\{T_n, n\geq 1\}$ be a sequence of positive integer-valued random variables such that*

$$T_n = \mathcal{O}_P(n). \tag{3.9}$$

Then if

$$nP\{||V_{11}|| > n\} = o(1), \tag{3.10}$$

then the WLLN

$$\frac{\sum_{j=1}^{T_n} V_{nj}}{n} - \frac{T_n}{n}E(V_{11}I(||V_{11}||\leq n)) \xrightarrow{P} 0 \tag{3.11}$$

holds.

Proof. Let $a_n = 1$ and $b_n = \alpha_n = n,\ n \geq 1$. Then (3.1) holds. Note that for all $0 < \lambda < \infty$, (3.3) holds with $\kappa_n = n,\ n \geq 1$. The conclusion (3.11) follows directly from Theorem 3.1. □

Remark 3.1. Apropos of Corollary 3.1, if $E\|V_{11}\| < \infty$ and $EV_{11} = 0$, then (3.10) holds and by the Lebesgue dominated convergence theorem

$$\lim_{n\to\infty} E(V_{11}I(||V_{11}|| \leq n)) = EV_{11} = 0.$$

Then (3.9) ensures that

$$\frac{T_n}{n} E(V_{11}I(||V_{11}|| \leq n)) \xrightarrow{P} 0,$$

which, when added to (3.11), yields

$$\frac{\sum_{j=1}^{T_n} V_{nj}}{n} \xrightarrow{P} 0. \tag{3.12}$$

However, $EV_{11} = 0$ can hold even when $E\|V_{11}\| = \infty$ (for an example, see Taylor, 1978, p. 41). In the next corollary, we show that under the hypotheses of Corollary 3.1, if $EV_{11} = 0$, then (3.12) holds irrespective of whether $E\|V_{11}\|$ is finite or infinite.

Corollary 3.2. *Under the hypotheses of Corollary 3.1, if* $EV_{11} = 0$, *then (3.12) holds.*

Proof. If

$$\lim_{n\to\infty} E(V_{11}I(||V_{11}|| \leq n)) = 0, \tag{3.13}$$

then recalling (3.9)

$$\frac{T_n}{n} E(V_{11}I(||V_{11}|| \leq n)) \xrightarrow{P} 0$$

and the conclusion (3.12) follows from (3.11). Thus we need to verify (3.13). To this end, we appeal to the Orlicz-Pettis theorem (see, e.g., Pettis, 1938; or Hille and Phillips, 1957, p. 78; or, for a simpler proof, Brooks, 1969), which asserts for a random element V_{11} such that EV_{11} exists that the Banach space valued set function ν defined by

$$\nu(A) = E(V_{11}I(A)),\ A \in \mathcal{F}$$

is countably additive. Consequently, it follows from $EV_{11} = 0$ that

$$\begin{aligned}
\lim_{n\to\infty} E(V_{11}I(||V_{11}|| \le n)) &= \lim_{n\to\infty} \nu(||V_{11}|| \le n) \\
&= \lim_{n\to\infty}\left[\nu(\|V_{11}\| = 0) + \sum_{j=1}^{n} \nu(j-1 < \|V_{11}\| \le j)\right] \\
&= \nu(\|V_{11}\| = 0) + \sum_{j=1}^{\infty} \nu(j-1 < \|V_{11}\| \le j) \\
&= \nu\left([\|V_{11}\| = 0] \cup \bigcup_{j=1}^{\infty}[j-1 < \|V_{11}\| \le j]\right) \\
&= \nu(\Omega) = EV_{11} = 0,
\end{aligned}$$

thereby establishing (3.13). □

Remark 3.2. The special case $T_n \equiv n$ of Corollary 3.2 should be compared with Corollary 1 of Rosalsky and Taylor (2004).

The example of Adler and Rosalsky (1991), which was also considered by Adler et al. (1997), shows that Theorem 3.1 can fail if the norming sequence $\{b_{\lfloor \alpha_n \rfloor},\ n \ge 1\}$ is replaced by $\{b_{T_n},\ n \ge 1\}$. In the ensuing corollary, additional conditions are provided under which the norming sequence can be taken to be $\{b_{T_n},\ n \ge 1\}$ in Theorem 3.1. Note that the pair of conditions (3.2) and (3.14) is equivalent to the single condition

$$\lim_{\lambda_1\to 0,\lambda_2\to\infty} \inf_{n\ge 1} P\left\{\lambda_1 \le \frac{T_n}{\alpha_n} \le \lambda_2\right\} = 1,$$

which is substantially weaker than $T_n/\alpha_n \xrightarrow{P} c$ for some constant $0 < c < \infty$.

Corollary 3.3. *Let* $\{V, V_{nj},\ j \ge 1, n \ge 1\}$, $\{a_n, n \ge 1\}$, $\{b_n, n \ge 1\}$, $\{\alpha_n, n \ge 1\}$, *and* $\{T_n, n \ge 1\}$ *satisfy the hypotheses of Theorem 3.1 and suppose, additionally, that* $b_n \uparrow$,

$$\frac{\alpha_n}{T_n} = \mathcal{O}_P(1), \tag{3.14}$$

and

$$b_{\lfloor \alpha_n \rfloor} = \mathcal{O}(b_{\lfloor \lambda\alpha_n \rfloor}) \text{ for all } 0 < \lambda < 1. \tag{3.15}$$

Then the WLLN

$$\frac{\sum_{j=1}^{T_n} a_j \left(V_{nj} E(V_{nj} I(\|V_{nj}\| \le b_{\lfloor \alpha_n \rfloor}/|a_{\lfloor \alpha_n \rfloor}|))\right)}{b_{T_n}} \xrightarrow{P} 0$$

holds.

Proof. In view of Theorem 3.1, it suffices to show that

$$\frac{b_{\lfloor \alpha_n \rfloor}}{b_{T_n}} = \mathcal{O}_P(1). \tag{3.16}$$

Let $\varepsilon > 0$ be arbitrary. By (3.14), there exists a constant $0 < \lambda_1 < 1$ such that

$$\sup_{n \ge 1} P\{T_n < \lambda_1 \alpha_n\} \le \varepsilon.$$

By (3.15), there exists a constant C such that

$$b_{\lfloor \alpha_n \rfloor} \le C b_{\lfloor \lambda_1 \alpha_n \rfloor}, \ n \ge 1.$$

Then for all $\lambda \ge C$ and $n \ge 1$,

$$\begin{aligned} \left[b_{\lfloor \alpha_n \rfloor} > \lambda b_{T_n}\right] \cap [T_n \ge \lambda_1 \alpha_n] \quad &\subseteq \quad \left[b_{\lfloor \alpha_n \rfloor} > \lambda b_{\lfloor \lambda_1 \alpha_n \rfloor}\right] \ \text{(since } b_n \uparrow) \\ &= \quad \emptyset. \end{aligned}$$

Thus,

$$\begin{aligned} &\lim_{\lambda \to \infty} \sup_{n \ge 1} P\left\{\frac{b_{\lfloor \alpha_n \rfloor}}{b_{T_n}} > \lambda\right\} \\ &\quad \le \lim_{\lambda \to \infty} \sup_{n \ge 1} \left(P\left\{\left[b_{\lfloor \alpha_n \rfloor} > \lambda b_{T_n}\right] \cap [T_n \ge \lambda_1 \alpha_n]\right\} + P\{T_n < \lambda_1 \alpha_n\}\right) \\ &\quad \le \varepsilon \end{aligned}$$

proving (3.16) since $\varepsilon > 0$ is arbitrary. □

4 Final Remarks

We close with several remarks pertaining to Theorem 3.1.

Remark 4.1. If $p = 1$, the hypothesis of independence in Theorem 3.1 is not needed. The argument is the same as that for the similar remark in Adler et al. (1997) pertaining to Theorem 1 there.

Remark 4.2. The following example of Beck (1963) (also considered by Adler et al., 1991; and Adler et al., 1997) shows that Theorem 3.1 can fail if the Rademacher type p hypothesis is dispensed with.

Consider the real separable Banach space ℓ_1 of absolutely summable real sequences $v = \{v_i, i \geq 1\}$ with norm $\|v\| = \sum_{i=1}^{\infty} |v_i|$. Let $v^{(j)}$ be the element having 1 in its j^{th} position and 0 elsewhere. Define a sequence $\{V_j, j \geq 1\}$ of random elements in ℓ_1 by requiring the $\{V_j, j \geq 1\}$ to be a collection of independent random elements with

$$P\{V_j = v^{(j)}\} = P\{V_j = -v^{(j)}\} = \frac{1}{2}, \quad j \geq 1.$$

Set

$$V_{nj} = V_j, \ 1 \leq j \leq n, \quad V_{nj} = 0, \ j \geq n+1, \ n \geq 1.$$

Let $p \in (1, 2]$, $a_n = 1$, $b_n = \alpha_n = T_n = n$, $n \geq 1$. Then (2.1) (with $V = V_1$ and $D = 1$), (3.1), (3.2), (3.3), and (3.4) hold. Since

$$P\left\{ \frac{\| \sum_{j=1}^{n} V_{nj} \|}{n} = 1 \right\} = 1, \ n \geq 1,$$

the conclusion (3.5) of Theorem 3.1 fails. It is well known (see, e.g., Adler et al., 1991) that ℓ_1 is not of Rademacher type p for any $p \in (1, 2]$. Thus the Rademacher type p hypothesis cannot be dispensed with in Theorem 3.1. We also note that while ℓ_1 is of Rademacher type 1, with the above choice of $\{a_n, \ n \geq 1\}$ and $\{b_n, \ n \geq 1\}$, the first and the third conditions of (3.1) fail when $p = 1$.

Remark 4.3. A perusal of Example 1 of Adler et al. (1997) reveals that in Theorem 3.1, the corresponding strong law of large numbers is not valid; that is, almost sure convergence does not necessarily hold in the conclusion (3.5) of Theorem 3.1.

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Andrew Rosalsky
Department of Statistics
University of Florida
Gainesville, FL 32611, USA
E-mail: rosalsky@stat.ufl.edu

Andrei Volodin
Department of Mathematics and Statistics
University of Regina
Regina, Saskatchewan S4S 0A2, Canada
E-mail: andrei@math.uregina.ca

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