MATH 110: Calculus I
Detailed Solution to Practice Problems
University of Regina

Paul Arnaud Songhafoouo Tsopméné
Contents

1 Practice Problems

1.1 PB1: The Tangent Problem and the Limit of a Function ........................................... 5
1.2 PB2: Calculating Limits Using the Limits Laws ......................................................... 6
1.3 PB3: Calculating Limits Using the Limits Laws (Continued) and Continuity ............. 7
1.4 PB4: Continuity (Continued), Derivatives (using the limit definition) .................... 9
1.5 PB5: Differentiation Formulas .................................................................................... 10
1.6 PB6: Derivatives of Trigonometric Functions and The Chain Rule ......................... 12
1.7 PB7: The Chain Rule (Continued) and Implicit Differentiation ................................. 13
1.8 PB8: Related Rates .................................................................................................... 15
1.9 PB9: The Mean Value Theorem, Increasing/Decreasing, and Limits at Infinity .......... 15
1.10 PB10: Limits at Infinity (Continued) and Summary of Curve Sketching ................. 16
1.11 PB11: Optimization Problems .................................................................................. 18
1.12 PB12: Indefinite Integrals and The Substitution Rule (SR) ...................................... 18
1.13 PB13: Definite Integrals, The FTC (parts 1 and 2), The SR, and Areas Between Curves ... 20

2 Solution to Practice Problems

2.1 Solution to PB1 ........................................................................................................... 23
2.2 Solution to PB2 ........................................................................................................... 25
2.3 Solution to PB3 ........................................................................................................... 28
2.4 Solution to PB4 ........................................................................................................... 33
2.5 Solution to PB5 ........................................................................................................... 38
2.6 Solution to PB6 ........................................................................................................... 52
2.7 Solution to PB7 ........................................................................................................... 58
2.8 Solution to PB8 ........................................................................................................... 65
2.9 Solution to PB9 ........................................................................................................... 71
2.10 Solution to PB10 ........................................................................................................ 78
2.11 Solution to PB11 ................................................................. 86
2.12 Solution to PB12 ................................................................. 91
2.13 Solution to PB13 ................................................................. 98
Chapter 1

Practice Problems

1.1 PB1: The Tangent Problem and the Limit of a Function

Sections covered:

- Section 1.4: The Tangent Problem.
- Section 1.5: The Limit of a Function.

Section 1.4: The Tangent Problem

1. Let \( f \) be the function defined by \( f(x) = 4x^2 \). Let \( x \) be different from 3. What is the slope \( m_x \) of the line through the points \((3,36)\) and \((x,4x^2)\)? Simplify your answer as much as possible.

2. Let \( f \) be the function defined by \( f(x) = \frac{2}{3x} \). Let \( x \) be different from 0 and 1. What is the slope \( m_x \) of the line through the points \((1,\frac{2}{3})\) and \((x,\frac{2}{3x})\)? Simplify your answer as much as possible.

3. The point \( P(2, -1) \) lies on the curve \( y = \frac{1}{1-x} \).
   
   (a) If \( Q \) is the point \((x, \frac{1}{1-x})\), use your calculator to find the slope of the secant line \( PQ \) for the following values of \( x \): 1.5, 1.9, 1.99, 1.999, 2.5, 2.1, 2.01, 2.001.
   
   (b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at \( P(2, -1) \).
   
   (c) Using the slope from part (b), find an equation of the tangent line to the curve at \( P(2, -1) \).

4. Let \( f \) be the function defined by \( f(x) = -\frac{1}{x^2} \). Let \( x \) be different from 0 and 2.
   
   (a) What is the slope \( m_x \) of the line through the points \((2, -\frac{1}{4})\) and \((x, -\frac{1}{2x})\)? Simplify your answer as much as possible.
   
   (b) Guess the value of \( \lim_{x \to 2} m_x \), and determine an equation for the line tangent to the graph of \( f \) at \((2, -\frac{1}{4})\).

Section 1.5: The Limit of a Function
CHAPTER 1. PRACTICE PROBLEMS

1. Use the given graph of \( f \) (see Figure 1.1) to state the value of each quantity, if it exists. If it does not exist, explain why.

(a) \( \lim_{x \to 2^-} f(x) \); (b) \( \lim_{x \to 2^+} f(x) \); (c) \( \lim_{x \to 2} f(x) \); (d) \( f(2) \); (e) \( \lim_{x \to 4} f(x) \); (f) \( f(4) \).

![Figure 1.1](image.png)

2. For the function \( g \) whose graph is given (see Figure 1.2), state the value of each quantity, if it exists. If it does not exist, explain why.

(a) \( \lim_{x \to 0^-} g(t) \); (b) \( \lim_{x \to 0^+} g(t) \); (c) \( \lim_{x \to 0} g(t) \); (d) \( \lim_{x \to 2^-} g(t) \); (e) \( \lim_{x \to 2^+} g(t) \); (f) \( \lim_{x \to 2} g(t) \); (g) \( g(2) \); (h) \( \lim_{x \to 4} g(t) \).

1.2 PB2: Calculating Limits Using the Limits Laws

Section covered: Section 1.6: Calculating Limits Using the Limits Laws.

**Section 1.6: Calculating Limits Using the Limits Laws**

Evaluate the following limits.

1. \( \lim_{x \to 1} \frac{9x^2-4}{x+1} \)
2. \( \lim_{x \to 3} \frac{x^2-2x-3}{x^2-9} \)
3. \( \lim_{x \to 0} \frac{1-\sqrt{1-x^2}}{x} \)
4. \( \lim_{x \to -5} \frac{x^2+3x-10}{x^2+6x+5} \)
5. \( \lim_{x \to 4} \frac{\sqrt{x}-2}{\frac{x}{3x+7}} \)
6. \( \lim_{x \to 2} \frac{\sqrt{2x+5} - \sqrt{x+7}}{x-2} \)

7. \( \lim_{x \to -2} \frac{-5x^2 - 10x}{6+3x} \)

8. \( \lim_{x \to 0} \sqrt{-x^2 + 4x + 2} \)

9. \( \lim_{x \to -7} \frac{x}{x-1} \)

10. \( \lim_{x \to 3} \frac{x-3}{\sqrt{x^2 - 5x + 10} - 2} \)

11. \( \lim_{x \to 1} \frac{x-1}{\sqrt{x^2 - 1}} \)

1.3 PB3: Calculating Limits Using the Limits Laws (Continued) and Continuity

Sections covered:

- Section 1.6: Calculating Limits Using the Limits Laws (Continued).
- Section 1.8: Continuity.

Section 1.6: Calculating Limits Using the Limits Laws (Continued)

Evaluate the following limits (if they exist).
1. \( \lim_{x \to 3} \frac{2x+1}{x^2+9} \)

2. \( \lim_{x \to 2} \frac{4-x}{x^2+1} \)

3. \( \lim_{x \to -2} \frac{-5}{(x+2)^2} \)

4. \( \lim_{x \to 1} \frac{x^2+3x-4}{x^2-2x+1} \)

5. \( \lim_{x \to 0} \frac{1}{x^2} \)

6. \( \lim_{x \to 2} f(x) \), where \( f(x) = \begin{cases} \sqrt{-2x+4} & \text{if } x \leq 2 \\ x-2 & \text{if } x > 2 \end{cases} \)

7. \( \lim_{x \to 5} f(x) \), where \( f(x) = \begin{cases} x^2+1 & \text{if } x < 5 \\ x+20 & \text{if } x \geq 5 \end{cases} \)

8. (a) \( \lim_{x \to -3} f(x) \), (b) \( \lim_{x \to 0} f(x) \), where \( f(x) = \begin{cases} \frac{7}{x-3} & \text{if } x < -3 \\ 2x+3 & \text{if } -3 < x < 0 \\ \frac{x^2+2x-3}{x-1} & \text{if } x \geq 0 \end{cases} \)

9. \( \lim_{x \to 0} f(x) \), where \( f(x) = \begin{cases} \sqrt{x-1} & \text{if } x > 0 \\ \frac{x^2}{x^4} & \text{if } x \leq 0 \end{cases} \)

10. \( \lim_{x \to 2} \sin(x^2-3x+2) \)

11. \( \lim_{x \to -3} \cos\left(\frac{x^2-9}{x+3}\right) \)

12. \( \lim_{x \to \frac{\pi}{2}} \left(\sin(2x) - 7\cos(4x)\right) \)

13. \( \lim_{x \to 0} \frac{\tan x}{x} \)

14. \( \lim_{x \to 0} \frac{9\sin^2 x}{2x} \)

15. \( \lim_{x \to 0} \left(\frac{4x^2-5\sin x}{7x}\right) \)

16. \( \lim_{x \to 0} x^2 \sin\left(\frac{1}{x^2}\right) \)

17. \( \lim_{x \to 0} |x| \cos\left(\frac{3}{x^3}\right) \)

Section 1.8: Continuity

1. Let \( f(x) = \frac{x^2-16}{x-4} \). Is \( f \) continuous at 4?

2. Let \( f(x) = \begin{cases} x^2 + 2x & \text{if } x \leq -2 \\ \sqrt{x+3} & \text{if } x > -2 \end{cases} \). Is \( f \) continuous at -2?

3. Let \( f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1-x^2 & \text{if } x > 0 \end{cases} \). Is \( f \) continuous at 0?
1.4 PB4: Continuity (Continued), Derivatives (using the limit definition)

sections covered:

- Section 1.8: Continuity (Continued).
- Section 2.1: Derivatives at a Specific Point.
- Section 2.2: The Derivatives as a Function.

Section 1.8: Continuity (Continued)

1. Show that the function \( f(x) = \sqrt{25 - x^2} \) is continuous on \([-5, 5]\).

2. Let \( f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \). Show that \( f \) is continuous everywhere.

3. Let \( f(x) = \begin{cases} x + 2 & \text{if } x \leq a \\ x^2 & \text{if } x > a \end{cases} \). Find all values of \( a \in \mathbb{R} \) that make \( f \) continuous on \( \mathbb{R} = (-\infty, +\infty) \).

4. Find all constants \( c \) such that the function

\[
    f(c) = \begin{cases} 3x^2 + cx & \text{if } x \leq 1 \\ x + c & \text{if } x > 1 \end{cases}
\]

is continuous on \(( -\infty, +\infty )\).

5. Where are the following functions continuous? (a) \( f(x) = \frac{x+1}{x^2+x-2} \); (b) \( g(x) = \sqrt{-3x+12} \); (c) \( h(x) = \cos(x^3+1) \); (d) \( l(x) = \frac{1}{\sqrt{x+2}-3} \).

6. Show that there is a root of the equation \( x^4 - x - 1 = 0 \) between \(-1\) and 0.

7. Show that there is a root of the equation \( 2x^3 - 4x^2 = -1 \) between 0 and 1.

Section 2.1: Derivatives at a Specific Point

1. Find an equation of the tangent line to \( f(x) = -x^2 \) at \((1, -1)\).
2. Find an equation of the tangent line to \( f(x) = \sqrt{x} \) at \((4, 2)\).

3. Let \( f(x) = -3x^2 + 4x \). Find the derivative of \( f \) at \( a = 0 \) using the limit definition of derivative.

4. Let \( f(x) = x^3 - x \). Find the derivative of \( f \) at \( a = -1 \) using the limit definition of derivative.

5. Let \( f(x) = \frac{x}{x+1} \). Find the derivative of \( f \) at \( a = 2 \) using the limit definition of derivative.

Section 2.2: The Derivatives as a Function

For the following derivatives, use the limit definition.

1. Let \( f(x) = -3x + 4 \). Find \( f'(x) \).
2. Let \( f(x) = -5x^2 + 7 \). Find \( f'(x) \).
3. Let \( f(x) = \frac{1}{x} \). Find \( f'(x) \).
4. Let \( f(x) = \frac{1}{\sqrt{x}} \). Find \( f'(x) \).

1.5 PB5: Differentiation Formulas

Section covered: Section 2.3: Differentiation Formulas.

Section 2.3: Differentiation Formulas

(i) Find the derivative of each of the following functions.

1. \( f(x) = 2018 \)
2. \( f(x) = 3^x \)
3. \( f(x) = \pi^2 \)
4. \( f(x) = x^6 \)
5. \( f(x) = 3x^{-2} \)
6. \( f(x) = \frac{x^3}{3} \)
7. \( f(x) = \frac{1}{4} x^{\frac{4}{3}} \)
8. \( f(x) = \sqrt{x^3} \)
9. \( f(x) = \frac{2}{x^2} \)
10. \( f(x) = \frac{1}{\sqrt{x}} \)
11. \( f(x) = \sqrt{x^5} \)
12. \( f(x) = \frac{\sqrt{x^5}}{2} \)
13. \( f(x) = \frac{x^2}{x} \)
14. \( f(x) = x + 1 \)
15. $f(x) = x^2 - \frac{2x}{3}$
16. $f(x) = \frac{x-\sqrt{2}}{2}$
17. $f(x) = -3x^4 - 2x^3 + x^2 - 1$
18. $f(x) = \frac{2x^3-3x^2}{4}$
19. $f(x) = x^{\frac{3}{2}} - x^{\frac{3}{4}}$
20. $f(x) = x^2 - \frac{1}{x}$
21. $f(x) = 1.4x^5 - 2.5x^2 + 3.8$
22. $f(x) = \frac{x}{\sqrt{x}}$
23. $f(x) = \frac{\sqrt{x}+x}{x^2}$
24. $f(x) = (3x + 4)(x - 5)$
25. $f(x) = (5x^2 - 2)(x^3 + 3x)$
26. $f(x) = (x^3 + 1)(2x^2 - 4x - 1)$
27. $f(x) = \frac{x^2+4x+3}{\sqrt{x}}$
28. $f(x) = (\frac{1}{x^2} - \frac{3}{x})(x + 5x^3)$
29. $f(x) = \frac{5x+1}{5x-1}$
30. $f(x) = \frac{1+2x}{3-4x}$
31. $f(x) = \frac{x^2+1}{x^3-1}$
32. $f(x) = \frac{3x^3}{x^2-4x+3}$
33. $f(x) = \frac{1}{x^2+2x^2-1}$
34. $f(x) = \frac{\sqrt{x}}{3+x}$
35. $f(x) = \frac{2x^5+x^4-6x}{x}$
36. $f(x) = \frac{x}{x^2+1}$

(ii) Find an equation of the tangent line to the curve at the given point.

1. $y = \frac{2x}{x+1}, P(1,1)$.
2. $y = 2x^3 - x^2 + 2, P(1,3)$.

(iii) Find the points on the curve $y = \frac{x^2-2x+1}{x-3}$ where the tangent line is horizontal.

(iv) If $h(x) = \sqrt{x}g(x)$, where $g(4) = 8$ and $g'(4) = 7$, find $h'(4)$. 

1.6 PB6: Derivatives of Trigonometric Functions and The Chain Rule

Sections covered:

- Section 2.4: Derivatives of Trigonometric Functions.
- Section 2.5: The Chain Rule.

Section 2.4: Derivatives of Trigonometric Functions

Find the derivative of each of the following functions.

1. \( f(x) = -5 \cos x \)
2. \( f(x) = \frac{\sin x}{x} - x^3 \)
3. \( f(x) = 3 \sin x - 8 \cos x + 2 \tan x \)
4. \( f(x) = 7 \sec x - \csc x + \cot x - 6 \)
5. \( f(x) = x \cot x \)
6. \( f(x) = \sin x \cos x \)
7. \( f(x) = \frac{\sin x}{x} \)
8. \( f(x) = \frac{\sec x}{\csc x} \)
9. \( f(x) = x \sin x + \frac{\cos x}{x} \)
10. \( f(x) = x^9 + \frac{9\sqrt{x}}{x} + \tan x \)
11. \( f(x) = x^2 \cos x - 2 \tan x + 3 \)
12. \( f(x) = x \cos x + x^2 \sin x \)
13. \( f(x) = \frac{\sin x}{1 + \tan x} \)
14. \( f(x) = \frac{1}{\cos x} + \frac{\csc x}{3} - \frac{1}{\sqrt{x}} \)
15. \( f(x) = x \cos x \sin x \)

Section 2.5: The Chain Rule

Find the derivative of each of the following functions.

1. \( f(x) = (2x)^5 \)
2. \( f(x) = (-3x + 4)^8 \)
3. \( f(x) = 3(5x - 1)^7 \)
4. \( f(x) = (x^2 - 1)^\frac{2}{5} \)
5. \( f(x) = (-x^3 + 2x + 1) \)
6. \( f(x) = -2(5x^6 - 2x)^4 \)
7. \( f(x) = (2 - \sin x)^\frac{3}{2} \)
8. \( f(x) = \sqrt{x^2 - x} \)
9. \( f(x) = \frac{1}{\sqrt{x-1}} \)
10. \( f(x) = (5x + 4)(x^3 + 1)^6 \)
11. \( f(x) = x\sqrt{2 - x^2} \)
12. \( f(x) = (x^2 + 1)^3(x^2 + 2)^6 \)
13. \( f(x) = \frac{(x+1)^5}{x^3+1} \)
14. \( f(x) = x^2 \sqrt{x^3+1} \)
15. \( f(x) = \sqrt{x^2+3} \)
16. \( f(x) = \left( \frac{x^4+1}{x^3+1} \right)^5 \)
17. \( f(x) = \sqrt{\frac{1+\sin x}{1+\cos x}} \)
18. \( f(x) = \sin^2 x \)
19. \( f(x) = \cos^4 x - \sec^2 x + 3\csc x \)
20.  

\[ f(x) = -3x^4 + (-2x^4 + 1)^{10} \quad \frac{1}{\sqrt{x-\cos x}} \]

1.7 PB7: The Chain Rule (Continued) and Implicit Differentiation

sections covered:

- Section 2.5: The Chain Rule (Continued).
- Section 2.6: Section 2.6: Implicit Differentiation.

Section 2.5: The Chain Rule (Continued)

1. Find the derivative of each of the following functions.

(a) \( f(x) = \sin(2x) \)
(b) \( f(x) = \cos(-3x + 1) \)
(c) \( f(x) = \tan(x^3 - x^2) \)
(d) \( f(x) = x \sin(x^2 + 1) \)
(e) \( f(x) = 3x^2 \cos(\pi x - 1) \)
(f) \( f(x) = \sin(\cos x) \)
(g) \( f(x) = \cos(\sec 4x) \)

(h) \( f(x) = \sin \sqrt{1 + x^2} \)

(i) \( f(x) = x \sin \frac{1}{x} \)

(j) \( f(x) = \sec^4(x^3 + 1) \)

2. Find an equation of the tangent line to the curve \( y = \sin(\sin x) \) at \((\pi, 0)\).

Section 2.6: Implicit Differentiation

1. Find \( \frac{dy}{dx} \) for each of the following functions.

(a) \( x^2 + y^2 = 25 \)

(b) \( x^3 + y^3 = 1 \)

(c) \( 2x^2 - y^2 = x \)

(d) \( x^4 + 3y^3 = 5y \)

(e) \( \sqrt{x} - y = y^2 + 3 \)

(f) \( xy = 5 \)

(g) \( 3x^2 + 2xy + y^2 = 2 \)

(h) \( -5x^2 + xy - y^3 = 1 \)

(i) \( x^3y^2 + y^4 = x \)

(j) \( \frac{x+y}{x-y} = 1 \)

(k) \( \sin y = 3 \)

(l) \( \cos y = x + 1 \)

(m) \( 3y^2 - \cos y = x^3 \)

(n) \( \sin x \cos y + y = -7 \)

(o) \( \cos(xy) = 1 + \sin y \)

(p) \( x \sin y + y \sin x = 1 \)

2. Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

(a) \( x^2 - xy - y^2 = 1 \) at \((2, 1)\)

(b) \( (x^2 + y^2)^2 = 2(x^3 + y^2) \) at \((1, 1)\)

3. Find \( y'' = \frac{d^2y}{dx^2} = \frac{d}{dx} \left[ \frac{dy}{dx} \right] \) by implicit differentiation.

(a) \( x^2 - y^2 = 4 \)

(b) \( x^2 + xy + y^2 = 3 \)
1.8  PB8: RELATED RATES

1.8  PB8: Related Rates

Section covered: Section 2.8: Related Rates.

**Section 2.8: Related Rates**

1. If $V$ is the volume of a cube with edge length $x$ and the cube expands as time passes, find $\frac{dV}{dt}$ in terms of $\frac{dx}{dt}$.

2. The radius of a sphere is increasing at a rate of 4 mm/s. How fast is the volume increasing when the diameter is 80 mm?

3. The radius of a spherical ball is increasing at a rate of 2 cm/min. At what rate is the surface area of the ball increasing when the radius is 8 cm?

4. A cylindrical tank with radius 5 m is being filled with water at a rate of 3 $m^3$/min. How fast is the height of the water increasing?

5. A water tank has the shape of an inverted circular cone with base radius 4 m and height 6 m. If water is being pumped into the tank at a rate of 8 $m^3$/min, find the rate at which the water level is rising when the water is 2 m deep.

6. A ladder 10 m long is leaning against a vertical wall with its other end on the ground. The top end of the ladder is sliding down the wall. When the bottom of the ladder is 6 m from the wall, it is sliding at 1 m/s.
   (a) How fast is the angle between the ladder and the ground changing at this instant?
   (b) How fast is the top of the ladder sliding down at the same instant?

7. A plane flying horizontally at an altitude of 3 mi and a speed of 500 mi/h passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 5 mi away from the station.

8. A hot air balloon rising vertically is viewed by an observer who is 2 km from the lift-off point. At a certain moment, the angle between the observer’s line of sight and the horizontal is $\frac{\pi}{6}$ radians, and this angle is increasing at the rate of $\frac{1}{5}$ radians per minute. How fast is the balloon rising (in km/min) at that moment?

1.9  PB9: The Mean Value Theorem, Increasing/Decreasing, and Limits at Infinity

sections covered:

- Section 3.2: The Mean Value Theorem.
- Section 3.3: How Derivatives Affect the Shape of a Graph.
- Section 3.2: Limits at Infinity; Horizontal Asymptotes.

**Section 3.2: The Mean Value Theorem**

Verify that the following functions satisfy the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers $c$ that satisfy the conclusion of the Mean Value Theorem.
1. $f(x) = -3x^2 + x$ on $[0, 1]$
2. $f(x) = x^3 - 3x + 2$ on $[-2, 2]$
3. $f(x) = x^3 - 2x$ on $[-1, 0]$
4. $f(x) = \frac{1}{x}$ on $[1, 3]$

Section 3.3: How Derivatives Affect the Shape of a Graph

For each of the following functions, find where it is increasing and where it is decreasing. Also find the local maximum and minimum values. Also find the intervals of concavity and the inflection points.

1. $f(x) = -x^2 + 4x - 3$
2. $f(x) = x^3 - 3x^2 - 9x + 4$
3. $f(x) = -x^4 + 4x^3 + 1$
4. $f(x) = \frac{x}{x^2 + 1}$

Section 3.4: Limits at Infinity; Horizontal Asymptotes

Find the following limits.

1. $\lim_{x \to -\infty} \frac{1}{x^3}$
2. $\lim_{x \to \infty} \frac{1}{\sqrt{x}}$
3. $\lim_{x \to -\infty} \frac{-x^2 + 1}{3x^2 + x + 1}$
4. $\lim_{x \to -\infty} \frac{7x^2}{2x^2 + 1}$
5. $\lim_{x \to -\infty} \frac{3x^2 - 2x}{5x^2 + x - x}$
6. $\lim_{x \to \infty} \sqrt{x^2 + 3 - x}$
7. $\lim_{x \to \infty} \sqrt{x^2 + x + 1} - x$
8. $\lim_{x \to \infty} \frac{\sqrt{x + x^3}}{2x - x^2}$
9. Find the horizontal asymptotes of $f(x) = \frac{x^2}{\sqrt{x^2 + 1}}$
10. Find the horizontal asymptotes of $f(x) = \frac{\sqrt{x + 1}}{2x - 1}$.

1.10 PB10: Limits at Infinity (Continued) and Summary of Curve Sketching

Sections covered:

- Section 3.4: Limits at infinity; Horizontal Asymptotes (continued).
1.10. PB10: LIMITS AT INFINITY (CONTINUED) AND SUMMARY OF CURVE SKETCHING

- Section 3.5: Summary of curve sketching.

Section 3.4: Limits at infinity; Horizontal Asymptotes (continued)

Find the following limits.

1. \[ \lim_{x \to \infty} -2x^{10} \]
2. \[ \lim_{x \to -\infty} -5x^{10} \]
3. \[ \lim_{x \to -\infty} 5x^{9} \]
4. \[ \lim_{x \to \infty} 3x^{2} \]
5. \[ \lim_{x \to \infty} -4x^{3} + x - 1 \]
6. \[ \lim_{x \to -\infty} 1 - 8x^{3} + 2x^{2} \]
7. \[ \lim_{x \to -\infty} 2x - x^{3} + 7x^{5} - 1 \]
8. \[ \lim_{x \to -\infty} \frac{3x^{2} - x^{3}}{1 + x^{2}} \]
9. \[ \lim_{x \to \infty} \frac{1-x+2x^{3}}{2-4x^{2}-6x} \]
10. \[ \lim_{x \to \infty} \sqrt{x} \]
11. \[ \lim_{x \to \infty} \sqrt{x^{2} - x + 1} \]
12. \[ \lim_{x \to \infty} \frac{1}{x} \sin x \]

Section 3.5: Summary of curve sketching

1. Sketch the graph of the following functions.
   
   (a) \( f(x) = -x^{3} + 3x^{2} - 1 \)
   
   (b) \( f(x) = x^{4} - 4x^{3} + 21 \)

2. Consider the following functions.
   
   (a) \( f(x) = \frac{-2x^{2}}{x^{2}-1} \), \( f'(x) = \frac{4x}{(x^{2}-1)^{2}} \), and \( f''(x) = \frac{-(12x^{2}+4)}{(x^{2}-1)^{3}} \).
   
   (b) \( f(x) = \frac{x^{2}+3x-4}{x^{2}-4} \), \( f'(x) = \frac{-3(x^{2}+4)}{(x^{2}-1)^{2}} \), and \( f''(x) = \frac{6x(x^{2}+12)}{(x^{2}-1)^{3}} \).
   
   (c) \( f(x) = \frac{1}{x^{2}+1} \), \( f'(x) = \frac{1-x^{2}}{(1+x^{2})^{2}} \), and \( f''(x) = \frac{2x(x^{2}-3)}{(x^{2}+1)^{3}} \).

In each case, determine the following features of \( f \): domain, even, odd, neither, intercepts, asymptotes (show the relevant limits), intervals of increase and decrease, local maxima and minima, intervals of upward and downwards concavity, points of inflections. Use all those features to sketch the graph of \( f \).
1.11 PB11: Optimization Problems

Section covered: Section 3.7: Optimization Problems.

Section 3.7: Optimization Problems

1. A farmer has 2000 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

2. A farmer wants to fence in a rectangular field beside a river. No fencing is required along the river and the farmer’s neighbour will pay half of the cost of one of the sides perpendicular to the river. If fencing costs 20 per linear meter and the field must have an area of 600 m², what are the dimensions of the field that will minimize the cost to the farmer?

3. A piece of wire 1 m long is cut into two pieces. One piece is bent in a square and the other is bent into an equilateral triangle. How much wire should be used for the square in order to minimize the sum of the areas of the square and triangle? [Recall that the area of an equilateral triangle with side s is \( A = \left( \frac{\sqrt{3}}{4} \right) s^2 \).]

4. A rectangular picture frame (see Figure 1.3) encloses an area of 600 cm². The top edge of the frame is constructed out of heavier material than the other three sides. If the material for the top edge weighs 200 gram/cm and the other three sides are made from material weighing 100 gram/cm, find the dimensions of the frame that would minimize the total weight of the material used.

![Figure 1.3:](image)

5. The volume \( V \) of a cylinder of height \( h \) and radius \( r \) is \( V = \pi r^2 h \), whereas the area of the cylinder’s surface, including top and bottom, is \( A = 2\pi r^2 + 2\pi rh \). Of all cylinders of volume \( V = 1 \), determine the height and radius of the cylinder that has minimal surface area.

6. The entrance to a tent is in the shape of an isosceles triangle as shown Figure 1.4. Zippers run vertically along the middle of the triangle and horizontally along the bottom of it. If the designers of the tent want to have a total zipper length of 5 metres, find the lengths \( x \) and \( y \) (see diagram) that will maximize the area of the entrance. Also find this maximum area. (Note the area of a triangle equals \( \frac{1}{2} \) (base)(height).)

7. If 1200 cm² of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

1.12 PB12: Indefinite Integrals and The Substitution Rule (SR)

Sections covered:
Integration table. Evaluate the following indefinite integrals.

1. \( \int 5 \, dx \)
2. \( \int x^8 \, dx \)
3. \( \int \frac{1}{x^2} \, dx \)
4. \( \int \sqrt{x} \, dx \)
5. \( \int \frac{1}{\sqrt{x}} \, dx \)
6. \( \int \frac{x}{\sqrt{x}} \, dx \)
7. \( \int \sqrt{x^5} \, dx \)
8. \( \int \frac{\sqrt{x}}{x} \, dx \)

Integration table + Properties. Evaluate the following indefinite integrals.

9. \( \int 4x \, dx \)
10. \( \int \sqrt{3x} \, dx \)
11. \( \int (-3x^2 + 5x - 1) \, dx \)
12. \( \int (x^2 + \sqrt{x} - 1) \, dx \)
13. \( \int (-2\sqrt{x} + \cos x) \, dx \)
14. \( \int (3 \sec^2 x - 4 \sin x - \csc^2 x) \, dx \)
15. \( \int (7 \sec x \tan x + 11 \csc x \cot x + 1) \, dx \)
16. \( \int \frac{4x - 3}{\sqrt{x}} \, dx \)
17. \( \int \frac{2 - \sqrt{1 + t^2}}{\sqrt{t}} \, dt \)
1. **Integration by Substitution.** Evaluate the following indefinite integrals.

20. \( \int (1 - 2x)^9 dx \)
21. \( \int 2x(x^2 + 3)^{15} dx \)
22. \( \int x^2(x^3 - 7)^6 dx \)
23. \( \int x\sqrt{1-x^2} dx \)
24. \( \int \sin(3x) dx \)
25. \( \int -5x^2 \cos(x^3) dx \)
26. \( \int 7 \tan^2 x \sec^2 x dx \)
27. \( \int \frac{\sec^2 x}{(3 + \tan x)^3} dx \)
28. \( \int \sin \theta \cos \theta \sqrt{\sin^2 \theta + 1} d\theta \)
29. \( \int (x^2 + 1) \cos(x^3 + 3x) dx \)
30. \( \int \frac{-3x^4}{(x^2 + 1)^3} dx \)
31. \( \int \frac{-4x + 4}{\sqrt{x^2 - 2x + 1}} dx \)
32. \( \int x\sqrt{x + 7} dx \)
33. \( \int \frac{x}{\sqrt{1 + 2x}} dx \)
34. \( \int -3 \cot^2 x \csc^2 x dx \)

1.13 **PB13: Definite Integrals, The FTC (parts 1 and 2), The SR, and Areas Between Curves**

Sections covered:

- Section 4.3: The Fundamental Theorem of Calculus.
- Section 4.4: Indefinite Integrals and the Net Change Theorem.
- Section 4.5: The Substitution Rule (SR).
- Section 5.1: Areas Between Curves.

**Definite Integrals and Area Between Two Curves**

1. **The Fundamental Theorem of Calculus, Part 2 (FTC2).** Evaluate the following definite integrals.

(a) \( \int_{-3}^{5} 5dx \)

(b) \( \int_{3}^{1} 3x \)

(c) \( \int_{4}^{0} \frac{1}{\sqrt{x}} dx \)

(d) \( \int_{1}^{3} (-x^2 + 3x - 1)dx \)

(e) \( \int_{1}^{3} \left( \frac{1}{x} - \frac{4}{x^2} \right) dx \)

(f) \( \int_{-2}^{2} (x^2 - 3)dx \)

(g) \( \int_{1}^{4} \frac{4 + 6x}{\sqrt{x}} dx \)

(h) \( \int_{1}^{8} \frac{2 + 4t}{\sqrt{t^2 + 3}} dt \)

(i) \( \int_{0}^{2} (2x - 3)(4x^2 + 1)dx \)

(j) \( \int_{0}^{4} (4 - x)\sqrt{x} dx \)

(k) \( \int_{0}^{2} (2 - 3\sin \theta) d\theta \)

(l) \( \int_{0}^{\frac{2}{7}} \sec^2 x dx \)

(m) \( \int_{0}^{\frac{2}{7}} (\sec x - \tan x) \sec x dx \)

2. Definite integrals with respect to Substitution. Evaluate the following definite integrals.

(a) \( \int_{0}^{2} 2x(x^2 - 2)^3 dx \)

(b) \( \int_{0}^{1} (4t - 1)^{50} dt \)

(c) \( \int_{0}^{1} \sqrt{26x + 1} dx \)

(d) \( \int_{0}^{\frac{\pi}{2}} \cos x \sin^4 x dx \)

(e) \( \int_{0}^{\frac{\pi}{2}} \frac{\sin t}{\cos t} dt \)

(f) \( \int_{0}^{\frac{\pi}{2}} \frac{1}{\sqrt{1 + 2x}} dx \)

(g) \( \int_{0}^{2} t^2 \sqrt{1 + t^3} dt \)

(h) \( \int_{0}^{\frac{\pi}{2}} \cos (3x) dx \)

(i) \( \int_{1}^{2} \frac{x + 1}{\sqrt{x^2 + 2x - 1}} dx \)

(j) \( \int_{\frac{\pi}{2}}^{\pi} \csc^2 (\frac{1}{2}t) dt \)

(k) \( \int_{0}^{4} \frac{x}{\sqrt{1 + 2x}} dx \)

(l) \( \int_{0}^{\frac{\pi}{2}} x \sin (x^2) dx \)

3. The Fundamental Theorem of Calculus, Part 1. Find the derivative of each of the following functions.

(a) \( g(x) = \int_{1}^{x} \cos (t^2) \ dt \)

(b) \( g(x) = \int_{x}^{2} t^3 \sin t \ dt \)

(c) \( g(x) = \int_{1}^{3x^2 + 2} \frac{t}{t + 1} \ dt \)

(d) \( g(x) = \int_{x^4}^{1} \cos^2 t \ dt \)

(e) \( g(x) = \int_{x^2}^{\sqrt{x}} (t^2 + 1) \ dt \)
4. Area Between Two Curves.

(a) Find the area bounded above by \( y = 2x + 5 \) and below by \( y = x^3 \) on \([0, 2]\).

(b) Find the area of the region enclosed by the curves \( y = x^2 \) and \( y = x + 2 \).

(c) Find the area of the region enclosed by the curves \( y = x^2 - 8 \) and \( y = -x^2 + 10 \). Include a sketch of the relevant region as part of your solution.

(d) Find the area of the region enclosed by the curves \( y = x^2 \) and \( y = 4x - x^2 \). Include a sketch of the relevant region as part of your solution.

(e) Find the area of the region enclosed by the curves \( y = x^3 \) and \( y = x \). Include a sketch of the relevant region as part of your solution.
Chapter 2

Solution to Practice Problems

2.1 Solution to PB1

Section 1.4: The Tangent Problem

1. Let \( f \) be the function defined by \( f(x) = 4x^2 \). Let \( x \) be different from 3. What is the slope \( m_x \) of the line through the points \((3, 36)\) and \((x, 4x^2)\)? Simplify your answer as much as possible.

**Solution.** First recall that the slope of a line through \( P_1(x_1, y_1) \) and \( P_2(x_2, y_2) \) is given by the formula

\[
\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1}.
\]

Let \( x \neq 3 \). The slope of the line through \((3, 36)\) and \((x, 4x^2)\) is then

\[
m_x = \frac{4x^2 - 36}{x - 3}.
\]

To simplify it, we need to factor the numerator and the denominator. The numerator is the difference of two squares as \( 4x^2 - 36 = (2x)^2 - (6)^2 \). So by the formula \( a^2 - b^2 = (a - b)(a + b) \), we have \( 4x^2 - 36 = (2x - 6)(2x + 6) \). The denominator is already on the factored form. Thus

\[
m_x = \frac{(2x - 6)(2x + 6)}{x - 3} = \frac{2(x - 3)(x + 3)}{x - 3} = 4(x + 3).
\]

We have simplified by \( x - 3 \) because \( x \neq 3 \), so that \( x - 3 \neq 0 \).

2. Let \( f \) be the function defined by \( f(x) = \frac{2}{3x} \). Let \( x \) be different from 0 and 1. What is the slope \( m_x \) of the line through the points \((1, \frac{2}{3})\) and \((x, \frac{2}{3x})\)? Simplify your answer as much as possible.

**Solution.** First recall some algebra about fractions.

\[
\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}, \quad \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}.
\]

Let \( x \) be different from 0 and 1. The slope of the line through \((1, \frac{2}{3})\) and \((x, \frac{2}{3x})\) is

\[
m_x = \frac{\frac{2}{3x} - \frac{2}{3}}{x - 1}.
\]
To simplify this, we first need to make the denominators of $\frac{2}{3x} - \frac{2}{3}$ the same. By using $\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$, we get $\frac{2}{3x} - \frac{2}{3} = \frac{2-2x}{9x}$. So

$$m_x = \frac{6-6x}{x-1} = \frac{6-6x}{9x(x-1)} = -\frac{6(x-1)}{9x(x-1)} = -\frac{6}{9x} = -\frac{2}{3x}.$$ 

We have simplified by $x - 1$ because $x \neq 1$, so that $x - 1 \neq 0$.

3. The point $P(2, -1)$ lies on the curve $y = \frac{1}{1-x}$.

(a) If $Q$ is the point $(x, \frac{1}{1-x})$, use your calculator to find the slope of the secant line $PQ$ for the following values of $x$: 1.5, 1.9, 1.99, 1.999, 2.5, 2.1, 2.01, 2.001.

(b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P(2, -1)$.

(c) Using the slope from part (b), find an equation of the tangent line to the curve at $P(2, -1)$.

**Solution.** (a) For $x \neq 2$, the slope of the line through $P(2, -1)$ and $Q(x, \frac{1}{1-x})$ is

$$m_x = \frac{\frac{1}{1-x} - (-1)}{x - 2} = \frac{\frac{1 + (1-x)}{1-x}}{x - 2} = \frac{2 - x}{(1-x)(x-2)} = \frac{-(x-2)}{(1-x)(x-2)} = \frac{-1}{1-x}.$$

<table>
<thead>
<tr>
<th>$x$</th>
<th>1.5</th>
<th>1.9</th>
<th>1.99</th>
<th>1.999</th>
<th>2</th>
<th>2.001</th>
<th>2.01</th>
<th>2.1</th>
<th>2.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_x$</td>
<td>2</td>
<td>1.111</td>
<td>1.01</td>
<td>1.001</td>
<td>0.999</td>
<td>0.990</td>
<td>0.909</td>
<td>0.666</td>
<td></td>
</tr>
</tbody>
</table>

(b) From the table, we can make the guess that the slope of the tangent line to the curve $y = \frac{1}{1-x}$ at $P(2, -1)$ is

$$m = \lim_{x \to 2} m_x = 1.$$ 

(c) An equation of the tangent line to the curve at $P(2, -1)$ is: $y - (-1) = m(x - 2)$, which is equivalent to $y + 1 = 1(x - 2)$, so that $y = x - 3$.

4. Let $f$ be the function defined by $f(x) = -\frac{1}{x^2}$. Let $x$ be different from 0 and 2.

(a) What is the slope $m_x$ of the line through the points $(2, -\frac{1}{4})$ and $(x, -\frac{1}{x^2})$? Simplify your answer as much as possible.

(b) Guess the value of $\lim_{x \to 2} m_x$, and determine an equation for the line tangent to the graph of $f$ at $(2, -\frac{1}{4})$.

**Solution.**

(a) Let $x \neq 0$ and $x \neq 2$. The slope of the line through the points $(2, -\frac{1}{4})$ and $(x, -\frac{1}{x^2})$ is

$$m_x = -\frac{1}{x^2} - \left(-\frac{1}{4}\right) = -\frac{1}{x^2} + \frac{1}{4} = \frac{-4 + x^2}{4x^2} = \frac{x^2 - 4}{4x^2(x - 2)} = \frac{(x-2)(x+2)}{4x^2(x-2)} = \frac{x+2}{4x^2}.$$ 

(b) In order to guess the limit, we make a table.

<table>
<thead>
<tr>
<th>$x$</th>
<th>1.5</th>
<th>1.9</th>
<th>1.99</th>
<th>1.999</th>
<th>2</th>
<th>2.001</th>
<th>2.01</th>
<th>2.1</th>
<th>2.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_x$</td>
<td>0.388</td>
<td>0.27</td>
<td>0.2518</td>
<td>0.2501</td>
<td>2</td>
<td>0.2498</td>
<td>0.2481</td>
<td>0.232</td>
<td>0.18</td>
</tr>
</tbody>
</table>

From that table we can make the guess that $m = \lim_{x \to 2} m_x = 0.25 = \frac{1}{4}$. An equation of the tangent line is $y - (-\frac{1}{4}) = m(x - 2)$, that is, $y + \frac{1}{4} = \frac{1}{4}(x - 2)$ or $y = \frac{1}{4}x - \frac{3}{4}$.

**Section 1.5: The Limit of a Function**
1. Use the given graph of $f$ (see Figure 2.1) to state the value of each quantity, if it exists. If it does not exist, explain why.

(a) \( \lim_{x \to 2^-} f(x) \); (b) \( \lim_{x \to 2^+} f(x) \); (c) \( \lim_{x \to 2} f(x) \); (d) \( f(2) \); (e) \( \lim_{x \to 4} f(x) \); (f) \( f(4) \).

![Figure 2.1:](image)

**Solution.** From the graph, we have:

(a) \( \lim_{x \to 2^-} f(x) = 3 \); (b) \( \lim_{x \to 2^+} f(x) = 1 \); and (c) \( \lim_{x \to 2} f(x) \) does not exist since the limit from the left is not the same as the limit from the right.

(d) \( f(2) = 3 \); (e) \( \lim_{x \to 4} f(x) = 4 \); and (f) \( f(4) \) does not exist because of the hole.

2. For the function $g$ whose graph is given (see Figure 2.2), state the value of each quantity, if it exists. If it does not exist, explain why.

(a) \( \lim_{t \to 0^-} g(t) \); (b) \( \lim_{t \to 0^+} g(t) \); (c) \( \lim_{t \to 0} g(t) \); (d) \( \lim_{t \to 2^-} g(t) \); (e) \( \lim_{t \to 2^+} g(t) \); (f) \( \lim_{t \to 2} g(t) \); (g) \( g(2) \); (h) \( \lim_{t \to 4} g(t) \).

**Solution.**

(a) \( \lim_{t \to 0^-} g(t) = -1 \); (b) \( \lim_{t \to 0^+} g(t) = -2 \); (c) \( \lim_{t \to 0} g(t) \) does not exist (DNE) since the limit from the left is different from the limit from the right.

(d) \( \lim_{t \to 2^-} g(t) = 2 \); (e) \( \lim_{t \to 2^+} g(t) = 0 \); (f) \( \lim_{t \to 2} g(t) \) DNE for the same reason as before.

(g) \( g(2) = 1 \) and \( \lim_{t \to 4} g(t) = 3 \).

### 2.2 Solution to PB2

**Section 1.6: Calculating Limits Using the Limits Laws**

Evaluate the following limits.

1. \( \lim_{x \to 1} \frac{9x^2-4}{x+1} \).
SOLUTION TO PRACTICE PROBLEMS

2. \( \lim_{x \to 1} \frac{9x^2 - 4}{x^2 + 1} \)  

Solution. We have \( \lim_{x \to 1} \frac{9x^2 - 4}{x^2 + 1} = \frac{9 - 4}{1 + 1} = \frac{5}{2} \).

2. \( \lim_{x \to 3} \frac{x^2 - 2x - 3}{x^2 - 9} \)

Solution. After substituting, we get \( \frac{0}{0} \), which is a problem. One way of getting rid of that is to factor the numerator and the denominator, and then simplify. In class I recalled the general method of factoring. Here I am going to give you another method (in fact it is a TRICK), which is easier.

To factor a polynomial of degree two \( p(x) = ax^2 + bx + c \) knowing a root \( r \) (recall that a root is a number that makes \( p(x) \) equal to zero), we can proceed as follows. First of all, since \( r \) is a root, we have \( p(x) = (x - r)(\alpha x + \beta) \), where \( \alpha \) and \( \beta \) are to be found. The unknown \( \alpha \) is the solution to the equation \( \alpha \times 1 = a \); so \( \alpha = a \). The unknown \( \beta \) is the solution to the equation \( \beta \times (-r) = c \); so \( \beta = -\frac{c}{r} \). Thus \( p(x) = (x - r)(ax - \frac{c}{r}) \).

- Factoring the numerator \( p(x) = x^2 - 2x - 3 \). Here \( a = 1, b = -2, \) and \( c = -3 \). Since \( p(3) = 0 \), we have \( r = 3 \). So \( p(x) = (x - 3)(\alpha x + \beta) \) with \( \alpha = a = 1 \) and \( \beta = -\frac{c}{r} = -\frac{3}{3} = 1 \). Thus \( p(x) = (x - 3)(x + 1) \).
- Factoring the denominator \( x^2 - 9 \). This is the difference of two squares as \( x^2 - 9 = (x)^2 - (3)^2 \). By the remarkable identity \( A^2 - B^2 = (A - B)(A + B) \), we have \( x^2 - 9 = (x - 3)(x + 3) \).
- Simplifying. Using the above factorizations, we get \( \frac{x^2 - 2x - 3}{x^2 - 9} = \frac{(x-3)(x+1)}{(x-3)(x+3)} = \frac{x+1}{x+3} \) for \( x \neq 3 \).

Now the limit is:

\[ \lim_{x \to 3} \frac{x^2 - 2x - 3}{x^2 - 9} = \lim_{x \to 3} \frac{x + 1}{x + 3} = \frac{3 + 1}{3 + 3} = \frac{4}{6} = \frac{2}{3}. \]

3. \( \lim_{x \to 0} \frac{1 - \sqrt{1 - x^2}}{x} \)

Solution. After substituting, we get \( \frac{0}{0} \), which is a problem. Since there is a square root, one way to get rid of the problem is to rationalize the numerator (because the square root appears there). Recall...
2.2. SOLUTION TO PB2

some rationalization formulas:

\[
\frac{\sqrt{A} + B}{C} = \frac{(\sqrt{A} + B)(\sqrt{A} - B)}{C(\sqrt{A} - B)} = \frac{A - B^2}{C(\sqrt{A} - B)},
\]

\[
\frac{\sqrt{A} - B}{C} = \frac{(\sqrt{A} - B)(\sqrt{A} + B)}{C(\sqrt{A} + B)} = \frac{A - B^2}{C(\sqrt{A} + B)},
\]

\[
\frac{A}{\sqrt{B} + C} = \frac{A(\sqrt{B} - C)}{(\sqrt{B} + C)(\sqrt{B} - C)} = \frac{A(\sqrt{B} - C)}{B - C^2},
\]

\[
\frac{A}{\sqrt{B} - C} = \frac{A(\sqrt{B} + C)}{(\sqrt{B} - C)(\sqrt{B} + C)} = \frac{A(\sqrt{B} + C)}{B - C^2}.
\]

Also recall that \((A - B)(A + B) = A^2 - B^2\). We come back to the limit. We have

\[
\lim_{x \to 0} \frac{1 - \sqrt{1 - x^2}}{x} = \lim_{x \to 0} \frac{(1 - \sqrt{1 - x^2})(1 + \sqrt{1 - x^2})}{x(1 + \sqrt{1 - x^2})} = \lim_{x \to 0} \frac{1^2 - (\sqrt{1 - x^2})^2}{x(1 + \sqrt{1 - x^2})} = \lim_{x \to 0} \frac{x^2}{x(1 + \sqrt{1 - x^2})} = \lim_{x \to 0} \frac{x}{1 + \sqrt{1 - x^2}} = \frac{0}{1 + \sqrt{0}} = 0.
\]

4. \(\lim_{x \to 5} \frac{x^2 + 3x - 10}{x^2 + 6x + 5}\).

Solution. \(\lim_{x \to 5} \frac{x^2 + 3x - 10}{x^2 + 6x + 5} = \lim_{x \to 5} \frac{(x+5)(x-2)}{(x+5)(x+1)} = \lim_{x \to 5} \frac{x-2}{x+1} = \frac{-5-2}{-5+1} = \frac{-7}{-4} = \frac{7}{4}\).

5. \(\lim_{x \to 4} \frac{x-2}{x^2+3x+5}\).

Solution. We have \(\lim_{x \to 4} \frac{x-2}{x^2+3x+5} = \frac{\sqrt{x-2}}{\sqrt{3x+5}} = \frac{2-2}{12+7} = \frac{0}{19} = 0\).

6. \(\lim_{x \to 2} \frac{\sqrt{2x+5} - \sqrt{x+7}}{x-2}\).

Solution. Rationalizing, we get

\[
\lim_{x \to 2} \frac{\sqrt{2x+5} - \sqrt{x+7}}{x-2} = \lim_{x \to 2} \frac{(\sqrt{2x+5} - \sqrt{x+7})(\sqrt{2x+5} + \sqrt{x+7})}{(x-2)(\sqrt{2x+5} + \sqrt{x+7})} = \lim_{x \to 2} \frac{(\sqrt{2x+5})^2 - (\sqrt{x+7})^2}{(x-2)(\sqrt{2x+5} + \sqrt{x+7})} = \lim_{x \to 2} \frac{(2x+5) - (x+7)}{(x-2)(\sqrt{2x+5} + \sqrt{x+7})} = \lim_{x \to 2} \frac{x-2}{(x-2)(\sqrt{2x+5} + \sqrt{x+7})} = \lim_{x \to 2} \frac{1}{\sqrt{2x+5} + \sqrt{x+7}} = \frac{1}{\sqrt{9} + \sqrt{9}} = \frac{1}{3+3} = \frac{1}{6}.
\]

7. \(\lim_{x \to -2} \frac{-5x^2 - 10x}{6 + 3x}\).

Solution. We have \(\lim_{x \to -2} \frac{-5x^2 - 10x}{6 + 3x} = \lim_{x \to -2} \frac{(x+2)(-5x)}{3(x+2)} = \lim_{x \to -2} \frac{-5x}{3} = \frac{-5(-2)}{3} = \frac{10}{3}\).

8. \(\lim_{x \to 0} \sqrt{x^2 + 4x + 2}\).

Solution. \(\lim_{x \to 0} \sqrt{x^2 + 4x + 2} = \lim_{x \to 0} \sqrt{(x+2)^2 + 4} = \sqrt{0+0+2} = \sqrt{2}\).
9. \( \lim_{x \to 7} \frac{7 - x}{x - 7} \).
   **Solution.** We have
   \[
   \lim_{x \to 7} \frac{7 - x}{x - 7} = \lim_{x \to 7} \frac{7 - x}{(x - 7)} = \lim_{x \to 7} \frac{-(x - 7)}{7(x - 7)} = \lim_{x \to 7} \frac{-1}{7} = -\frac{1}{7} = -\frac{1}{49}.
   \]

10. \( \lim_{x \to 3} \frac{x - 3}{\sqrt{x^2 - 5x + 10} - 2} \).
   **Solution.** After substituting, we get \( \frac{0}{0} \). To get rid of this problem, we rationalize the denominator.
   \[
   \lim_{x \to 3} \frac{x - 3}{\sqrt{x^2 - 5x + 10} - 2} = \lim_{x \to 3} \frac{(x - 3)(\sqrt{x^2 - 5x + 10} + 2)}{(x^2 - 5x + 10) - 4} = \lim_{x \to 3} \frac{(x - 3)(\sqrt{x^2 - 5x + 10} + 2)}{x^2 - 5x + 6} = \lim_{x \to 3} \frac{\sqrt{x^2 - 5x + 10} + 2}{x - 2}.
   \]
   \[
   \frac{\sqrt{3^2 - 5(3) + 10} + 2}{3 - 1} = \frac{\sqrt{9 - 15 + 10} + 2}{1} = \sqrt{4} + 2 = 2 + 2 = 4.
   \]

11. \( \lim_{x \to 1} \frac{x^4 - 1}{x^3 - 1} \).
   **Solution.** First recall the following identity:
   \[a^3 - b^3 = (a - b)(a^2 + ab + b^2).\]
   Now let’s calculate the limit.
   \[
   \lim_{x \to 1} \frac{x^4 - 1}{x^3 - 1} = \lim_{x \to 1} \frac{(x^2)^2 - (1)^2}{(x^3 - 1)(x^3 - 1)} = \lim_{x \to 1} \frac{(x^2 - 1)(x^2 + 1)}{(x - 1)(x + 1)(x^2 + x + 1)} = \lim_{x \to 1} \frac{(x + 1)(x^2 + 1)}{x^2 + x + 1} = \frac{2(2)}{1 + 1 + 1} = \frac{4}{3}.
   \]

2.3 Solution to PB3

Section 1.6: Calculating Limits Using the Limits Laws (Continued)

Evaluate the following limits (if they exist).

1. \( \lim_{x \to 3} \frac{2x + 1}{3x + 9} \).
   **Solution.** Set \( f(x) = \frac{2x + 1}{3x + 9} \). Substituting by 3, we get \( \frac{7}{0} \), which is a problem of the form \( \frac{k}{0} \) with \( k \neq 0 \). To get rid of that, we find one-sided limits and see if they are equal. To find one-sided limits, we make a sign analysis test. First put on the \( x \)-axis numbers that make the numerator and the denominator equal to zero. For the numerator, we have \( 2x + 1 = 0 \) if and only if \( x = -\frac{1}{2} \). For the denominator, we have \(-3x + 9 = 0 \) if and only if \( x = 3 \). So we have numbers \(-\frac{1}{2} \) and \( 3 \). Now pick any number between \(-\frac{1}{2} \) and \( 3 \), for example 0, and plug-in \( f(x) \), we get \( \frac{1}{9} \), which is a positive number. So on the left of 3 the sign is + (as shown Figure 2.3). To get the sign on the right, we pick any number greater than 3, for example 4, and plug-in \( f(x) \), we get \( \frac{2(4) + 1}{-3(4) + 9} = \frac{9}{9} = -3 \), which is a negative number. So the sign on the right is -. By the sign analysis test, we deduce that \( \lim_{x \to 3^-} f(x) = +\infty \) and \( \lim_{x \to 3^+} f(x) = -\infty \). Since those limits are no equal, it follows that \( \lim_{x \to 3} \frac{2x + 1}{3x + 9} \) does not exist.
2.3. SOLUTION TO PB3

2. \( \lim_{x \to 2} \left( \frac{4-x}{x^2-4x+4} \right) \).

Solution. We have \( \lim_{x \to 2} \left( \frac{4-x}{x^2-4x+4} \right) = +\infty \) by the sign analysis test.

3. \( \lim_{x \to -2} \frac{-5}{(x+2)^2} \).

Solution. Substituting by \(-2\), we get \( -\frac{5}{4} \), which is a problem of the form \( \frac{k}{0} \) with \( k \neq 0 \). So we need to make the sign analysis test, which is given by Figure 2.4. It follows from that figure that \( \lim_{x \to -2^-} \frac{-5}{(x+2)^2} = -\infty \) and \( \lim_{x \to -2^+} \frac{-5}{(x+2)^2} = -\infty \). Since those limits are equal, it follows that \( \lim_{x \to -2} \frac{-5}{(x+2)^2} = -\infty \).

Figure 2.4: Sign Analysis Test for \( \frac{-5}{(x+2)^2} \)

4. \( \lim_{x \to 1} \frac{x^2+3x-4}{x^2-2x+1} \).

Solution. That limit does not exist since \( \lim_{x \to 1^-} \frac{x^2+3x-4}{x^2-2x+1} = -\infty \) and \( \lim_{x \to 1^+} \frac{x^2+3x-4}{x^2-2x+1} = +\infty \) by the sign analysis test.

5. \( \lim_{x \to 0} \frac{1}{x^2} \).

Solution. From Figure 2.5, which is the sign analysis test for \( \frac{1}{x^2} \), we deduce that \( \lim_{x \to 0} \frac{1}{x^2} = +\infty \).

6. \( \lim_{x \to 2^-} f(x) \), where \( f(x) = \begin{cases} \sqrt{-2x+4} & \text{if } x \leq 2 \\ x-2 & \text{if } x > 2. \end{cases} \)

Solution. We first need to find one-sided limits. (Recall that “\( x \to a^- \)” means that \( x \) approaches \( a \) and \( x < a \). And “\( x \to a^+ \)” means that \( x \) approaches \( a \) and \( x > a \).) So as \( x \to 2^- \), we have \( x < 2 \), and therefore \( f(x) = \sqrt{-2x+4} \). As \( x \to 2^- \), we have \( x > 2 \), and therefore \( f(x) = x-2 \).

\[
\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \sqrt{-2x+4} = \sqrt{-2(2)+4} = \sqrt{0} = 0.
\]
CHAPTER 2. SOLUTION TO PRACTICE PROBLEMS

Figure 2.5: Sign Analysis Test for $\frac{1}{x^2}$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} x - 2 = 2 - 2 = 0.$$  
Since those limits are equal, it follows that $\lim_{x \to 2} f(x) = 0.$

7. $\lim_{x \to 5} f(x)$, where $f(x) = \begin{cases}  
x^2 + 1 & \text{if } x < 5 \\
x + 20 & \text{if } x \geq 5. 
\end{cases}$

**Solution.** The one-sided limits are:

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} (x^2 + 1) = (5)^2 + 1 = 25 + 1 = 26$$

and

$$\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} (x + 20) = 5 + 20 = 25.$$  
Since those limits are not equal, it follows that $\lim_{x \to 5} f(x)$ does not exist.

8. (a) $\lim_{x \to -3^-} f(x)$, (b) $\lim_{x \to 0^+} f(x)$, where $f(x) = \begin{cases}  
\sqrt{x - 1} & \text{if } x > 0 \\
x^2 - x + \frac{1}{2} & \text{if } x \leq 0. 
\end{cases}$

**Solution.** (a) By the sign analysis test, one can see that $\lim_{x \to -3^-} f(x) = +\infty.$ Moreover $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (2x + 3) = 2(3) + 3 = 9.$ Since the limit from the right is different from the limit from the left, we conclude that the limit $\lim_{x \to 3} f(x)$ does not exist. (b) The one-sided limits are: $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (2x + 3) = 2(0) + 3 = 3.$ And $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{x^2 + 2x - 3}{x - 1} = \frac{0^2 + 2(0) - 3}{0 - 1} = \frac{-3}{-1} = 3.$ Since both limits are equal to 3, it follows that $\lim_{x \to 0} f(x) = 3.$

9. $\lim_{x \to 0} f(x)$, where $f(x) = \begin{cases}  
\frac{\sqrt{x} - 1}{x - 1} & \text{if } x > 0 \\
x^2 - x + \frac{1}{2} & \text{if } x \leq 0. 
\end{cases}$

**Solution.** The one-sided limits are:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x^2 - x + \frac{1}{2}) = 0^2 - 0 + \frac{1}{2} = \frac{1}{2}$$

and

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \sqrt{x} - 1 = \lim_{x \to 0^+} (\sqrt{x} - 1)(\sqrt{x} + 1) = \lim_{x \to 0^+} (x - 1)(\sqrt{x} + 1) = \lim_{x \to 0^+} \frac{x - 1}{\sqrt{x} + 1} = \frac{1}{\sqrt{0} + 1} = 1.$$  
Since the limit from the left is not equal to the limit from the right, it follows that $\lim_{x \to 0} f(x)$ does not exist.
10. \( \lim_{x \to 2} \sin(x^2 - 3x + 2) \)

**Solution.** We have \( \lim_{x \to 2} \sin(x^2 - 3x + 2) = \sin \left( \lim_{x \to 2} (x^2 - 3x + 2) \right) = \sin \left( (2)^2 - 3(2) + 2 \right) = \sin (4 - 6 + 2) = \sin(0) = 0. \)

11. \( \lim_{x \to -3} \cos \left( \frac{x^2 - 9}{x + 3} \right) \)

**Solution.** We have \( \lim_{x \to -3} \cos \left( \frac{x^2 - 9}{x + 3} \right) = \cos \left( \lim_{x \to -3} \frac{x^2 - 9}{x + 3} \right) = \cos \left( \lim_{x \to -3} (x - 3) \right) = \cos(-3 - 3) = \cos(-6). \)

12. \( \lim_{x \to \frac{\pi}{12}} (\sin(2x) - 7\cos(4x)) \)

**Solution.** We have \( \lim_{x \to \frac{\pi}{12}} (\sin(2x) - 7\cos(4x)) = \sin \left( 2 \left( \frac{\pi}{12} \right) \right) - 7 \cos \left( 4 \left( \frac{\pi}{12} \right) \right) = \sin \left( \frac{\pi}{6} \right) - 7 \cos \left( \frac{\pi}{3} \right) = \frac{1}{2} - 7 \left( \frac{1}{2} \right) = \frac{1}{2} - \frac{7}{2} = -\frac{6}{2} = -3. \)

13. \( \lim_{x \to 0} \tan \frac{x}{x} \)

**Solution.** We have \( \lim_{x \to 0} \tan \frac{x}{x} = \lim_{x \to 0} \tan \frac{x}{x} = \lim_{x \to 0} \frac{\sin \frac{x}{x}}{\cos \frac{x}{x}} = \lim_{x \to 0} \left( \frac{1}{\cos x} \cdot \frac{\sin \frac{x}{x}}{x} \right) = \left( \lim_{x \to 0} \frac{\sin \frac{x}{x}}{x} \right) \left( \lim_{x \to 0} \frac{1}{\cos x} \right) = \frac{1}{\cos 0} \times 1 = \frac{1}{1} 	imes 1 = 1. \) Recall the basic limit:

\[ \lim_{x \to 0} \frac{\sin x}{x} = 1. \]

14. \( \lim_{x \to 0} \frac{9\sin^2 x}{x^2} \)

**Solution.** We have \( \lim_{x \to 0} \frac{9\sin^2 x}{x^2} = 9 \lim_{x \to 0} \frac{\sin^2 x}{x^2} = 9 \lim_{x \to 0} \left( \frac{\sin x}{x} \right)^2 = 9 \left( \lim_{x \to 0} \frac{\sin x}{x} \right)^2 = 9(1)^2 = 9. \) Note that \( \sin^2(x) = (\sin x)^2, \) which is not the same (in general) as \( \sin(x^2) \).

15. \( \lim_{x \to 0} \left( \frac{4x^2 - 5\sin x}{7x} \right) \)

**Solution.** We have

\[ \lim_{x \to 0} \left( \frac{4x^2 - 5\sin x}{7x} \right) = \lim_{x \to 0} \left( \frac{4x^2}{7x} - \frac{5\sin x}{7x} \right) = \lim_{x \to 0} \left( \frac{4x^2}{7x} \right) - \lim_{x \to 0} \left( \frac{5\sin x}{7x} \right) = \lim_{x \to 0} \left( \frac{4x}{7} \right) - \lim_{x \to 0} \left( \frac{5\sin x}{7x} \right) = \frac{4}{7}(0) - \frac{5}{7} \lim_{x \to 0} \left( \frac{\sin x}{x} \right) = 0 - \frac{5}{7} = -\frac{5}{7}. \]

16. \( \lim_{x \to 0} x^2 \sin \left( \frac{1}{x^2} \right) \)

**Solution.** Since \( -1 \leq \sin \left( \frac{1}{x^2} \right) \leq 1 \), it follows that \( -x^2 \leq x^2 \sin \left( \frac{1}{x^2} \right) \leq x^2 \) (we multiply everything by \( x^2 \)). Since \( \lim_{x \to 0} (-x^2) = 0 = \lim_{x \to 0} (x^2) \), it follows by the Squeeze Theorem that \( \lim_{x \to 0} x^2 \sin \left( \frac{1}{x^2} \right) = 0. \)

17. \( \lim_{x \to 0} |x| \cos \left( \frac{3}{x^3} \right) \)

**Solution.** Since \( -1 \leq \cos \left( \frac{3}{x^3} \right) \leq 1 \), it follows that (we multiply everything by \( |x| \))

\[ -|x| \leq |x| \cos \left( \frac{3}{x^3} \right) \leq |x|. \]

Since \( \lim_{x \to 0} (-|x|) = 0 = \lim_{x \to 0} |x| \), it follows that \( \lim_{x \to 0} |x| \cos \left( \frac{3}{x^3} \right) = 0 \) by the Squeeze Theorem.
Section 1.8: Continuity

Recall that a function \( f \) is continuous at a point \( a \) if \( \lim_{x \to a} f(x) = f(a) \). Notice that this definition implicitly requires three things if \( f \) is continuous at \( a \):

1. \( f(a) \) is defined (that is, \( a \) is in the domain of \( f \))
2. \( \lim_{x \to a} f(x) \) exists
3. \( \lim_{x \to a} f(x) = f(a) \).

1. Let \( f(x) = \frac{x^2 - 16}{x - 4} \). Is \( f \) continuous at 4?

**Solution.** No since it is not defined at 4 (plug-in 4, we get \( \frac{0}{0} \), which is not defined).

2. Let \( f(x) = \begin{cases} x^2 + 2x & \text{if } x \leq -2 \\ \sqrt{x + 3} & \text{if } x > -2. \end{cases} \) Is \( f \) continuous at -2?

**Solution.** Condition (1) is satisfied since \( f \) is defined at \(-2 \) (in fact, \( f(-2) = (-2)^2 + 2(-2) = 4 - 4 = 0 \)). Let see whether condition (2) is satisfied. We need to find one-sided limits and see if they are equal.

\[
\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} (x^2 + 2x) = (-2)^2 + 2(-2) = 4 - 4 = 0, \\
\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} \sqrt{x + 3} = \sqrt{-2 + 3} = \sqrt{1} = 1.
\]

Since those limits are not equal, it follows that \( \lim_{x \to -2} f(x) \) does not exist, which implies that (2) is not satisfied. So \( f \) is not continuous at \(-2 \).

3. Let \( f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 - x^2 & \text{if } x > 0. \end{cases} \) Is \( f \) continuous at 0?

**Solution.** Condition (1) is clearly satisfied since \( f(0) = 0 \), and then exists. For condition (2), we need to find one-sided limits. We have \( \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \cos x = \cos 0 = 1 \), and \( \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 1 - x^2 = 1 - (0)^2 = 1 - 0 = 1 \). So the limit \( \lim_{x \to 0} f(x) \) exists and is equal to 1, which implies that condition (2) is satisfied. The last condition does not hold since \( \lim_{x \to 0^+} f(x) \neq f(0) \). Thus \( f \) is not continuous at 0.

4. Let \( f(x) = \begin{cases} \frac{2 \sin x}{x} & \text{if } x < 0 \\ 2 & \text{if } x = 0 \\ \frac{x^2 + 2x}{x} & \text{if } x > 0. \end{cases} \) Is \( f \) continuous at 0?

**Solution.** Clearly the first condition is satisfied \( f(0) = 2 \). For the second condition, we have \( \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{2 \sin x}{x} = 2 \lim_{x \to 0^-} \frac{\sin x}{x} = 2(1) = 2 \). And the right-sided limit is

\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{x^2 + 2x}{x} = \lim_{x \to 0^+} \frac{x(x + 2)}{x} = \lim_{x \to 0^+} x + 2 = 0 + 2 = 2.
\]

So \( \lim_{x \to 0^+} f(x) = 2 \), which implies that condition (2) is satisfied. The third condition is also satisfied since \( \lim_{x \to 0^-} f(x) = f(0) \). Hence \( f \) is continuous at 0.
5. Find $c$ such that the function $f(x) = \begin{cases} cx^2 - 1 & \text{if } x \leq 3 \\ x + c & \text{if } x > 3 \end{cases}$ is continuous at 3.

**Solution.** If we want $f$ to be continuous at 3, then the three conditions above have to be satisfied. For condition (1), we have $f(3) = c(3)^2 - 1 = 9c - 1$. For (2), we have $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (cx^2 - 1) = c(3)^2 - 1 = 9c - 1$, and $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x + c) = 3 + c$. The limit exists if and only if the left-sided limit is equal to the right-sided limit, that is, if and only if $9c - 1 = 3 + c$. This latter equation is equivalent to $8c = 4$, so that $c = \frac{4}{8} = \frac{1}{2}$. Condition (3) is clearly satisfied when $c = \frac{1}{2}$. So for $c = \frac{1}{2}$, the function $f$ is continuous at 3.

6. Find $c$ such that the function $f(x) = \begin{cases} x^3 - x - 2 & \text{if } x < 1 \\ c^2 - 3c & \text{if } x = 1 \\ -2x & \text{if } x > 1 \end{cases}$ is continuous at 1.

**Solution.** It is the same reasoning as before. First we have $f(1) = c^2 - 3c$. Next the one-sided limits are: $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^3 - x - 2) = (1)^3 - 1 - 2 = 1 - 3 = -2$, and $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (-2x) = -2(1) = -2$. So $\lim_{x \to 1} f(x) = -2$. Condition (3) holds if and only if $c^2 - 3c = -2$, that is, $c^2 - 3c + 2 = 0$, or $(c - 1)(c - 2) = 0$. Solving this latter equation for $c$, we get $c = 1$ or $c = 2$.

### 2.4 Solution to PB4

Section 1.8: Continuity (Continued)

1. Show that the function $f(x) = \sqrt{25 - x^2}$ is continuous on $[-5, 5]$.

**Solution.** Recall that a function $f$ is **continuous at a number** $a$ if $\lim f(x) = f(a)$. Also recall that a function is **continuous on an interval** if it is continuous at every number in that interval. If the interval is closed, we need to check the continuity on the associated open interval (here it is $(-5, 5)$) and at the endpoints (here $-5$ and $5$).

- Continuity on $(-5, 5)$. Let $a \in (-5, 5)$. We have $\lim_{x \to a} f(x) = \sqrt{25 - a^2} = f(a)$. This shows that $f$ is continuous at $a$.

- Continuity at the endpoints. At $a = -5$, we have $\lim_{x \to -5^+} f(x) = \sqrt{25 - (-5)^2} = 0 = f(-5)$, which shows that $f$ is continuous at $-5$. Similarly, $\lim_{x \to 5^-} f(x) = f(5)$, which shows that $f$ is continuous at 5.

Hence $f$ is continuous on $[-5, 5]$.

2. Let $f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$. Show that $f$ is continuous everywhere.

**Solution.** On the interval $(-\infty, 0)$, $x < 0$ and $f(x) = -x$ is continuous since it is a polynomial. For the same reason, $f$ is continuous on the interval $(0, +\infty)$. How about the continuity at 0? We have $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (-x) = 0$ and $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x) = 0$. So $\lim f(x)$ exists and is equal to 0. Moreover, $f(0) = 0$. Thus $f$ is continuous at 0. Hence, $f$ is continuous everywhere.
3. Let \( f(x) = \begin{cases} x + 2 & \text{if } x \leq a \\ x^2 & \text{if } x > a \end{cases} \). Find all values of \( a \in \mathbb{R} \) that make \( f \) continuous on \( \mathbb{R} = (-\infty, +\infty) \).

**Solution.** Since \( x^2 + 2 \) and \( x^2 \) are polynomials, and then continuous everywhere, to have \( f \) continuous everywhere, it suffices to have the continuity at \( a \). For this, we need to have \( \lim_{x \to a} f(x) = f(a) \). First \( f(a) = a + 2 \). Next \( \lim_{x \to a} f(x) = a + 2 \) and \( \lim_{x \to a} f(x) = a^2 \). So the limit \( \lim_{x \to a} f(x) \) exists if and only if \( a^2 = a + 2 \), that is, \( a^2 - a - 2 = 0 \) or \( (a + 1)(a - 2) = 0 \). This latter equation implies that \( a = -1 \) or \( a = 2 \). So the values of \( a \) that make \( f \) continuous on \( \mathbb{R} \) are: \(-1 \) and \( 2 \).

4. Find all constants \( c \) such that the function

\[
 f(c) = \begin{cases} 3x^2 + cx & \text{if } x \leq 1 \\ x + c^2 & \text{if } x > 1 \end{cases}
\]

is continuous on \( (-\infty, +\infty) \).

**Solution.** This problem is very similar to the previous one. By using the same approach, we get the equation \( 1 + c^2 = 3 + c \), that is, \( c^2 - c - 2 = 0 \). \( \) is the same equation as before (here \( "1 + c^2" \) is the limit \( \lim_{x \to 1^+} f(x) \) from the right and \( "3 + c" \) is the limit \( \lim_{x \to 1^-} f(x) \) from the left). So the values of \( c \) are: \(-1 \) and \( 2 \).

5. Where are the following functions continuous? (a) \( f(x) = \frac{x+1}{x^2+x-2} \); (b) \( g(x) = \sqrt{-3x+12} \); (c) \( h(x) = \cos(x^3 + 1) \); (d) \( l(x) = \sqrt{x+2-3} \).

**Solution.**

(a) Since \( f \) is a rational function, it is continuous where the denominator is different from zero. That is, where \( x^2 + x - 2 = (x - 1)(x + 2) \neq 0 \). So \( f \) is continuous where \( x \neq 1 \) and \( x \neq -2 \). In other words \( f \) is continuous on \( \mathbb{R} \setminus \{-2, 1\} \).

(b) The function \( g \) is continuous where \(-3x + 12 \geq 0 \), that is, \(-3x \geq -12 \). Dividing this latter inequality by \(-3 \), we get \( x \leq 4 \) (Warning! When dividing or multiplying an inequality by a negative number, we have to change the direction of the inequality). So \( g \) is continuous on \( (-\infty, 4] \).

(c) We can write \( h \) as the composition \( h(x) = (f \circ g)(x) \) where \( f(x) = \cos(x) \) and \( g(x) = x^3 + 1 \). Since \( f \) and \( g \) are both continuous everywhere, it follows that \( h \) is also continuous everywhere, that is, on \( \mathbb{R} \).

(d) The function \( l \) is continuous in its domain \( D = \{ x \in \mathbb{R} \mid x + 2 \geq 0 \text{ and } \sqrt{x+2-3} \neq 0 \} \). The first condition, \( x + 2 \geq 0 \), is equivalent to \( x \geq -2 \), or \( x \in [-2, +\infty) \). The second condition, \( \sqrt{x+2-3} \neq 0 \), is equivalent to \( \sqrt{x+2} \neq 3 \), that is, \( x + 2 \neq 9 \) or \( x \neq 7 \). So the domain is \( D = \{ x \in \mathbb{R} \mid x \geq -2 \text{ and } x \neq 7 \} = [-2, +\infty) \setminus \{ 7 \} \). Hence \( l \) is continuous at every number greater that or equal to \(-2 \) except \( 7 \).

6. Show that there is a root of the equation \( x^4 - x - 1 = 0 \) between \(-1 \) and \( 0 \).

**Solution.** For this question and the next one, we are going to use the Intermediate Value Theorem, which states if

1. \( f \) is continuous on a closed interval \([a, b]\), and
2. if \( N \) is a number between \( f(a) \) and \( f(b) \),
then there exists a number \(c\) in \([a, b]\) such that \(f(c) = N\).

Conditions (1) and (2) are called the hypothesis. To apply the theorem, we first need to check (1) and (2). If the equation is \(f(x) = \alpha\), then we take \(N = \alpha\). For example, if the equation is \(f(x) = 0\), then \(N = 0\). If the equation is \(f(x) = 7\), then \(N = 7\), and so on...

We come back to the question. Let \(f(x) = x^4 - x - 1\), and consider the interval \([-1, 0]\). Clearly \(f\) is continuous on that interval (since it is a polynomial). So condition (1) is satisfied. Moreover, we have \(f(-1) = (-1)^4 - (-1) - 1 = 1 + 1 - 1 = 1\) and \(f(0) = (0)^4 - (0) - 1 = -1\). Take \(N = 0\). Clearly \(N\) is between \(f(-1)\) and \(f(0)\). So condition (2) is satisfied. By the intermediate value theorem, there exists \(c\) in \([-1, 0]\) such that \(f(c) = 0\). This implies that \(c\) is a root of the equation \(x^4 - x - 1 = 0\).

7. Show that there is a root of the equation \(2x^3 - 4x^2 = -1\) between 0 and 1.

**Solution.** Let \(f(x) = 2x^3 - 4x^2\), and consider the closed interval \([0, 1]\). Clearly \(f\) is continuous on that interval (since it is a polynomial). Moreover, we have \(f(0) = 2(0)^3 - 4(0)^2 = 0\) and \(f(1) = 2(1)^3 - 4(1)^2 = 2 - 4 = -2\). Take \(N = -1\). Clearly \(N\) is between \(f(0)\) and \(f(1)\). So by the intermediate value theorem, there exists \(c\) in \([0, 1]\) such that \(f(c) = -1\). This implies that \(c\) is a root of the equation \(2x^3 - 4x^2 = -1\).

### Section 2.1: Derivatives at a Specific Point

Recall that the tangent line to \(f(x)\) at a certain point \(P(a, f(a))\) is the line through \(P\) with slope \(m\), where \(m\) is given by the formula:

\[
m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.
\]

**How to get \(f(a + h)\)?** To get \(f(a + h)\) we substitute \(x\) by \(a + h\). This means that anytime you see “\(x\)”, replace it by “\(a + h\)”. For example,

- if \(f(x) = x^2\), then \(f(a + h) = (a + h)^2\).
- If \(f(x) = x^2 - 3x\), then \(f(a + h) = (a + h)^2 - 3(a + h)\).
- Another example: if \(f(x) = \frac{-x^3 + 2x - 5}{2x^2 - x}\), then \(f(a + h) = \frac{-(a+h)^3 + 2(a+h) - 5}{2(a+h)^2 - (a+h)}\). And so on...

**Useful identities:**

- \(\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}\).
- \(\frac{a}{b} = \frac{a}{b^2}\).
- \((\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}) = a - b\). This is useful when we are rationalizing.

1. Find an equation of the tangent line to \(f(x) = -x^2\) at \((1, -1)\).

**Solution.**

Here \(P(1, -1)\). So \(a = 1\), \(f(a + h) = f(1 + h) = -(1 + h)^2\), and \(f(1) = -(1)^2 = -1\). Recalling that \((a + b)^2 = a^2 + 2ab + b^2\), we have

\[
\frac{f(1 + h) - f(1)}{h} = \frac{-(1 + h)^2 - (-1)}{h} = \frac{-(1^2 + 2h + h^2) + 1}{h} = \frac{-h^2 - 2h}{h}.
\]
\[
\frac{-1 - 2h - h^2 + 1}{h} = \frac{-2h - h^2}{h} = \frac{h(-2 - h)}{h}.
\]

So
\[
m = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0} \frac{h(-2 - h)}{h} = \lim_{h \to 0}(-2 - h) = -2 - 0 = -2.
\]

Having one point \((x_1, y_1)\) and the slope \(m\), the point-slope form of the equation of the line through that point with the slope \(m\) is given by:
\[
y - y_1 = m(x - x_1).
\]

Here the point is \(P(1, -1)\) (so \(x_1 = 1\) and \(y_1 = -1\)) and the slope is \(m = -2\). The desired equation is then: \(y - (-1) = -2(x - 1)\) or \(y + 1 = -2(x - 1)\). (We don’t need to go further.)

2. Find an equation of the tangent line to \(f(x) = \sqrt{x}\) at \((4, 2)\).

**Solution.** First we need to find the slope:
\[
m = \lim_{h \to 0} \frac{f(4 + h) - f(4)}{h} = \lim_{h \to 0} \frac{\sqrt{4 + h} - \sqrt{4}}{h} = \lim_{h \to 0} \frac{\sqrt{4 + h} - \sqrt{4}}{h} = \lim_{h \to 0} \frac{(\sqrt{4 + h} - \sqrt{4})(\sqrt{4 + h} + \sqrt{4})}{h(\sqrt{4 + h} + \sqrt{4})} = \lim_{h \to 0} \frac{(4 + h) - 4}{h(\sqrt{4 + h} + \sqrt{4})} = \lim_{h \to 0} \frac{1}{\sqrt{4 + h} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{4}.
\]

The required equation is then: \(y - 2 = \frac{1}{4}(x - 4)\).

3. Let \(f(x) = -3x^2 + 4x\). Find the derivative of \(f\) at \(a = 0\) using the limit definition of derivative.

**Solution.** Recall that the derivative of \(f\) at a certain point \(a\) is defined to be
\[
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.
\]

Here \(a = 0\) and \(f(x) = -3x^2 + 4x\). So \(f(a + h) = f(0 + h) = f(h) = -3h^2 + 4h\) and \(f(a) = f(0) = -3(0)^2 + 4(0) = 0\). Now we have
\[
f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{-3h^2 + 4h - 0}{h} = \lim_{h \to 0} \frac{h(-3h + 4)}{h} = \lim_{h \to 0}(-3h + 4) = -3(0) + 4 = 4.
\]

4. Let \(f(x) = x^3 - x\). Find the derivative of \(f\) at \(a = -1\) using the limit definition of derivative.

**Solution.** First recall the formula: \((a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\). We have
\[
f'(-1) = \lim_{h \to 0} \frac{f(-1 + h) - f(-1)}{h} = \lim_{h \to 0} \frac{[(-1 + h)^3 - (-1 + h)] - [(-1)^3 - (-1)]}{h} = \lim_{h \to 0} \frac{[(-1)^3 + 3(-1)^2h + 3(-1)h^2 + h^3 - (-1 + h)] - [-1 + 1]}{h} = \lim_{h \to 0} \frac{(-1 + 3h - 3h^2 + h^3 + 1 - h) - (0)}{h} = \lim_{h \to 0} \frac{2h - 3h^2 + h^3}{h} = \lim_{h \to 0} \frac{h(2 - 3h + h^2)}{h} = \lim_{h \to 0} (2 - 3h + h^2) = 2 - 3(0) + 0^2 = 2 - 0 + 0 = 2.
\]
2.4. SOLUTION TO PB4

5. Let \( f(x) = \frac{x}{x+1} \). Find the derivative of \( f \) at \( a = 2 \) using the limit definition of derivative.

**Solution.** We have

\[
 f'(2) = \lim_{{h \to 0}} \frac{f(2 + h) - f(2)}{h} = \lim_{{h \to 0}} \frac{\frac{2+h}{2+h+1} - \frac{2}{2+1}}{h} = \lim_{{h \to 0}} \frac{2+h - 2}{3h} = \frac{1}{3}.
\]

Section 2.2: The Derivatives as a Function

For the following derivatives, use the limit definition.

1. Let \( f(x) = -3x + 4 \). Find \( f'(x) \).

**Solution.** First recall that the derivative of a function \( f \) at any point \( x \) is given by:

\[
 f'(x) = \lim_{{h \to 0}} \frac{f(x + h) - f(x)}{h}.
\]

**How to get \( f(x + h) \)?** To get \( f(x + h) \) we substitute \( x \) by \( x + h \). This means that anytime you see “\( x \)”, replace it by “\( x + h \)”. For example,

- If \( f(x) = x^2 \), then \( f(x + h) = (x + h)^2 \).
- If \( f(x) = x^2 - 3x \), then \( f(x + h) = (x + h)^2 - 3(x + h) \).
- Another example: if \( f(x) = \frac{-x^3+2x-5}{2x^2-x} \), then \( f(x + h) = \frac{-(x+h)^3+2(x+h)-5}{2(x+h)^2-(x+h)} \). And so on...

We come back to the question. We have

\[
 f'(x) = \lim_{{h \to 0}} \frac{f(x + h) - f(x)}{h} = \lim_{{h \to 0}} \frac{-3(x + h) + 4}{h} = \lim_{{h \to 0}} \frac{-3x - 3h + 4}{h} = \lim_{{h \to 0}} \frac{-3h}{h} = \lim_{{h \to 0}} (-3) = -3.
\]

2. Let \( f(x) = -5x^2 + 7 \). Find \( f'(x) \).

**Solution.** We have

\[
 f'(x) = \lim_{{h \to 0}} \frac{-5(x + h)^2 + 7}{h} - \frac{-5x^2 + 7}{h} = \lim_{{h \to 0}} \frac{-5(x^2 + 2xh + h^2) + 7}{h} = \lim_{{h \to 0}} \frac{-10xh - 5h^2}{h} = \lim_{{h \to 0}} \frac{-10x - 5h}{h} = \lim_{{h \to 0}} (-10x - 5h) = -10x - 5(0) = -10x - 0 = -10x.
\]
3. Let \( f(x) = \frac{1}{\sqrt{x}} \). Find \( f'(x) \).

**Solution.** We have

\[
f'(x) = \lim_{h \to 0} \frac{1}{h} \left( \frac{1}{(x+h)^{\frac{1}{2}}} - \frac{1}{x^{\frac{1}{2}}} \right) = \lim_{h \to 0} \frac{x^{\frac{1}{2}} - (x+h)^{\frac{1}{2}}}{hx^{\frac{1}{2}}(x+h)^{\frac{1}{2}}} \]

\[
= \lim_{h \to 0} \frac{x^{\frac{1}{2}} - (x+h)^{\frac{1}{2}}}{hx^{\frac{1}{2}}(x+h)^{\frac{1}{2}}}(x^{\frac{1}{2}} + (x+h)^{\frac{1}{2}}) = \lim_{h \to 0} \frac{x - (x+h)}{hx^{\frac{1}{2}} + h(x+h)^{\frac{1}{2}}} = \frac{-h}{hx^{\frac{1}{2}} + h(x+h)^{\frac{1}{2}}} = \frac{-1}{\sqrt{x^{\frac{3}{2}} + x^2}} = \frac{-1}{x(2\sqrt{x})} \frac{1}{2x\sqrt{x}}.
\]

### 2.5 Solution to PB5

Section 2.3: Differentiation Formulas

(i) Find the derivative of each of the following functions.

1. \( f(x) = 2018 \)

   **Solution.** First recall that the derivative of the constant function is 0. That is, if \( c \) is a constant, then

   \[
   \frac{d}{dx}[c] = 0.
   \]

   Since \( f(x) = 2018 \) is constant, it follows that \( f'(x) = 0 \).

2. \( f(x) = 3^4 \)

   **Solution.** \( f'(x) = 0 \) since \( 3^4 \) is a constant (\( 3^4 \) is a constant because it does not depend on \( x \)).

3. \( f(x) = \pi^2 \)

   **Solution.** Since \( \pi = 3.14159 \cdots \) is a constant, it follows that \( \pi^2 = \pi \times \pi \) is also a constant. Therefore, we have \( f'(x) = 0 \) (and NOT \( 2\pi \)).
4. \( f(x) = x^6 \)

**Solution.** First recall the following formula

\[ \frac{d}{dx}[x^n] = nx^{n-1} \] for any real number \( n \)

Here \( n = 6 \). Applying the power rule, we get \( f'(x) = 6x^{6-1} = 6x^5 \).

5. \( f(x) = 3x^{-2} \)

**Solution.** First recall the following property. Let \( c \) be a constant, and let \( u \) be a differentiable function.

\[ \frac{d}{dx}[cu(x)] = c\frac{d}{dx}[u(x)]. \]

Here \( c = 3 \) and \( u(x) = x^{-2} \). By applying the constant multiple rule, we get \( f'(x) = 3\frac{d}{dx}[x^{-2}] \). By the power rule, we have \( \frac{d}{dx}[x^{-2}] = -2x^{-2-1} = -2x^{-3} \). So \( f'(x) = 3(-2x^{-3}) = -6x^{-3} \).

6. \( f(x) = \frac{x^2}{7} \)

**Solution.** First we can rewrite \( f(x) \) as \( f(x) = \frac{1}{7}x^2 \). Now we have

\[ f'(x) = \frac{d}{dx} \left[ \frac{1}{7}x^2 \right] = \frac{1}{7} \frac{d}{dx}[x^2] = \frac{1}{7} (2x^{2-1}) = \frac{1}{7} (2x^1) = \frac{2}{7} x. \]

7. \( f(x) = \frac{1}{2}x^\frac{4}{3} \)

**Solution.** Recall the addition and subtraction of fractions.

\[ \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}, \quad \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}, \quad \frac{a}{b} - 1 = \frac{a - b}{b}. \]

We have

\[ f'(x) = \frac{d}{dx} \left[ \frac{1}{2}x^\frac{4}{3} \right] = \frac{1}{2} \frac{d}{dx} \left[ x^\frac{4}{3} \right] = \frac{1}{2} \left( \frac{4}{3} \cdot x^{\frac{4}{3}-1} \right) = \frac{4}{6} x^\frac{1}{3} = \frac{2}{3} x^\frac{1}{3}. \]

8. \( f(x) = \sqrt{x^3} \)

**Solution.** The idea is to first rewrite \( f(x) \) on the form \( x^n \), and then apply the power rule. To do that, recall the following identity:

\[ \sqrt{x^m} = x^{\frac{m}{2}}. \] (2.5.1)

Applying that identity, we get \( f(x) = x^{\frac{3}{2}} \), so that

\[ f'(x) = \frac{d}{dx} \left[ x^{\frac{3}{2}} \right] = \frac{3}{2} x^{\frac{3}{2}-1} = \frac{3}{2} x^{\frac{1}{2}} = \frac{3}{2} \sqrt{x}. \]

9. \( f(x) = \frac{2}{x^7} \)

**Solution.** First recall the identity:

\[ \frac{1}{x^m} = x^{-m}. \] (2.5.2)

Using that identity, we can rewrite \( f(x) \) as \( f(x) = 2\frac{1}{x^7} = 2x^{-3} \). So

\[ f'(x) = \frac{d}{dx} \left[ 2x^{-3} \right] = 2 \frac{d}{dx}[x^{-3}] = 2(-3)x^{-3-1} = -6x^{-4} = \frac{-6}{x^4}. \]
10. \( f(x) = \frac{1}{\sqrt{x}} \)

**Solution.** First recall the identity:

\[
\sqrt{x} = x^{\frac{1}{2}}.
\]

(2.5.3)

Using that identity, we can rewrite \( f(x) \) as \( f(x) = \frac{1}{x^{\frac{1}{2}}} \). Using the identity (2.5.2), we have \( \frac{1}{x^{\frac{1}{2}}} = x^{-\frac{1}{2}} \).

So

\[
f'(x) = \frac{d}{dx} \left[x^{-\frac{1}{2}}\right] = -\frac{1}{3}x^{-\frac{1}{2}-1} = -\frac{1}{3}x^{-\frac{3}{2}} = -\frac{1}{3x^\frac{3}{2}}.
\]

11. \( f(x) = \sqrt[3]{x} \)

**Solution.** First recall the identity:

\[
\sqrt[3]{x^m} = x^{\frac{m}{3}}.
\]

(2.5.4)

Using that identity, we can rewrite \( f(x) \) as \( f(x) = x^{\frac{1}{3}} \). So

\[
f'(x) = \frac{d}{dx} \left[x^{\frac{1}{3}}\right] = \frac{5}{3}x^{\frac{5}{3}-1} = \frac{5}{3}x^{\frac{2}{3}}.
\]

12. \( f(x) = \frac{\sqrt{x}}{x} \)

**Solution.** First, by using the identity (2.5.3) above, we have \( f(x) = \frac{\sqrt{x}}{x} = \frac{1}{2}x^{\frac{1}{2}} \). So

\[
f'(x) = \frac{d}{dx} \left[\frac{1}{2}x^{\frac{1}{2}}\right] = \frac{1}{2} \frac{d}{dx} \left[x^{\frac{1}{2}}\right] = \frac{1}{2} \left(x^{\frac{1}{2}-1}\right) = \frac{1}{16}x^{-\frac{3}{8}} = \frac{1}{16} \frac{1}{x^{\frac{3}{8}}} = \frac{1}{16x^\frac{3}{8}}.
\]

13. \( f(x) = \frac{x^2}{x} \)

**Solution.** We have

\[
f'(x) = \frac{d}{dx} \left[\frac{\pi^2}{x}\right] = \pi^2 \frac{d}{dx} \left[\frac{1}{x}\right] = \pi^2 \frac{d}{dx} \left[x^{-1}\right] = \pi^2 \left((-1)x^{-1-1}\right) = \\
\pi^2(-x^{-2}) = \pi^2 \left(-\frac{1}{x^2}\right) = -\frac{\pi^2}{x^2}.
\]

14. \( f(x) = x + 1 \)

**Solution.** First recall the following formula.

The Sum Rule \( \frac{d}{dx} [u(x) + v(x)] = \frac{d}{dx} [u(x)] + \frac{d}{dx} [v(x)]. \)

Here \( u(x) = x \) and \( v(x) = 1 \). Applying the sum rule, we get

\[
f'(x) = \frac{d}{dx} [x + 1] = \frac{d}{dx} [x] + \frac{d}{dx}[1] = 1 + 0 = 1.
\]

15. \( f(x) = x^2 - \frac{2x}{5} \)

**Solution.** First recall the following formula.

The Difference Rule \( \frac{d}{dx} [u(x) - v(x)] = \frac{d}{dx} [u(x)] - \frac{d}{dx} [v(x)]. \)

Here \( u(x) = x^2 \) and \( v(x) = \frac{2x}{5} \). Applying the difference rule, we get

\[
f'(x) = \frac{d}{dx} \left[x^2 - \frac{2x}{5}\right] = \frac{d}{dx} \left[x^2\right] - \frac{d}{dx} \left[\frac{2x}{5}\right] = 2x - \frac{2}{5} \frac{d}{dx}[x] = 2x - \frac{2}{5}(1) = 2x - \frac{2}{5}.
\]
16. \( f(x) = \frac{x - \sqrt{x}}{2} \)

**Solution.** First we can rewrite \( f(x) \) as \( f(x) = \frac{1}{2} (x - \sqrt{x}) \).

\[
f'(x) = \frac{d}{dx} \left[ \frac{1}{2} (x - \sqrt{x}) \right] = \frac{1}{2} \frac{d}{dx} [x - \sqrt{x}] = \frac{1}{2} \left( \frac{d}{dx} [x] - \frac{d}{dx} [\sqrt{x}] \right) = \frac{1}{2} \left( 1 - \frac{d}{dx} \left[ x^{\frac{1}{2}} \right] \right) = \frac{1}{2} \left( 1 - \frac{1}{2} x^{-\frac{1}{2}} \right) = \frac{1}{2} \left( 1 - \frac{1}{2 \sqrt{x}} \right).
\]

17. \( f(x) = -3x^4 - 2x^3 + x^2 - 1 \)

**Solution.** We have

\[
f'(x) = \frac{d}{dx} \left[ -3x^4 - 2x^3 + x^2 - 1 \right] = \frac{d}{dx} [-3x^4] - \frac{d}{dx} [2x^3] + \frac{d}{dx} [x^2] - \frac{d}{dx} [1] = -3(4x^3) - 2(3x^2) + 2x - 0 = -12x^3 - 6x^2 + 2x.
\]

18. \( f(x) = \frac{2x^3 - 3x^2}{4} \)

**Solution.** We have

\[
f'(x) = \frac{d}{dx} \left[ \frac{1}{4} (2x^3 - 3x^2) \right] = \frac{1}{4} \frac{d}{dx} [2x^3] - \frac{d}{dx} [3x^2] = \frac{1}{4} \left( 2(3x^2) - 3(2x) \right) = \frac{1}{4} (6x^2 - 6x).
\]

19. \( f(x) = x^{\frac{5}{3}} - x^{\frac{2}{3}} \)

**Solution.** We have

\[
f'(x) = \frac{d}{dx} \left[ x^{\frac{5}{3}} - x^{\frac{2}{3}} \right] = \frac{5}{3} x^{\frac{5}{3} - 1} - \frac{2}{3} x^{\frac{2}{3} - 1} = \frac{5}{3} x^{\frac{2}{3}} - \frac{2}{3} x^{-\frac{1}{3}}.
\]

20. \( f(x) = x^2 - \frac{1}{x} \)

**Solution.** We have

\[
f'(x) = \frac{d}{dx} \left[ x^2 - \frac{1}{x} \right] = \frac{d}{dx} [x^2] - \frac{d}{dx} \left[ \frac{1}{x} \right] = 2x - \left( -\frac{1}{x^2} \right) = 2x + \frac{1}{x^2}.
\]

For the derivative \( \frac{d}{dx} \left[ \frac{1}{x} \right] \), see question 13.

21. \( f(x) = 1.4x^5 - 2.5x^2 + 3.8 \)

**Solution.** We have

\[
f'(x) = 1.4(5x^4) - 2.5(2x) + 0 = 7x^4 - 5x.
\]

22. \( f(x) = \sqrt{x} \)

**Solution.** First recall the identity

\[
\frac{x^k}{x^m} = x^{k-m}. \tag{2.5.5}
\]

We have

\[
f'(x) = \frac{d}{dx} \left[ \frac{x}{x^{\frac{1}{2}}} \right] = \frac{d}{dx} \left[ x^{1-\frac{1}{2}} \right] = \frac{d}{dx} \left[ x^{\frac{1}{2}} \right] = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2 \sqrt{x}}.
\]
23. \( f(x) = \frac{\sqrt{x} + x}{x^2} \)

**Solution.** We have

\[
f'(x) = \frac{d}{dx} \left[ \frac{\sqrt{x} + x}{x^2} \right] = \frac{d}{dx} \left[ \frac{\sqrt{x}}{x^2} + \frac{x}{x^2} \right] = \frac{d}{dx} \left[ x^{-\frac{1}{2}} + x^{-1} \right] = \]

\[
\frac{1}{2} x^{-\frac{3}{2}} + \frac{-1}{x^2} = \frac{\sqrt{x} - 2}{2x^3}. \]

24. \( f(x) = (3x + 4)(x - 5) \)

**Solution.** First we recall the following formula.

**The Product Rule:**

\[
\frac{d}{dx} [u(x)v(x)] = u'(x)v(x) + u(x)v'(x) = \left( \frac{d}{dx} [u(x)] \right) v(x) + u(x) \left( \frac{d}{dx} [v(x)] \right). \]

Here \( u(x) = 3x + 4 \) and \( v(x) = x - 5 \). Applying the product rule, we have

\[
f'(x) = \left( \frac{d}{dx} [3x + 4] \right) (x - 5) + (3x + 4) \left( \frac{d}{dx} [x - 5] \right) = \]

\[
(3 + 0)(x - 5) + (3x + 4)(1 - 0) = 3x - 15 + 3x + 4 = 6x - 11. \]

25. \( f(x) = (5x^2 - 2)(x^3 + 3x) \)

**Solution.** Applying the product rule, we have

\[
f'(x) = \left( \frac{d}{dx} [5x^2 - 2] \right) (x^3 + 3x) + (5x^2 - 2) \left( \frac{d}{dx} [x^3 + 3x] \right) = \]

\[
10x(x^3 + 3x) + (5x^2 - 2)(3x^2 + 3) = 10x^4 + 30x^2 + 15x^2 - 6x^2 - 6 = 25x^4 + 39x^2 - 6. \]

26. \( f(x) = (x^3 + 1)(2x^2 - 4x - 1) \)

**Solution.** Applying the product rule, we have

\[
f'(x) = 3x^2(2x^2 - 4x - 1) + (x^3 + 1)(4x - 4) = (6x^4 - 12x^3 - 3x^2) + (4x^4 - 4x^3 + 4x - 4) = \]

\[
10x^4 - 16x^3 - 3x^2 + 4x - 4. \]

27. \( f(x) = \frac{x^2 + 4x + 3}{\sqrt{x}} \)

**Solution.** There are many ways of calculating \( f'(x) \): we can use the quotient rule or separate first and use the power rule or use the product rule. We will use the separation method because it is easier. We have

\[
f(x) = \frac{x^2 + 4x + 3}{\sqrt{x}} = \frac{x^2}{\sqrt{x}} + \frac{4x}{\sqrt{x}} + \frac{3}{\sqrt{x}} = x^{\frac{3}{2}} + 4x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}. \]

Now, by applying the sum rule and the power rule, the derivative is:

\[
f'(x) = 3\frac{1}{2}x^{\frac{1}{2}} + 4\left( \frac{1}{2}x^{-\frac{3}{2}} \right) + 3\left( -\frac{1}{2}x^{-\frac{3}{2}} \right) = 3\frac{1}{2}x^{\frac{1}{2}} + 2x^{-\frac{1}{2}} - 3\frac{1}{2}x^{-\frac{3}{2}}. \]
28. \( f(x) = (\frac{1}{x^2} - \frac{3}{x^3})(x + 5x^3) \)

**Solution.** Applying the product rule, we get

\[
f'(x) = \left( \frac{d}{dx} \left[ \frac{1}{x^2} - \frac{3}{x^3} \right] \right) (x + 5x^3) + \left( \frac{d}{dx} \left[ x + 5x^3 \right] \right) \left( \frac{1}{x^2} - \frac{3}{x^3} \right)
\]

\[
\left( \frac{d}{dx} \left[ x^{-2} - 3x^{-3} \right] \right) (x + 5x^3) + \left( \frac{d}{dx} \left[ x + 5x^3 \right] \right) = \]

\[
\left( -2x^{-3} - 3(-4x^{-4}) \right) (x + 5x^3) + (x^{-2} - 3x^{-3} - 15x^2) = \]

\[
-2x^{-2} - 10x^0 + 12x^{-4} + 60x^{-2} + x^{-2} + 15x^0 - 3x^{-4} - 45x^{-2} = \]

\[
14x^{-2} + 5x^0 + 9x^{-4} = \frac{14}{x^2} + 5 + \frac{9}{x^4}.
\]

Note that \( x^0 = 1 \).

29. \( f(x) = \frac{5x+1}{5x-1} \)

**Solution.** First recall the following formula.

**The Quotient Rule:** \( \frac{d}{dx} \left[ \frac{u(x)}{v(x)} \right] = \frac{u'(x)v(x) + u(x)v'(x)}{(v(x))^2} \).

Here \( u(x) = 5x + 1 \) and \( v(x) = 5x - 1 \). So \( u'(x) = 5 \) and \( v'(x) = 5 \). Applying the quotient rule, we have

\[
f'(x) = \frac{5(5x-1) - (5x+1)(5)}{(5x-1)^2} = \frac{25x - 5 - 25x + 5}{(5x-1)^2} = \frac{25x - 25}{(5x-1)^2} = \frac{-10}{(5x-1)^2}.
\]

30. \( f(x) = \frac{1+2x}{3-4x} \)

**Solution.** Applying the quotient rule, we have

\[
f'(x) = \frac{(1 + 2x)'(3 - 4x) - (1 + 2x)(3 - 4x)'}{(3 - 4x)^2} = \frac{2(3 - 4x) - (1 + 2x)(-4)}{(3 - 4x)^2} = \]

\[
\frac{6 - 8x - (-4 - 8x)}{(3 - 4x)^2} = \frac{6 - 8x + 4 + 8x}{(3 - 4x)^2} = \frac{10}{(3 - 4x)^2}.
\]

31. \( f(x) = \frac{x^2+1}{x^3-1} \)

**Solution.** Applying the quotient rule, we have

\[
f'(x) = \frac{(x^2 + 1)'(x^3 - 1) - (x^2 + 1)(x^3 - 1)'}{(x^3 - 1)^2} = \frac{2x(x^3 - 1) - (x^2 + 1)(3x^2)}{(x^3 - 1)^2} = \]

\[
\frac{2x^4 - 2x - 3x^4 + 3x^2}{(x^3 - 1)^2} = \frac{2x^4 - 2x - 3x^4 + 3x^2}{(x^3 - 1)^2} = \frac{-x^4 - 2x - 3x^2}{(x^3 - 1)^2}.
\]

32. \( f(x) = \frac{x^3+3x}{x^2-4x+3} \)

**Solution.** Applying the quotient rule, we have

\[
f'(x) = \frac{(x^3 + 3x)'(x^2 - 4x + 3) - (x^3 + 3x)(x^2 - 4x + 3)'}{(x^2 - 4x + 3)^2} = \]

\[
\frac{3x^2 + 3)(x^2 - 4x + 3) - (x^3 + 3x)(2x - 4)}{(x^2 - 4x + 3)^2} = \]
\( \frac{(3x^4 - 12x^3 + 9x^2 + 3x^2 - 12x + 9) - (2x^4 - 4x^3 + 6x^2 - 12x)}{(x^2 - 4x + 3)^2} = \frac{3x^4 - 12x^3 + 9x^2 + 3x^2 - 12x + 9 - 2x^4 + 4x^3 - 6x^2 + 12x}{(x^2 - 4x + 3)^2} = \frac{x^4 - 8x^3 + 6x^2 + 9}{(x^2 - 4x + 3)^2}. \)

33. \( f(x) = \frac{1}{x^3 + 2x - 1} \)

**Solution.** Applying the quotient rule, we have

\[
\frac{d}{dx} \left( \frac{1}{x^3 + 2x - 1} \right) = \frac{\frac{d}{dx} (x^3 + 2x - 1)}{(x^3 + 2x - 1)^2} = \frac{3x^2 + 2}{(x^3 + 2x - 1)^2}.
\]

34. \( f(x) = \frac{\sqrt{x}}{x+3} \)

**Solution.** Applying the quotient rule, we have

\[
\frac{d}{dx} \left( \frac{\sqrt{x}}{x+3} \right) = \frac{(\sqrt{x})' (x+3) - \sqrt{x} (x+3)'}{(x+3)^2} = \frac{\frac{1}{2\sqrt{x}} (x+3) - \sqrt{x}}{(x+3)^2} = \frac{\frac{x+3}{2\sqrt{x}} - \sqrt{x}}{(x+3)^2} = \frac{3 - x}{2\sqrt{x}(x+3)^2}.
\]

35. \( f(x) = \frac{2x^5 + x^4 - 6x}{x^3 + x} \)

**Solution.** Here it is easier to first separate, and then use the sum and power rules (instead of using the quotient rule). We have

\[
\frac{d}{dx} \left[ \frac{2x^5 + x^4 - 6x}{x^3 + x} \right] = \frac{d}{dx} \left[ \frac{2x^5 + x^4 - 6x}{x^3 + x} \right] = \frac{d}{dx} \left[ \frac{2x^4 + x^3 - 6}{x^3 + x} \right] = \frac{d}{dx} \left[ \frac{2x^4 + x^3 - 6}{x^3 + x} \right] = \frac{2(4x^3) + 3x^2 - 0}{x^3 + x} = 8x^3 + 3x^2.
\]

36. \( f(x) = \frac{x}{x+1} \)

**Solution.** First we have \( f(x) = \frac{x}{x+1} = \frac{x}{x+1} = \frac{x}{x+1} = \frac{x^2}{x^2 + 1}. \) Now we have the following derivative (by applying the quotient rule):

\[
\frac{d}{dx} \left[ \frac{x^2}{x^2 + 1} \right] = \frac{(x^2)'(x^2 + 1) - x^2(x^2 + 1)'}{(x^2 + 1)^2} = \frac{2x(x^2 + 1) - x^2(2x)}{(x^2 + 1)^2} = \frac{2x^3 + 2x - 2x^3}{(x^2 + 1)^2} = \frac{2x}{(x^2 + 1)^2}.
\]

(ii) Find an equation of the tangent line to the curve at the given point.

1. \( y = \frac{2x}{x+1}, P(1,1) \)

**Solution.** Let \( f(x) = \frac{2x}{x+1} \). We need two things: the point and the slope. We already have the point, which is \( P(1,1) \). For the slope, we need to find the derivative \( f'(x) \).

- Applying the quotient rule, we have

\[
f'(x) = \frac{2(x+1) - 2x(1)}{(x+1)^2} = \frac{2x + 2 - 2x}{(x+1)^2} = \frac{2}{(x+1)^2}.
\]
2.5. SOLUTION TO PB5

- So the slope at $P(1, 1)$ is: $m = f'(1) = \frac{2}{(1+1)^2} = \frac{2}{4} = \frac{1}{2}$.
- Now the desired equation is: $y - y_1 = m(x - x_1)$, where $x_1 = 1$, $y_1 = 1$, and $m = \frac{1}{2}$. So we have $y - 1 = \frac{1}{2}(x - 1)$.

2. $y = 2x^3 - x^2 + 2, P(1, 3)$.

Solution. Let $f(x) = 2x^3 - x^2 + 2$. As before, we have three steps:

- The derivative is: $f'(x) = 6x^2 - 2x + 0 = 6x^2 - 2x + 0 = 6x^2 - 2x$.
- The slope at $P(1, 3)$ is: $m = f'(1) = 6(1)^2 - 2(1) = 6 - 2 = 4$.
- The equation of the tangent line to the curve $y = f(x) = 2x^3 - x^2 + 2$ at $P(1, 3)$ is: $y - 3 = 4(x - 1)$.

(iii) Find the points on the curve $y = \frac{x^2 - 2x + 1}{x - 3}$ where the tangent line is horizontal.

Solution. Let $f(x) = \frac{x^2 - 2x + 1}{x - 3}$. Horizontal tangents occur where the derivative is zero. Applying the quotient rule, we have

$$f'(x) = \frac{(2x - 2)(x - 3) - (x^2 - 2x + 1)(1)}{(x - 3)^2} = \frac{2x^2 - 6x - 2x + 6 - x^2 + 2x - 1}{(x - 3)^2} = \frac{x^2 - 6x + 5}{(x - 3)^2} = \frac{(x - 1)(x - 5)}{(x - 3)^2}.$$  

We are looking for points that make the derivative $f'(x)$ equal to 0. That is, we are looking for points $x$ such that $\frac{(x - 1)(x - 5)}{(x - 3)^2} = 0$. This latter equation is equivalent to $(x - 1)(x - 5) = 0$. So $x = 1$ or $x = 5$. (Note that $x = 3$ is not a solution since $f'(x)$ is undefined when $x = 3$).

- If $x = 1$, then $f(x) = f(1) = \frac{1^2 - 2(1) + 1}{1 - 3} = 0 - \frac{2}{2} = 0$. This gives us the point $P(1, 0)$.
- If $x = 5$, then $f(5) = \frac{5^2 - 2(5) + 1}{5 - 3} = \frac{16}{2} = 8$. This gives us the point $Q(5, 8)$.

Conclusion: The tangent line is horizontal at the points $P(1, 0)$ and $Q(5, 8)$.

(iv) If $h(x) = \sqrt{x}g(x)$, where $g(4) = 8$ and $g'(4) = 7$, find $h'(4)$.

Solution. Applying the product rule, we have

$$h'(x) = (\sqrt{x})'g(x) + \sqrt{x}g'(x) = \frac{1}{2\sqrt{x}}g(x) + \sqrt{x}g'(x).$$

Substituting $x$ by 4, we get

$$h'(4) = \frac{1}{2\sqrt{4}}g(4) + \sqrt{4}g'(4).$$

But $g(4)$ and $g'(4)$ are both given ($g(4) = 8$ and $g'(4) = 7$). So

$$h'(4) = \frac{1}{2\sqrt{4}}(8) + \sqrt{4}(7) = \frac{1}{4}(8) + 2(7) = 2 + 14 = 16.$$  

Section 2.4: Derivatives of Trigonometric Functions

Find the derivative of each of the following functions.
CHAPTER 2. SOLUTION TO PRACTICE PROBLEMS

1. \( f(x) = -5 \cos x \).

**Solution.** First recall that \( \frac{d}{dx} \cos x = -\sin x \).

We have

\[
 f'(x) = \frac{d}{dx}[-5 \cos x] = -5 \frac{d}{dx} \cos x = -5(-\sin x) = 5 \sin x.
\]

2. \( f(x) = \frac{\sin x}{4} - x^3 \)

**Solution.** First recall that \( \frac{d}{dx} \sin x = \cos x \).

We have

\[
 f'(x) = \frac{d}{dx} \left[ \frac{\sin x}{4} - x^3 \right] = \frac{d}{dx} \frac{\sin x}{4} - \frac{d}{dx} x^3 = \frac{1}{4} \frac{d}{dx} \sin x - 3x^2 = \frac{1}{4} \cos x - 3x^2.
\]

3. \( f(x) = 3 \sin x - 8 \cos x + 2 \tan x \)

**Solution.** First recall that \( \frac{d}{dx} \tan x = \sec^2 x = \frac{1}{\cos^2 x} \).

We have

\[
 f'(x) = \frac{d}{dx} [3 \sin x - 8 \cos x + 2 \tan x] = \frac{d}{dx} [3 \sin x] - \frac{d}{dx} [8 \cos x] + \frac{d}{dx} [2 \tan x] = \
 3 \frac{d}{dx} [\sin x] - 8 \frac{d}{dx} [\cos x] + 2 \frac{d}{dx} [\tan x] = 3 \cos x - 8(-\sin x) + 2 \sec^2 x = 3 \cos x + 8 \sin x + 2 \sec^2 x.
\]

4. \( f(x) = 7 \sec x - \csc x + \cot x - 6 \)

**Solution.** First recall the following trigonometric functions and their derivatives.

- The **cotangent function** denoted \( \cot x \) is defined to be \( \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x} \). Its derivative is given by
  \[
  \frac{d}{dx} [\cot x] = -\csc^2 x = -\frac{1}{\sin^2 x}.
  \]

- The **secant function** denoted \( \sec x \) is defined to be \( \sec x = \frac{1}{\cos x} \). Its derivative is given by
  \[
  \frac{d}{dx} [\sec x] = \sec x \tan x.
  \]

- The **cosecant function** denoted \( \csc x \) is defined to be \( \csc x = \frac{1}{\sin x} \). Its derivative is given by
  \[
  \frac{d}{dx} [\csc x] = -\csc x \cot x.
  \]

Now we have

\[
 f'(x) = 7 \frac{d}{dx} [\sec x] - \frac{d}{dx} [\csc x] + \frac{d}{dx} [\cot x] - \frac{d}{dx} [6] = 7 \sec x \tan x - (-\csc x \cot x) + (-\csc^2 x) - 0 = 
  7 \sec x \tan x + \csc x \cot x - \csc^2 x.
\]

5. \( f(x) = x \cot x \)

**Solution.** Using the Product Rule, we get

\[
 f'(x) = \frac{d}{dx} [x] \cot x + x \frac{d}{dx} [\cot x] = (1) \cot x + x(-\csc^2 x) = \cot x - x \csc^2 x.
\]
6. \( f(x) = \sin x \cos x \)

**Solution.** Again by the Product Rule, we have

\[
\frac{d}{dx}[\sin x \cos x] = (\cos x) \cos x + \sin x \frac{d}{dx}[\cos x] = \cos^2 x - \sin^2 x.
\]

7. \( f(x) = \frac{\sin x}{x} \)

**Solution.** Using the Quotient Rule, we have

\[
\frac{d}{dx}\left[\frac{\sin x}{x}\right] = \frac{x \cos x - \sin x}{x^2}.
\]

8. \( f(x) = \frac{\sec x}{\csc x} \)

**Solution.** Again by using the Quotient Rule, we have

\[
\frac{d}{dx}\left[\frac{\sec x}{\csc x}\right] = \frac{\sec x \csc x + \sec x \csc x \cot x}{\csc^2 x}.
\]

9. \( f(x) = x \sin x + \frac{\cos x}{x} \)

**Solution.** We have

\[
\frac{d}{dx}[x \sin x] + \frac{d}{dx}\left[\frac{\cos x}{x}\right] = \frac{d}{dx}[\sin x] + x \frac{d}{dx}[\sin x] + \frac{d}{dx}[\cos x] - \cos \frac{d}{dx}[x] = (1) \sin x + x \cos x + \frac{-\sin x(1)}{x^2} = \sin x + x \cos x + \frac{-\sin x - \cos x}{x^2}.
\]

10. \( f(x) = x^9 + \frac{\sqrt{x}}{2} + \tan x \)

**Solution.** We have

\[
\frac{d}{dx}[x^9] + \frac{d}{dx}\left[\frac{\sqrt{x}}{2}\right] + \frac{d}{dx}[\tan x] = 9x^8 + \frac{d}{dx}\left[\frac{\sqrt{x}}{2}\right] + \frac{d}{dx}[\sec^2 x] = 9x^8 + \frac{1}{2} \frac{d}{dx}[\sqrt{x}] + \sec^2 x = \sec^2 x.
\]

11. \( f(x) = x^2 \cos x - 2 \tan x + 3 \)

**Solution.** We have

\[
\frac{d}{dx}[x^2 \cos x] - 2 \frac{d}{dx}[\tan x] + \frac{d}{dx}[3] = \frac{d}{dx}[x^2 \cos x] + x^2 \frac{d}{dx}[\cos x] - 2 \sec^2 x + 0 = 2x \cos x + 2 \cos x - 2 \sec^2 x = 2 \cos x - x^2 \sin x - 2 \sec^2 x.
\]

12. \( f(x) = x \cos x + x^2 \sin x \)

**Solution.** We have

\[
\frac{d}{dx}[x \cos x] + \frac{d}{dx}[x^2 \sin x] = \frac{d}{dx}[x \cos x] + x \frac{d}{dx}[\cos x] + \frac{d}{dx}[x^2] \sin x + x^2 \frac{d}{dx}[\sin x] = (1) \cos x + x(- \sin x) + 2x \sin x + x^2 \cos x = \cos x - x \sin x + 2x \sin x + x^2 \cos x = \cos x + x \sin x + x^2 \cos x.
\]
13. \( f(x) = \frac{\sin x}{1 + \tan x} \)

**Solution.** We have

\[
f'(x) = \frac{\frac{d}{dx} \sin x (1 + \tan x) - \sin x \frac{d}{dx} [1 + \tan x]}{(1 + \tan x)^2} = \frac{\cos x (1 + \tan x) - \sin x (0 + \sec^2 x)}{(1 + \tan x)^2} = \frac{\cos x + \cos x \tan x - \sin x \sec x}{(1 + \tan x)^2}.
\]

14. \( f(x) = \frac{1}{\cos x} + \csc x - \sqrt{x} \)

**Solution.** We have

\[
f'(x) = \frac{d}{dx} \left[ \frac{1}{\cos x} \right] + \frac{d}{dx} \left[ \csc x \right] - \frac{d}{dx} \left[ \sqrt{x} \right] = \frac{\frac{d}{dx} [1] \cos x - \frac{d}{dx} [\cos x]}{\cos^2 x} + \frac{1}{3} \frac{d}{dx} [\csc x] - \frac{d}{dx} \left[ \frac{1}{7} \right] = \frac{(0) \cos x - (-\sin x)}{\cos^2 x} + \frac{1}{3} (-\csc x \cot x) - \frac{1}{7} x^{-\frac{6}{7}} = \frac{\sin x}{\cos^2 x} - \frac{1}{3} \csc x \cot x - \frac{1}{7} x^{-\frac{6}{7}}.
\]

15. \( f(x) = x \cos x \sin x \)

**Solution.** First we have \( f(x) = (x \cos x) \sin x \). Now we have

\[
f'(x) = \frac{d}{dx} [x \cos x \sin x] + x \cos x \frac{d}{dx} [\sin x] = \left( \frac{d}{dx} [x] \cos x + x \frac{d}{dx} \cos x \right) \sin x + x \cos x \cos x = (\cos x - x \sin x) \sin x + x \cos^2 x = \cos x \sin x - x \sin^2 x + x \cos^2 x.
\]

**Section 2.5: The Chain Rule**

Find the derivative of each of the following functions.

1. \( f(x) = (2x)^5 \)

**Solution.** First recall the General Power Rule, which is a combination of the Power Rule with the Chain Rule:

\[
\frac{d}{dx} [(g(x))^n] = n(g(x))^{n-1} \frac{d}{dx} [g(x)].
\]

Here \( g(x) = 2x \) and \( n = 5 \). So

\[
f'(x) = 5(2x)^5 \frac{d}{dx} [2x] = 5(2x)^4 (2) = (5)(2)(2x)^4 = 10(2x)^4.
\]

2. \( f(x) = (-3x + 4)^8 \)

**Solution.** We have

\[
f'(x) = 8(-3x + 4)^7 \frac{d}{dx} [-3x + 4] = 8(-3x + 4)^7 (-3) = 8(-3)(-3x + 4)^7 = -24(-3x + 4)^7.
\]

3. \( f(x) = 3(5x - 1)^7 \)

**Solution.** We have

\[
f'(x) = \frac{d}{dx} [3(5x - 1)^7] = 3 \frac{d}{dx} [(5x - 1)^7] = 3(7)(5x - 1)^6 \frac{d}{dx} [5x - 1] = 21(5x - 1)^6 (5) = 105(5x - 1)^6.
\]
4. \( f(x) = (x^2 - 1)^{\frac{3}{2}} \)

**Solution.** We have

\[
f'(x) = \frac{3}{2} (x^2 - 1)^{\frac{1}{2}} \cdot \frac{d}{dx} (x^2 - 1) = \frac{3}{2} (x^2 - 1)^{\frac{1}{2}} \cdot 2x = \frac{3}{2} (2x)(x^2 - 1)^{\frac{1}{2}} = 3x(x^2 - 1)^{\frac{1}{2}}.
\]

5. \( f(x) = (-x^3 + 2x + 1)^{15} \)

**Solution.** We have

\[
f'(x) = 15(-x^3 + 2x + 1)^{14} \cdot \frac{d}{dx} (-x^3 + 2x + 1) = 15(-x^3 + 2x + 1)^{14}(-3x^2 + 2) = 15(-3x^2 + 2)(-x^3 + 2x + 1)^{14}.
\]

6. \( f(x) = -2(5x^6 - 2x)^4 \)

**Solution.** We have

\[
f'(x) = \frac{d}{dx} [-2(5x^6 - 2x)^4] = -2 \frac{d}{dx} [(5x^6 - 2x)^4] = -2(4)(5x^6 - 2x)^3 \frac{d}{dx} [5x^6 - 2x] = -8(5x^6 - 2x)^3(30x^5 - 2) = -8(30x^5 - 2)(5x^6 - 2x)^3.
\]

7. \( f(x) = (2 - \sin x)^{\frac{3}{2}} \)

**Solution.** We have

\[
f'(x) = \frac{5}{2} (2 - \sin x)^{\frac{1}{2}} \cdot \frac{d}{dx} [2 - \sin x] = \frac{5}{2} (2 - \sin x)^{\frac{1}{2}} (0 - \cos x) = 5 \frac{2 - \sin x}{2} (-\cos x) = \frac{5}{2} \cos x(2 - \sin x)^{\frac{1}{2}}.
\]

8. \( f(x) = \sqrt{x^2 - x} \)

**Solution.** First we have \( f(x) = \sqrt{x^2 - x} = (x^2 - x)^{\frac{1}{2}} \). Now we have

\[
f'(x) = \frac{1}{2} (x^2 - x)^{\frac{1}{2}} \cdot \frac{d}{dx} [x^2 - x] = \frac{1}{2} (x^2 - x)^{-\frac{1}{2}} (2x - 1) = \frac{1}{2} (2x - 1) \cdot \frac{1}{(x^2 - x)^{\frac{1}{2}}} = \frac{1}{2} \cdot \frac{2x - 1}{\sqrt{x^2 - x}} = \frac{2x - 1}{2 \sqrt{x^2 - x}}.
\]

9. \( f(x) = \frac{1}{\sqrt{x^2 - 1}} \)

**Solution.** First we have \( f(x) = \frac{1}{(x^2 - 1)^{\frac{1}{2}}} = (x^2 - 1)^{-\frac{1}{2}} \). Now we have

\[
f'(x) = -\frac{1}{3} (x^2 - 1)^{-\frac{3}{2}} \cdot \frac{d}{dx} [x^2 - 1] = -\frac{1}{3} (x^2 - 1)^{-\frac{3}{2}} (2x) = -\frac{2}{3} x (x^2 - 1)^{-\frac{3}{2}}.
\]

10. \( f(x) = (-5x + 4)(x^3 + 1)^6 \)

**Solution.** If we look at \( f(x) \), we can see many rules including the General Power Rule (G.P.R), the Product Rule, etc. The G.P.R applies only to the second term, while the Product Rule applies to the whole function. So we are going to first use the Product Rule.

\[
f'(x) = \frac{d}{dx} [-5x + 4] (x^3 + 1)^6 + (-5x + 4) \frac{d}{dx} [(x^3 + 1)^6] = -5(x^3 + 1)^6 + (-5x + 4)6(x^3 + 1)^5(3x^2) = -5(x^3 + 1)^6 + 18x^2(-5x + 4)(x^3 + 1)^5 = (x^3 + 1)^5 (-5(x^3 + 1) + 18x^2(-5x + 4)) = (x^3 + 1)^5(-5x^3 - 5 - 90x^3 + 72x^2) = (x^3 + 1)^5(-95x^3 + 72x^2 - 5).
\]
11. \( f(x) = x\sqrt{2 - x^2} \)

**Solution.** First we have \( f(x) = x(2 - x^2)^{\frac{1}{2}} \). As before, we first use the Product Rule.

\[
\frac{d}{dx}[x(2 - x^2)^{\frac{1}{2}}] = \frac{d}{dx}[x(2 - x^2)^{\frac{1}{2}}] = (1)(2 - x^2)^{\frac{1}{2}} + x \frac{d}{dx}[(2 - x^2)^{\frac{1}{2}}] = (1)(2 - x^2)^{\frac{1}{2}} + x \left( \frac{1}{2} \right)(2 - x^2)^{-\frac{1}{2}}(-2x) = \\
(2 - x^2)^{\frac{1}{2}} - x^2 \frac{1}{(2 - x^2)^{\frac{1}{2}}} = \frac{\sqrt{2 - x^2} - \frac{x^2}{\sqrt{2 - x^2}}}{\frac{2(1-x^2)}{\sqrt{2 - x^2}}} = \frac{2(1-x^2) - x^2}{\sqrt{2 - x^2}} = \frac{\sqrt{2 - x^2} - x^2}{2(1-x^2)}. 
\]

12. \( f(x) = (x^2 + 1)^3(x^2 + 2)^6 \)

**Solution.** We have

\[
f'(x) = \frac{d}{dx}[(x^2 + 1)^3(x^2 + 2)^6] + (x^2 + 1)^3 \frac{d}{dx}[(x^2 + 2)^6] = \\
3(x^2 + 1)(2x)(x^2 + 2)^6 + (x^2 + 1)^3(6(x^2 + 2)(2x)) = 6x(x^2 + 1)^2(x^2 + 2)^6 + 12x(x^2 + 1)^3(x^2 + 2)^5 = \\
6x(x^2 + 1)^2(x^2 + 2)^5(x^2 + 2 + 2(x^2 + 1)) = 6x(x^2 + 1)^2(x^2 + 2)^5(x^2 + 2 + 2x^2 + 2) = \\
6x(x^2 + 1)^2(x^2 + 2)^5(3x^2 + 4). 
\]

13. \( f(x) = \frac{(x+1)^5}{x^3+1} \)

**Solution.** Using the Quotient Rule, we get

\[
f'(x) = \frac{d}{dx}(x^5 + 1) - (x+1)^5 = \frac{5(x^5 + 1) - (x+1)^5}{} = \frac{(x+1)^4(5x^5 + 5 - 5x^5 - 5x^4)}{x^3 + 1} = \frac{(x+1)^4(5 - 5x^4)}{x^3 + 1}.
\]

14. \( f(x) = \frac{x^2}{\sqrt{x^3+1}} \)

**Solution.** First we have \( f(x) = \frac{x^2}{(x^3+1)^{\frac{1}{2}}} \). Now, by using the Quotient Rule, we get

\[
f'(x) = \frac{2x(x^3 + 1)^{\frac{1}{2}} - x^2 \frac{1}{2}(x^3 + 1)^{-\frac{1}{2}}(3x^2)}{(x^3 + 1)} = \\
\frac{2x \sqrt{x^3 + 1} - \frac{3}{2} x^4 \sqrt{x^3 + 1}}{x^3 + 1} = \frac{2x(x^3 + 1) - \frac{3}{2} x^4}{x^3 + 1} = \\
2x^4 + 2x - \frac{3}{2} x^4 = \frac{1}{2} x^4 + 2x = \frac{1}{2} x^4 + 4 = \frac{x(x^3 + 4)}{2(x^3 + 1)}.
\]

15. \( f(x) = \sqrt{x^{x+3}} \)

**Solution.** First we have \( f(x) = \left( \frac{x}{x^3+1} \right)^{\frac{1}{2}} \). Now we have

\[
f'(x) = \frac{1}{2} \left( \frac{x}{x^2 + 3} \right)^{-\frac{1}{2}} \frac{d}{dx} \left[ \frac{x}{x^2 + 3} \right] = \frac{1}{2} \left( \frac{x}{x^2 + 3} \right)^{-\frac{1}{2}} \left( \frac{(x^2 + 3)^2 - x(x^2 + 3)}{(x^2 + 3)^2} \right) = \\
\frac{1}{2} \left( \frac{x}{x^2 + 3} \right)^{-\frac{1}{2}} \left( \frac{-x^2 + 3}{(x^2 + 3)^2} \right) = \frac{-x^2 + 3}{2(x^2 + 3)^2} \left( \frac{x}{x^2 + 3} \right)^{-\frac{1}{2}}.
\]
16. \( f(x) = \left( \frac{x^4 + 1}{x^2 + 1} \right)^5 \)

**Solution.** We have
\[
f'(x) = 5 \left( \frac{x^4 + 1}{x^2 + 1} \right)^4 \frac{d}{dx} \left[ \frac{x^4 + 1}{x^2 + 1} \right] = 5 \left( \frac{x^4 + 1}{x^2 + 1} \right)^4 \left[ \frac{4x^3(x^2 + 1) - (x^4 + 1)2x}{(x^2 + 1)^2} \right] =
\]
\[
\begin{align*}
& 5 \left( \frac{x^4 + 1}{x^2 + 1} \right)^4 \left( \frac{4x^5 + 4x^3 - 2x^5 - 2x}{(x^2 + 1)^2} \right) = 5 \left( \frac{x^4 + 1}{x^2 + 1} \right)^4 \left( \frac{2x^5 + 4x^3 - 2x}{(x^2 + 1)^2} \right) \\
& 5 \left( \frac{2x(x^4 + 2x^2 - 1)}{(x^2 + 1)^2} \right) \left( \frac{x^4 + 1}{x^2 + 1} \right)^4 = \frac{10x(x^4 + 2x - 1)}{(x^2 + 1)^2} \left( \frac{x^4 + 1}{x^2 + 1} \right)^4.
\end{align*}
\]

17. \( f(x) = \sqrt{\frac{1 + \sin x}{1 + \cos x}} \)

**Solution.** First we have \( f(x) = \left( \frac{1 + \sin x}{1 + \cos x} \right)^{\frac{1}{2}} \). Now we have
\[
f'(x) = \frac{1}{2} \left( \frac{1 + \sin x}{1 + \cos x} \right)^{-\frac{1}{2}} \frac{d}{dx} \left[ \frac{1 + \sin x}{1 + \cos x} \right] = \]
\[
\frac{1}{2} \left( \frac{1 + \sin x}{1 + \cos x} \right)^{-\frac{1}{2}} \left( \cos x(1 + \cos x) - (1 + \sin x)(-\sin x) \right) = \]
\[
\frac{1}{2} \left( \frac{1 + \sin x}{1 + \cos x} \right)^{-\frac{1}{2}} \left( \cos x + \cos x^2 + \sin x + \sin^2 x \right) = \]
\[
\frac{1}{2} \left( \frac{1 + \sin x}{1 + \cos x} \right)^{-\frac{1}{2}} \left( \cos x + x + 1 \right) = \frac{\cos x + x + 1}{2(1 + \cos x)^2} \left( \frac{1 + \sin x}{1 + \cos x} \right)^{-\frac{1}{2}}.
\]

18. \( f(x) = \sin^2 x \)

**Solution.** First we have \( f(x) = (\sin x)^2 \). Now we have
\[
f'(x) = 2(\sin x) \frac{1}{dx} [\sin x] = 2 \sin x \cos x.
\]

19. \( f(x) = \frac{\cos^4 x}{8} - \sec^2 x + \sqrt[3]{\csc x} \)

**Solution.** We have
\[
f'(x) = \frac{1}{8} \frac{d}{dx} [\cos^4 x] - \frac{d}{dx} [\sec^2 x] + \frac{d}{dx} [\sqrt[3]{\csc x}] = \frac{1}{8} \frac{d}{dx} [(\cos x)^4] - \frac{d}{dx} [(\sec x)^2] + \frac{d}{dx} [(\csc x)^{\frac{1}{3}}] = \]
\[
\frac{1}{8}(4)(\cos x)^3 \frac{d}{dx} [\cos x] - 2(\sec x) \frac{d}{dx} [\sec x] + \frac{1}{3}(\csc x)^{-\frac{4}{3}} - \frac{1}{dx} [\csc x] = \]
\[
\frac{1}{2}(\cos x)^3(-\sin x) - 2 \sec x(\sec x \tan x) + \frac{1}{3}(\csc x)^{-\frac{4}{3}}(-\csc x \cot x) = \]
\[
-\frac{1}{2} \sin x \cos^3 x - 2 \sec^2 x \tan x - \frac{1}{3} \cot x(\csc x)^{\frac{1}{3}} = -\frac{1}{2} \sin x \cos^3 x - 2 \sec^2 x \tan x - \frac{1}{3} \cot x \sqrt[3]{\csc x}.
\]

20. \( f(x) = -3x^4 + (-2x^4 + 1)^{10} - \frac{1}{\sqrt{x - \cos x}} \)

**Solution.** We have
\[
f'(x) = \frac{d}{dx} [-3x^4] + \frac{d}{dx} [(-2x^4 + 1)^{10}] - \frac{d}{dx} [(x - \cos x)^{-\frac{1}{2}}] = \]
\[
-12x^3 + 10(-2x^4 + 1)^9(-8x^3) - \left( -\frac{1}{2} \right)(x - \cos x)^{-\frac{3}{2}}(1 + \sin x) = \]
\[
-12x^3 - 80x^3(-2x^4 + 1)^9 + \frac{1}{2}(1 + \sin x)(x - \cos x)^{-\frac{3}{2}}.
\]
2.6 Solution to PB6

Section 2.4: Derivatives of Trigonometric Functions

Find the derivative of each of the following functions.

1. \( f(x) = -5 \cos x \).

   **Solution.** First recall that \( \frac{d}{dx} \cos x = -\sin x \).

   We have
   \[
   f'(x) = \frac{d}{dx} [-5 \cos x] = -5 \frac{d}{dx} \cos x = -5(-\sin x) = 5 \sin x.
   \]

2. \( f(x) = \frac{\sin x}{4} - x^3 \)

   **Solution.** First recall that \( \frac{d}{dx} \sin x = \cos x \).

   We have
   \[
   f'(x) = \frac{d}{dx} \left[ \frac{\sin x}{4} - x^3 \right] = \frac{d}{dx} \left[ \frac{\sin x}{4} \right] - \frac{d}{dx} [x^3] = \frac{1}{4} \frac{d}{dx} [\sin x] - 3x^2 = \frac{1}{4} \cos x - 3x^2.
   \]

3. \( f(x) = 3 \sin x - 8 \cos x + 2 \tan x \)

   **Solution.** First recall that \( \frac{d}{dx} \tan x = \sec^2 x = \frac{1}{\cos^2 x} \).

   We have
   \[
   f'(x) = \frac{d}{dx} [3 \sin x - 8 \cos x + 2 \tan x] = \frac{d}{dx} [3 \sin x] - \frac{d}{dx} [8 \cos x] + \frac{d}{dx} [2 \tan x] =
   3 \frac{d}{dx} [\sin x] - 8 \frac{d}{dx} [\cos x] + 2 \frac{d}{dx} [\tan x] = 3 \cos x - 8(-\sin x) + 2 \sec^2 x = 3 \cos x + 8 \sin x + 2 \sec^2 x.
   \]

4. \( f(x) = 7 \sec x - \csc x + \cot x - 6 \)

   **Solution.** First recall the following trigonometric functions and their derivatives.

   - The **cotangent function** denoted \( \cot x \) is defined to be \( \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x} \). Its derivative is given by
     \[
     \frac{d}{dx} [\cot x] = -\csc^2 x = -\frac{1}{\sin^2 x}.
     \]

   - The **secant function** denoted \( \sec x \) is defined to be \( \sec x = \frac{1}{\cos x} \). Its derivative is given by
     \[
     \frac{d}{dx} [\sec x] = \sec x \tan x.
     \]

   - The **cosecant function** denoted \( \csc x \) is defined to be \( \csc x = \frac{1}{\sin x} \). Its derivative is given by
     \[
     \frac{d}{dx} [\csc x] = -\csc x \cot x.
     \]
2.6. SOLUTION TO PB6

Now we have
\[ f'(x) = 7 \frac{d}{dx} [\sec x] - \frac{d}{dx} [\csc x] + \frac{d}{dx} [\cot x] - \frac{d}{dx} [6] = 7 \sec x \tan x - (\csc x \cot x) + (-\csc^2 x) - 0 = 7 \sec x \tan x + \csc x \cot x - \csc^2 x. \]

5. \( f(x) = x \cot x \)

**Solution.** Using the Product Rule, we get
\[ f'(x) = \frac{d}{dx} [\cot x] = (1) \cot x + x \frac{d}{dx} [\cot x] = x - x \csc^2 x. \]

6. \( f(x) = \sin x \cos x \)

**Solution.** Again by the Product Rule, we have
\[ f'(x) = \frac{d}{dx} [\sin x \cos x] = (\cos x) \cos x + \sin x (\sin x) = \cos^2 x - \sin^2 x. \]

7. \( f(x) = \frac{\sin x}{x} \)

**Solution.** Using the Quotient Rule, we have
\[ f'(x) = \frac{\frac{d}{dx} [\sin x] \cdot x - \sin x \frac{d}{dx} [x]}{x^2} = \frac{\cos x - \sin x(1)}{x^2} = \frac{x \cos x - \sin x}{x^2}. \]

8. \( f(x) = \frac{\sec x}{\csc x} \)

**Solution.** Again by using the Quotient Rule, we have
\[ f'(x) = \frac{\frac{d}{dx} [\sec x] \csc x - \sec x \frac{d}{dx} [\csc x]}{(\csc x)^2} = \frac{\sec x \tan x \csc x - \sec x (-\csc x \cot x)}{(\csc x)^2} = \frac{\sec x \csc x (\tan x + \cot x)}{\csc^2 x}. \]

9. \( f(x) = x \sin x + \frac{\cos x}{x} \)

**Solution.** We have
\[ f'(x) = \frac{d}{dx} [x \sin x] + \frac{d}{dx} \left[ \frac{\cos x}{x} \right] = \frac{d}{dx} [\sin x] x + x \frac{d}{dx} [\sin x] + \frac{d}{dx} [\cos x] \frac{x}{x^2} = \]
\[ (1) \sin x + x \cos x + \frac{-\sin x(1) - \cos x}{x^2} = \sin x + x \cos x + \frac{-\sin x - \cos x}{x^2}. \]

10. \( f(x) = x^9 + \frac{\sqrt{x}}{2} + \tan x \)

**Solution.** We have
\[ f'(x) = \frac{d}{dx} [x^9] + \frac{d}{dx} \left[ \frac{\sqrt{x}}{2} \right] + \frac{d}{dx} [\tan x] = 9x^8 + \frac{1}{2} \frac{d}{dx} \left[ \sqrt{x} \right] + \sec^2 x = \]
\[ 9x^8 + \frac{1}{2} \frac{d}{dx} \left[ x^{\frac{1}{2}} \right] + \sec^2 x = 9x^8 + \frac{1}{2} \left( \frac{1}{9} \right) x^{-\frac{1}{2}} + \sec^2 x = 9x^8 + \frac{1}{18} x^{-\frac{1}{2}} + \sec^2 x. \]

11. \( f(x) = x^2 \cos x - 2 \tan x + 3 \)

**Solution.** We have
\[ f'(x) = \frac{d}{dx} [x^2 \cos x] - 2 \frac{d}{dx} [\tan x] + \frac{d}{dx} [3] = \frac{d}{dx} [x^2] \cos x + x^2 \frac{d}{dx} [\cos x] - 2 \sec^2 x + 0 = \]
\[ 2x \cos x + x^2 (-\sin x) - 2 \sec^2 x = 2x \cos x - x^2 \sin x - 2 \sec^2 x. \]
12. \( f(x) = x \cos x + x^2 \sin x \)

**Solution.** We have

\[
f'(x) = \frac{d}{dx}[\cos x] + \frac{d}{dx}[x^2 \sin x] = \frac{d}{dx}[\cos x] + \frac{d}{dx}[x^2] \sin x + x^2 \frac{d}{dx}[\sin x] =
\]

\[
(1) \cos x + x(\cos x) + 2x \sin x + x^2 \cos x = \cos x - x \sin x + 2x \sin x + x^2 \cos x = 
\]

\[
\cos x + x \sin x + x^2 \cos x.
\]

13. \( f(x) = \frac{\sin x}{1 + \tan x} \)

**Solution.** We have

\[
f'(x) = \frac{\frac{d}{dx}[\sin x](1 + \tan x) - \sin x \frac{d}{dx}[1 + \tan x]}{(1 + \tan x)^2} = \frac{\cos x(1 + \tan x) - \sin x(0 + \sec^2 x)}{(1 + \tan x)^2} = 
\]

\[
\cos x + \cos x \tan x - \sin x \sec^2 x
\]

\[
(1 + \tan x)^2
\]

14. \( f(x) = \frac{1}{\cos x} + \frac{\csc x}{x} - \sqrt{x} \)

**Solution.** We have

\[
f'(x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) + \frac{d}{dx}\left[\csc x \frac{\cos x}{x}\right] + \frac{d}{dx}\left[\sqrt{x}\right] = \frac{\frac{d}{dx}[\cos x] + \frac{d}{dx}[\csc x]}{\cos x} + \frac{1}{2} \frac{d}{dx}[\csc x] - \frac{d}{dx}[x^{\frac{1}{2}}] = 
\]

\[
(0) \frac{\cos x - (-\sin x)}{\cos^2 x} + \frac{1}{3} \left(-\csc x \cot x\right) - \frac{1}{2} x^{-\frac{3}{2}} = \frac{\sin x}{\cos^2 x} - \frac{1}{3} \csc x \cot x - \frac{1}{2} x^{-\frac{3}{2}}.
\]

15. \( f(x) = x \cos x \sin x \)

**Solution.** First we have \( f(x) = (x \cos x) \sin x \). Now we have

\[
f'(x) = \frac{d}{dx}[x \cos x] \sin x + x \cos x \frac{d}{dx}[\sin x] = \left(\frac{d}{dx}[x \cos x] + x \frac{d}{dx}[\cos x]\right) \sin x + x \cos x (\cos x) = 
\]

\[
(\cos x - x \sin x) \sin x + x \cos^2 x = \cos x \sin x - x \sin^2 x + x \cos^2 x.
\]

**Section 2.5: The Chain Rule**

Find the derivative of each of the following functions.

1. \( f(x) = (2x)^5 \)

**Solution.** First recall the General Power Rule, which is a combination of the Power Rule with the Chain Rule:

\[
\frac{d}{dx}[(g(x))^n] = n(g(x))^{n-1} \frac{d}{dx}[g(x)].
\]

Here \( g(x) = 2x \) and \( n = 5 \). So

\[
f'(x) = 5(2x)^{5-1} \frac{d}{dx}[2x] = 5(2)^4(2) = (5)(2)(2)^4 = 10(2)^4.
\]

2. \( f(x) = (-3x + 4)^8 \)

**Solution.** We have

\[
f'(x) = 8(-3x + 4)^7 \frac{d}{dx}[-3x + 4] = 8(-3x + 4)^7(-3) = 8(-3)(-3x + 4)^7 = -24(-3x + 4)^7.
\]
3. \( f(x) = 3(5x - 1)^7 \)

Solution. We have

\[
f'(x) = \frac{d}{dx} [3(5x - 1)^7] = 3 \frac{d}{dx} [(5x - 1)^7] =
\]

\[3(7)(5x - 1)^6 \frac{d}{dx} [5x - 1] = 21(5x - 1)^6(5) = 105(5x - 1)^6.\]

4. \( f(x) = (x^2 - 1)^{\frac{7}{2}} \)

Solution. We have

\[
f'(x) = \frac{3}{2} (x^2 - 1)^{\frac{5}{2}} - 1 \frac{d}{dx} [x^2 - 1] = \frac{3}{2} (x^2 - 1)^{\frac{5}{2}}(2x) = \frac{3}{2}(2x)(x^2 - 1)^{\frac{5}{2}} = 3x(x^2 - 1)^{\frac{5}{2}}.\]

5. \( f(x) = (-x^3 + 2x + 1)^{15} \)

Solution. We have

\[
f'(x) = 15(-x^3 + 2x + 1)^{14} \frac{d}{dx} [-x^3 + 2x + 1] =
\]

\[15(-x^3 + 2x + 1)^{14}(-3x^2 + 2) = 15(-3x^2 + 2)(-x^3 + 2x + 1)^{14}.\]

6. \( f(x) = -2(5x^6 - 2x)^4 \)

Solution. We have

\[
f'(x) = \frac{d}{dx} [-2(5x^6 - 2x)^4] = -2 \frac{d}{dx} [(5x^6 - 2x)^4] = -2(4)(5x^6 - 2x)^3 \frac{d}{dx} [5x^6 - 2x] =
\]

\[-8(5x^6 - 2x)^3(30x^5 - 2) = -8(30x^5 - 2)(5x^6 - 2x)^3.\]

7. \( f(x) = (2 - \sin x)^{\frac{3}{2}} \)

Solution. We have

\[
f'(x) = \frac{5}{2} (2 - \sin x)^{\frac{1}{2}} - 1 \frac{d}{dx} [2 - \sin x] = \frac{5}{2}(2 - \sin x)^{\frac{1}{2}}(0 - \cos x) =
\]

\[\frac{5}{2}(2 - \sin x)^{\frac{1}{2}}(\cos x) = \frac{5}{2}\cos x(2 - \sin x)^{\frac{1}{2}}.\]

8. \( f(x) = \sqrt{x^2 - x} \)

Solution. First we have \( f(x) = \sqrt{x^2 - x} = (x^2 - x)^{\frac{1}{2}}. \) Now we have

\[
f'(x) = \frac{1}{2} (x^2 - x)^{\frac{1}{2}} - 1 \frac{d}{dx} [x^2 - x] = \frac{1}{2}(x^2 - x)^{-\frac{1}{2}}(2x - 1) =
\]

\[\frac{1}{2}(2x - 1) \frac{1}{(x^2 - x)^{\frac{1}{2}}} = \frac{\frac{1}{2}(2x - 1)}{\sqrt{x^2 - x}} = \frac{2x - 1}{2\sqrt{x^2 - x}}.\]

9. \( f(x) = \frac{1}{\sqrt{x^2 - 1}} \)

Solution. First we have \( f(x) = \frac{1}{(x^2 - 1)^{\frac{1}{2}}} = (x^2 - 1)^{-\frac{1}{2}}. \) Now we have

\[
f'(x) = -\frac{1}{3}(x^2 - 1)^{-\frac{1}{2}} \frac{d}{dx} [x^2 - 1] = -\frac{1}{3}(x^2 - 1)^{-\frac{1}{2}}(2x) = -\frac{2}{3}x(x^2 - 1)^{-\frac{1}{2}}.\]
10. \( f(x) = (-5x + 4)(x^3 + 1)^6 \)

**Solution.** If we look at \( f(x) \), we can see many rules including the General Power Rule (G.P.R), the Product Rule, etc. The G.P.R applies only to the second term, while the Product Rule applies to the whole function. So we are going to first use the Product Rule.

\[
f'(x) = \frac{d}{dx} [-5x + 4] (x^3 + 1)^6 + (-5x + 4) \frac{d}{dx} [(x^3 + 1)^6] = -5(x^3 + 1)^6 + (-5x + 4)6(x^3 + 1)^5(3x^2) = \]
\[
-5(x^3 + 1)^6 + 18x^2(-5x + 4)(x^3 + 1)^5 = (x^3 + 1)^5(-5x^3 + 18x^2 - 5x + 4) =
\]
\[
(x^3 + 1)^5(-5x^3 - 5 - 90x + 72x^2) = (x^3 + 1)^5(-95x^3 + 72x^2 - 5).
\]

11. \( f(x) = x\sqrt{2 - x^2} \)

**Solution.** First we have \( f(x) = x(2 - x^2)^{\frac{1}{2}} \). As before, we first use the Product Rule.

\[
f'(x) = \frac{d}{dx} [x(2 - x^2)^{\frac{1}{2}}] = (1)(2 - x^2)^{\frac{1}{2}} + x \frac{d}{dx} [(2 - x^2)^{\frac{1}{2}}] = (1)(2 - x^2)^{\frac{1}{2}} + x \left(\frac{1}{2}\right)(2 - x^2)^{-\frac{1}{2}}(-2x) =
\]
\[
(2 - x^2)^{\frac{1}{2}} - x^2 \frac{1}{(2 - x^2)^{\frac{1}{2}}} = \sqrt{2 - x^2} - \frac{x^2}{\sqrt{2 - x^2}} = \frac{(2 - x^2) - x^2}{\sqrt{2 - x^2}} = \frac{2 - 2x^2}{\sqrt{2 - x^2}} = \frac{2(1 - x^2)}{\sqrt{2 - x^2}}.
\]

12. \( f(x) = (x^2 + 1)^3(x^2 + 2)^6 \)

**Solution.** We have

\[
f'(x) = \frac{d}{dx} [(x^2 + 1)^3] (x^2 + 2)^6 + (x^2 + 1)^3 \frac{d}{dx} [(x^2 + 2)^6] =
\]
\[
3(x^2 + 1)(2x)(x^2 + 2)^6 + (x^2 + 1)^36(x^2 + 2)^5(2x) = 6x(x^2 + 1)^2(x^2 + 2)^6 + 12x(x^2 + 1)^3(x^2 + 2)^5 =
\]
\[
6x(x^2 + 1)^2(x^2 + 2)^5(2x^2 + 2 + 2(x^2 + 1)) = 6x(x^2 + 1)^2(x^2 + 2)^5(x^2 + 2 + 2x^2 + 2) =
\]
\[
6x(x^2 + 1)^2(x^2 + 2)^5(3x^2 + 4).
\]

13. \( f(x) = \frac{(x + 1)^5}{x^2 + 1} \)

**Solution.** Using the Quotient Rule, we get

\[
f'(x) = \frac{d}{dx} \left[\frac{(x + 1)^5}{x^2 + 1}\right] = \frac{5(x + 1)^4(1)(x^2 + 1) - (x + 1)^5(2x)}{(x^2 + 1)^2} =
\]
\[
\frac{(x + 1)^4(5x^2 + 5 - 5x^2 + 5x^2)}{(x^2 + 1)^2} = \frac{(x + 1)^4(5 - 5x^2)}{(x^2 + 1)^2}.
\]

14. \( f(x) = \frac{x^2}{\sqrt{x^3 + 1}} \)

**Solution.** First we have \( f(x) = \frac{x^2}{(x^3 + 1)^{\frac{1}{2}}} \). Now, by using the Quotient Rule, we get

\[
f'(x) = \frac{2x(x^3 + 1)^{\frac{1}{2}} - x^2 \frac{1}{2}(x^3 + 1)^{-\frac{3}{2}}(3x^2)}{(\sqrt{x^3 + 1})^2} =
\]
\[
\frac{2x\sqrt{x^3 + 1} - \frac{3}{2}x^4 \frac{1}{\sqrt{x^3 + 1}}}{x^3 + 1} = \frac{2x(x^3 + 1) - \frac{3}{2}x^4}{(x^3 + 1)\sqrt{x^3 + 1}} =
\]
\[
\frac{2x^4 + 2x - \frac{3}{2}x^4}{(x^3 + 1)^2} = \frac{\frac{1}{2}x^4 + 2x + \frac{1}{2}x^4}{(x^3 + 1)^2} = \frac{x(x^3 + 1)}{2(x^3 + 1)\sqrt{x^3 + 1}}.
\]
15. \( f(x) = \sqrt{\frac{x}{x^2 + 3}} \)

**Solution.** First we have \( f(x) = \left( \frac{x}{x^2 + 3} \right)^{\frac{1}{2}} \). Now we have

\[
   f'(x) = \frac{1}{2} \left( \frac{x}{x^2 + 3} \right)^{-\frac{1}{2}} \frac{d}{dx} \left[ \frac{x}{x^2 + 3} \right] = \frac{1}{2} \left( \frac{x}{x^2 + 3} \right)^{-\frac{1}{2}} \left( \frac{1(x^2 + 3) - x(2x)}{(x^2 + 3)^2} \right) = \frac{1}{2} \left( \frac{x}{x^2 + 3} \right)^{-\frac{1}{2}} \frac{-x^2 + 3}{2(x^2 + 3)^2} \left( \frac{x}{x^2 + 3} \right)^{-\frac{1}{2}}.
\]

16. \( f(x) = \left( \frac{x^4 + 1}{x^2 + 1} \right)^5 \)

**Solution.** We have

\[
   f'(x) = 5 \left( \frac{x^4 + 1}{x^2 + 1} \right)^4 \frac{d}{dx} \left[ \frac{x^4 + 1}{x^2 + 1} \right] = 5 \left( \frac{x^4 + 1}{x^2 + 1} \right)^4 \left( \frac{4x^3(x^2 + 1) - (x^4 + 1)2x}{(x^2 + 1)^2} \right) = 5 \left( \frac{x^4 + 1}{x^2 + 1} \right)^4 \left( \frac{2x^5 + 4x^3 - 2x}{(x^2 + 1)^2} \right) = 5 \left( \frac{2x^4 + 2x^2 - 1}{(x^2 + 1)^2} \right) \left( \frac{x^4 + 1}{x^2 + 1} \right)^4 = \frac{10x(x^4 + 2x^2 - 1)}{(x^2 + 1)^2} \left( \frac{x^4 + 1}{x^2 + 1} \right)^4.
\]

17. \( f(x) = \sqrt{\frac{1 + \sin x}{1 + \cos x}} \)

**Solution.** First we have \( f(x) = \left( \frac{1 + \sin x}{1 + \cos x} \right)^{\frac{1}{2}} \). Now we have

\[
   f'(x) = \frac{1}{2} \left( \frac{1 + \sin x}{1 + \cos x} \right)^{-\frac{1}{2}} \frac{d}{dx} \left[ \frac{1 + \sin x}{1 + \cos x} \right] = \frac{1}{2} \frac{1 + \sin x}{1 + \cos x} \left( \frac{\cos x(1 + \cos x) - (1 + \sin x)(-\sin x)}{(1 + \cos x)^2} \right) = \frac{1}{2} \frac{1 + \sin x}{1 + \cos x} \left( \frac{\cos x + \cos^2 x + \sin x + \sin^2 x}{(1 + \cos x)^2} \right) = \frac{1}{2} \frac{1 + \sin x}{1 + \cos x} \left( \frac{\cos x + \sin x + 1}{(1 + \cos x)^2} \right) = \frac{\cos x + \sin x + 1}{2(1 + \cos x)^2} \left( \frac{1 + \sin x}{1 + \cos x} \right)^{-\frac{1}{2}}.
\]

18. \( f(x) = \sin^2 x \)

**Solution.** First we have \( f(x) = (\sin x)^2 \). Now we have

\[
   f'(x) = 2(\sin x)^1 \frac{d}{dx} [\sin x] = 2 \sin x \cos x.
\]

19. \( f(x) = \cos^4 x - \sec^2 x + \sqrt{\csc x} \)

**Solution.** We have

\[
   f'(x) = \frac{1}{8} \frac{d}{dx} [\cos^4 x] - \frac{d}{dx} [\sec^2 x] + \frac{d}{dx} [\sqrt{\csc x}] = \frac{1}{8} \frac{d}{dx} [(\cos x)^4] - \frac{d}{dx} [(\sec x)^2] + \frac{d}{dx} [(\csc x)^{\frac{1}{2}}] = \frac{1}{8} (4)(\cos x)^3 \frac{d}{dx} [\cos x] - (2)(\csc x)^2 \frac{d}{dx} [\csc x] + \frac{1}{3} (\csc x)^{-\frac{3}{2}} \frac{d}{dx} [\csc x] = \frac{1}{2} (\cos x)^3 (-\csc x x) - 2 \sec x (\sec x \tan x) + \frac{1}{3} (\csc x)^{-\frac{5}{2}} (-\csc x \cot x) = -\frac{1}{2} \sin x \cos^3 x - 2 \sec^2 x \tan x - \frac{1}{3} \csc x (\csc x)^{\frac{1}{2}} = -\frac{1}{2} \sin x \cos^3 x - 2 \sec^2 x \tan x - \frac{1}{3} \cot x \sqrt{\csc x}.
\]
20. \( f(x) = -3x^4 + (-2x^4 + 1)^{10} - \frac{1}{\sqrt{x - \cos x}} \).

**Solution.** We have

\[
\frac{d}{dx}[-3x^4] + \frac{d}{dx} [(-2x^4 + 1)^{10}] - \frac{d}{dx} [(x - \cos x)^{-\frac{1}{2}}] =
\]

\[-12x^3 + 10(-2x^4 + 1)^9(-8x^3) - \frac{1}{2}(x - \cos x)^{-\frac{3}{2}}(1 + \sin x) =
\]

\[-12x^3 - 80x^3(-2x^4 + 1)^9 + \frac{1}{2}(1 + \sin x)(x - \cos x)^{-\frac{3}{2}}.\]

### 2.7 Solution to PB7

**Section 2.5: The Chain Rule (Continued)**

1. Find the derivative of each of the following functions.

   (a) \( f(x) = \sin(2x) \)

   **Solution.** First recall the Chain Rule. Given two functions \( h \) and \( g \) such that the composition \( h(g(x)) \) is defined, \( g(x) \) is called the inside or inner function, while \( h \) is called the outer function. The Chain Rule says that:

   \[
   \frac{d}{dx}[h(g(x))] = h'(g(x))\frac{d}{dx}[g(x)].
   \]

   For \( f(x) = \sin(2x) \), the inner function is \( g(x) = 2x \), and the outer function \( h(x) = \sin x \) is the sine function. To find \( f'(x) \) by the Chain Rule, first we need to find \( h'(x) \), which is \( h'(x) = \cos x \) (for the derivatives of basic trigonometric functions, see the solution to Practice Problems 6). Then take \( h'(g(x)) = \cos(2x) \). And then multiply by the derivative of the inside (which is \( g'(x) = 2 \)).

   We end up with \( f'(x) = \cos(2x) \cdot 2 = 2 \cos(2x) \). In other words, we have

   \[
   f'(x) = \cos(2x)\frac{d}{dx}[2x] = \cos(2x)(2) = 2 \cos(2x).
   \]

   (b) \( f(x) = \cos(-3x + 1) \)

   **Solution.** Since \( \frac{d}{dx} \cos x = - \sin x \), we have

   \[
   f'(x) = - \sin(-3x + 1)\frac{d}{dx}[-3x + 1] = - \sin(-3x + 1)(-3) =
   \]

   \[ (-3) \sin(-3x + 1) = 3 \sin(-3x + 1). \]

   (c) \( f(x) = \tan(x^3 - x^2) \)

   **Solution.** Since \( \frac{d}{dx} \tan x = \sec^2 x \), we have

   \[
   f'(x) = \sec^2(x^3 - x^2)\frac{d}{dx}[x^3 - x^2] = \sec^2(x^3 - x^2)(3x^2 - 2x) = (3x^2 - 2x) \sec^2(x^3 - x^2).\]

   (d) \( f(x) = x \sin(x^2 + 1) \)

   **Solution.** Since \( f(x) \) is the product of two functions, we need to first apply the Product Rule:

   \[
   f'(x) = \frac{d}{dx}[x] \sin(x^2 + 1) + x \frac{d}{dx} \left[ \sin(x^2 + 1) \right] = (1) \sin(x^2 + 1) + x \cos(x^2 + 1) \frac{d}{dx} [x^2 + 1] =
   \]

   \[
   \sin(x^2 + 1) + x \cos(x^2 + 1)(2x) = \sin(x^2 + 1) + 2x^2 \cos(x^2 + 1).
   \]

   To get \( \frac{d}{dx} \left[ \sin(x^2 + 1) \right] \), we used the Chain Rule.

\(^1\)Warning: Do not forget to multiply by the derivative of the inside.
2.7. SOLUTION TO PB7

(e) \( f(x) = 3x^2 \cos(\pi x - 1) \)

**Solution.** Again by the Product Rule first, and then the Chain Rule, we have

\[
f'(x) = \frac{d}{dx}[3x^2] \cos(\pi x - 1) + 3x^2 \frac{d}{dx}[\cos(\pi x - 1)] =
\]

\[
6x \cos(\pi x - 1) + 3x^2 \left( -\sin(\pi x - 1) \frac{d}{dx} [\pi x - 1] \right) =
\]

\[
6x \cos(\pi x - 1) - 3x^2 \sin(\pi x - 1)(\pi) = 6x \cos(\pi x - 1) - 3\pi x^2 \sin(\pi x - 1).
\]

(f) \( f(x) = \sin(\cos x) \)

**Solution.** By the Chain Rule, we have

\[
f'(x) = \cos(\cos x) \cdot \frac{d}{dx}[\cos x] = \cos(\cos x)(-\sin x) = -\sin x \cos(\cos x).
\]

(g) \( f(x) = \cos(\sec(4x)) \)

**Solution.** By the Chain Rule, we have \( f'(x) = -\sin(\sec(4x)) \frac{d}{dx}[\sec(4x)] \). To find \( \frac{d}{dx}[\sec(4x)] \), we will use the Chain Rule again. First recall that \( \frac{d}{dx}[\sec x] = \sec x \tan x \). So the derivative of \( \sec x \) at the inside function, \( 4x \), is: \( \sec(4x) \tan(4x) \). Multiplying by the derivative of the inside, we get

\[
\frac{d}{dx}[\sec(4x)] = \sec(4x) \tan(4x) \frac{d}{dx}[4x] = \sec(4x) \tan(4x)(4) = 4 \sec(4x) \tan(4x).
\]

Now we have

\[
f'(x) = -\sin(\sec(4x)) (4 \sec(4x) \tan(4x)) = -4 \sec(4x) \tan(4x) \sin(\sec(4x)).
\]

(h) \( f(x) = \sin \sqrt{1 + x^2} \)

**Solution.** Using the Chain Rule, we have

\[
f'(x) = \cos \sqrt{1 + x^2} \frac{d}{dx} \left[ \sqrt{1 + x^2} \right] = \cos \sqrt{1 + x^2} \frac{d}{dx} \left[ (1 + x^2)^{\frac{1}{2}} \right] =
\]

\[
\cos \sqrt{1 + x^2} \left( \frac{1}{2} \right) (1 + x^2)^{-\frac{1}{2}} \frac{d}{dx} [1 + x^2] = \cos \sqrt{1 + x^2} \left( \frac{1}{2} (1 + x^2)^{-\frac{1}{2}} (2x) \right) =
\]

\[
\cos \sqrt{1 + x^2} \left( x(1 + x^2)^{-\frac{1}{2}} \right) = \frac{x}{\sqrt{1 + x^2}} \cos \sqrt{1 + x^2}.
\]

(i) \( f(x) = x \sin \frac{1}{x} \)

**Solution.** We have

\[
f'(x) = \frac{d}{dx} [x] \sin \left( \frac{1}{x} \right) + x \frac{d}{dx} \left[ \sin \left( \frac{1}{x} \right) \right] = (1) \sin \left( \frac{1}{x} \right) + x \left( \cos \left( \frac{1}{x} \right) \frac{d}{dx} \left[ \frac{1}{x} \right] \right) =
\]

\[
\sin \left( \frac{1}{x} \right) + x \left( \cos \left( \frac{1}{x} \right) \left( -\frac{1}{x^2} \right) \right) = \sin \left( \frac{1}{x} \right) - \frac{x}{x^2} \cos \left( \frac{1}{x} \right) =
\]

\[
\sin \left( \frac{1}{x} \right) - \frac{1}{x} \cos \left( \frac{1}{x} \right).
\]

(j) \( f(x) = \sec^4(x^3 + 1) \)

**Solution.** First we have \( f(x) = (\sec(x^3 + 1))^4 \). Using the Chain Rule, we have

\[
f'(x) = 4 (\sec(x^3 + 1))^3 \frac{d}{dx} [\sec(x^3 + 1)] =
\]

\[
4 \sec^3(x^3 + 1) \left( \sec(x^3 + 1) \tan(x^3 + 1) \frac{d}{dx} [x^3 + 1] \right) =
\]

\[
4 \sec^3(x^3 + 1) \sec(x^3 + 1) \tan(x^3 + 1)(3x^2) = 12x^2 \sec^4(x^3 + 1) \tan(x^3 + 1).
\]
2. Find an equation of the tangent line to the curve $y = \sin(\sin x)$ at $(\pi, 0)$.

**Solution.** Let $f(x) = \sin(\sin x)$.

- First we need to find the derivative. Using the Chain Rule, we have

$$f'(x) = \cos(\sin x) \frac{d}{dx}[\sin x] = \cos(\sin x)(\cos x) = \cos x \cos(\sin x).$$

- The slope at $(\pi, 0)$ is then

$$f'(\pi) = \cos\pi \cos(\sin \pi) = (-1) \cos(0) = (-1)(1) = -1.$$

Recall that $\cos(0) = 1$, $\cos(\pi) = -1$, $\sin(0) = 0$, and $\sin \pi = 0$.

- Equation of the tangent line: $y - \pi = -x(\pi - 0)$ or $y = -x + \pi$.

### Section 2.6: Implicit Differentiation

The method of implicit differentiation basically consists of two steps. Step 1: Derive BOTH sides of the equation with respect to $x$. Step 2: Solve the resulting equation for $\frac{dy}{dx}$. We will need the following formula, which is noting but the General Power Rule:

$$\frac{d}{dx}[y^n] = n y^{n-1} \frac{dy}{dx}.$$

1. Find $\frac{dy}{dx}$ for each of the following function.

   - (a) $x^2 + y^2 = 25$

     **Solution.** Taking the derivative of both sides with respect to $x$, we get
     $$\frac{d}{dx}[x^2 + y^2] = \frac{d}{dx}[25],$$
     which is equivalent to $\frac{d}{dx}[x^2] + \frac{d}{dx}[y^2] = 0$ or $2x + 2y \frac{dy}{dx} = 0$. Solving now this latter equation for $\frac{dy}{dx}$, we get
     $$\frac{dy}{dx} = -\frac{x}{y}.$$

   - (b) $x^3 + y^3 = 1$

     **Solution.** We have
     $$\frac{d}{dx}[x^3 + y^3] = \frac{d}{dx}[1] \leftrightarrow \frac{d}{dx}[x^3] + \frac{d}{dx}[y^3] = 0 \leftrightarrow 3x^2 + 3y^2 \frac{dy}{dx} = 0 \leftrightarrow 3y^2 \frac{dy}{dx} = -3x^2 \leftrightarrow$$
     $$\frac{dy}{dx} = -\frac{x^2}{3y^2} \leftrightarrow \frac{dy}{dx} = -\frac{x}{y^2}.$$

   - (c) $2x^2 - y^2 = x$

     **Solution.** We have
     $$\frac{d}{dx}[2x^2 - y^2] = \frac{d}{dx}[x] \leftrightarrow \frac{d}{dx}[2x^2] - \frac{d}{dx}[y^2] = 1 \leftrightarrow 4x - 2y \frac{dy}{dx} = 1 \leftrightarrow$$
     $$-2y \frac{dy}{dx} = 1 - 4x \leftrightarrow \frac{dy}{dx} = \frac{1 - 4x}{-2y}.$$

   - (d) $x^4 + 3y^3 = 5y$

     **Solution.** We have
     $$\frac{d}{dx}[x^4 + 3y^3] = \frac{d}{dx}[5y] \leftrightarrow \frac{d}{dx}[x^4] + 3 \frac{d}{dx}[y^3] = 5 \frac{dy}{dx} \leftrightarrow 4x^3 + 9y^2 \frac{dy}{dx} = 5 \frac{dy}{dx} \leftrightarrow$$
     $$9y^2 \frac{dy}{dx} - 5 \frac{dy}{dx} = -4x^3 \leftrightarrow (9y^2 - 5) \frac{dy}{dx} = -4x^3 \leftrightarrow \frac{dy}{dx} = \frac{-4x^3}{9y^2 - 5}.$$
2.7. SOLUTION TO PB7

(e) \( \sqrt{x} - y = y^2 + 3 \)

Solution. We have

\[
\frac{dy}{dx} \left[ \sqrt{x} - y \right] = \frac{dy}{dx} \left[ y^2 + 3 \right] \Leftrightarrow \frac{dy}{dx} \left[ \sqrt{x} \right] - \frac{dy}{dx} \left[ y \right] = \frac{dy}{dx} \left[ y^2 \right] + \frac{dy}{dx} \left[ 3 \right] \Leftrightarrow \frac{1}{2 \sqrt{x}} - \frac{dy}{dx} = 2y \frac{dy}{dx} + 0 \Leftrightarrow
\]

\[- \frac{dy}{dx} - 2y \frac{dy}{dx} = - \frac{1}{2 \sqrt{x}} \leftrightarrow \left( -1 - 2y \right) \frac{dy}{dx} = - \frac{1}{2 \sqrt{x}} \leftrightarrow
\]

\[
\frac{dy}{dx} = - \frac{1}{2 \sqrt{x}} \frac{1}{-1 - 2y} = \frac{1}{2 \sqrt{x}(1 + 2y)}.
\]

(f) \( xy = 5 \)

Solution. Taking the derivative of both sides, and applying the Product to the lefthand side, we get

\[
\frac{dy}{dx} \left[ xy \right] = \frac{dy}{dx} \left[ 5 \right] \Leftrightarrow \frac{dy}{dx} \left[ x \right] y + x \frac{dy}{dx} \left[ y \right] = 0 \leftrightarrow \left( 1 \right) y + x \frac{dy}{dx} = 0 \leftrightarrow x \frac{dy}{dx} = -y \leftrightarrow \frac{dy}{dx} = -\frac{y}{x}.
\]

(g) \( 3x^2 + 2xy + y^2 = 2 \)

Solution. We have

\[
\frac{dy}{dx} \left[ 3x^2 + 2xy + y^2 \right] = \frac{dy}{dx} \left[ 2 \right] \leftrightarrow 3 \frac{dy}{dx} \left[ x \right]^2 + 2 \frac{dy}{dx} \left[ x \right] y + \frac{dy}{dx} \left[ y^2 \right] = 0 \leftrightarrow 6x + 2 \left( y + \frac{dy}{dx} \right) + 2y \frac{dy}{dx} = 0 \leftrightarrow
\]

\[
6x + 2y + 2x \frac{dy}{dx} + 2y \frac{dy}{dx} = 0 \leftrightarrow (2x + 2y) \frac{dy}{dx} = -6x - 2y \leftrightarrow \frac{dy}{dx} = -\frac{6x + 2y}{2x + 2y} = -\frac{3x - y}{x + y}.
\]

(h) \( -5x^2 + xy - y^3 = 1 \)

Solution. We have

\[
\frac{dy}{dx} \left[ -5x^2 + xy - y^3 \right] = \frac{dy}{dx} \left[ 1 \right] \leftrightarrow -5 \frac{dy}{dx} \left[ x \right] + \frac{dy}{dx} \left[ xy \right] - \frac{dy}{dx} \left[ y^3 \right] = 0 \leftrightarrow
\]

\[
-10x + \left( y + x \frac{dy}{dx} \right) - 3y^2 \frac{dy}{dx} = 0 \leftrightarrow (x - 3y^2) \frac{dy}{dx} = 10x - y \leftrightarrow \frac{dy}{dx} = \frac{10x - y}{x - 3y^2}.
\]

(i) \( x^3y^2 + y^4 = x \)

Solution. We have

\[
\frac{dy}{dx} \left[ x^3y^2 + y^4 \right] = \frac{dy}{dx} \left[ x \right] \leftrightarrow \frac{dy}{dx} \left[ x^3y^2 \right] + \frac{dy}{dx} \left[ y^4 \right] = 1 \leftrightarrow \frac{dy}{dx} \left[ x^3y^2 \right] + x^3 \frac{dy}{dx} \left[ y^2 \right] + 4y^3 \frac{dy}{dx} = 0 \leftrightarrow
\]

\[
3x^2y^2 + x^3(2y) \frac{dy}{dx} + 4y^3 \frac{dy}{dx} = 1 \leftrightarrow (2x^3y + 4y^3) \frac{dy}{dx} = 1 - 3x^2y^2 \leftrightarrow \frac{dy}{dx} = \frac{1 - 3x^2y^2}{2x^3y + 4y^3}.
\]

(j) \( \frac{x + y}{x - y} = 1 \)

Solution. We have

\[
\frac{dy}{dx} \left[ \frac{x + y}{x - y} \right] = \frac{dy}{dx} \left[ 1 \right] \leftrightarrow \frac{dy}{dx} \left[ x + y \right] \left( x - y \right) - \left( x + y \right) \frac{dy}{dx} \left[ x - y \right] = 0 \leftrightarrow
\]

\[
\frac{dy}{dx} \left[ x + y \right] \left( x - y \right) - \left( x + y \right) \frac{dy}{dx} \left[ x - y \right] = 0 \leftrightarrow
\]

\[
(1 + \frac{dy}{dx}) \left( x - y \right) - (x + y) \left( 1 - \frac{dy}{dx} \right) = 0 \leftrightarrow x - y + x \frac{dy}{dx} - y \frac{dy}{dx} - \left( x + y - x \frac{dy}{dx} - x \frac{dy}{dx} \right) = 0 \leftrightarrow
\]

\[
x - y + x \frac{dy}{dx} - y \frac{dy}{dx} - y \frac{dy}{dx} + y \frac{dy}{dx} = 0 \leftrightarrow
\]

\[
x - y + x \frac{dy}{dx} - y \frac{dy}{dx} + y \frac{dy}{dx} = 0 \leftrightarrow 2x \frac{dy}{dx} = 2y \leftrightarrow \frac{dy}{dx} = \frac{2y}{2x} = \frac{y}{x}.
\]

From the first line to the second we used the fact that if a fraction \( \frac{A}{B} = 0 \), then \( A = 0 \).
(k) $\sin y = 3$

**Solution.** First recall that
\[
\frac{d}{dx} [\sin y] = \cos y \frac{dy}{dx} \quad \text{and} \quad \frac{d}{dx} [\sin x] = \cos x.
\]
We have
\[
\frac{d}{dx} [\sin y] = \frac{d}{dx} [3] \leftrightarrow \cos y \frac{dy}{dx} = 0 \leftrightarrow \frac{dy}{dx} = 0 \cos y = 0.
\]

(l) $\cos y = x + 1$

**Solution.** First recall that
\[
\frac{d}{dx} [\cos y] = -\sin y \frac{dy}{dx} \quad \text{and} \quad \frac{d}{dx} [\cos x] = -\sin x.
\]
We have
\[
\frac{d}{dx} [\cos y] = \frac{d}{dx} [x + 1] \leftrightarrow -\sin y \frac{dy}{dx} = 1 \leftrightarrow \frac{dy}{dx} = -\frac{1}{\sin y}.
\]

(m) $3y^2 - \cos y = x^3$

**Solution.** We have
\[
\frac{d}{dx} [3y^2 - \cos y] = \frac{d}{dx} [x^3] \leftrightarrow 3 \frac{d}{dx} [y^2] - \frac{d}{dx} [\cos y] = 3x^2 \leftrightarrow 6y \frac{dy}{dx} - (\sin y \frac{dy}{dx}) = 3x^2 \leftrightarrow 6y \frac{dy}{dx} + \sin y \frac{dy}{dx} = 3x^2 \leftrightarrow (6y + \sin y) \frac{dy}{dx} = 3x^2 \leftrightarrow \frac{dy}{dx} = \frac{3x^2}{6y + \sin y}.
\]

(n) $\sin x \cos y + y = -7$

**Solution.** We have
\[
\frac{d}{dx} [\sin x \sin y + y] = \frac{d}{dx} [-7] \leftrightarrow \frac{d}{dx} [\sin x \cos y] + \frac{d}{dx} [y] = 0 \leftrightarrow
\]
\[
\frac{d}{dx} [\sin x] \cos y + \sin x \frac{d}{dx} [\cos y] + \frac{dy}{dx} = 0 \leftrightarrow
\]
\[
\cos x \cos y + \sin x (-\sin y \frac{dy}{dx}) + \frac{dy}{dx} = 0 \leftrightarrow -\sin x \sin y \frac{dy}{dx} + \frac{dy}{dx} = -\cos x \cos y \leftrightarrow
\]
\[
(-\sin x \sin y + 1) \frac{dy}{dx} = -\cos x \cos y \leftrightarrow \frac{dy}{dx} = -\frac{\cos x \cos y}{-\sin x \sin y + 1}.
\]

(o) $\cos(xy) = 1 + \sin y$

**Solution.** We have
\[
\frac{d}{dx} [\cos(xy)] = \frac{d}{dx} [1 + \sin y] \leftrightarrow -\sin(xy) \frac{d}{dx} [xy] = \frac{d}{dx} [1] + \frac{d}{dx} [\sin y] \leftrightarrow
\]
\[
-\sin(xy) \left( y + x \frac{dy}{dx} \right) = 0 + \cos y \frac{dy}{dx} \leftrightarrow
\]
\[
-y \sin(xy) - x \sin(xy) \frac{dy}{dx} = \cos y \frac{dy}{dx} \leftrightarrow -x \sin(xy) \frac{dy}{dx} - \cos y \frac{dy}{dx} = y \sin(xy) \leftrightarrow
\]
\[
(-x \sin(xy) - \cos y) \frac{dy}{dx} = y \sin(xy) \leftrightarrow \frac{dy}{dx} = -\frac{y \sin(xy)}{-x \sin(xy) - \cos y}.
\]
2.7. SOLUTION TO PB7

(p) \( x \sin y + y \sin x = 1 \)

Solution. We have

\[
\frac{d}{dx} [x \sin y + y \sin x] = \frac{d}{dx} [x \sin y] + \frac{d}{dx} [y \sin x] = 0 \leftrightarrow \\
\left( \frac{d}{dx} [x \sin y + x \frac{dy}{dx} \sin y] \right) + \left( \frac{d}{dx} [y \sin x + y \frac{d}{dx} \sin x] \right) = 0 \leftrightarrow \\
(1) \sin y + x \cos y \frac{dy}{dx} + \frac{dy}{dx} \sin x + y \cos x = 0 \leftrightarrow \\
(x \cos y + \sin x) \frac{dy}{dx} = -\sin y - y \cos x \leftrightarrow \frac{dy}{dx} = \frac{-\sin y - y \cos x}{x \cos y + \sin x}.
\]

2. Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

(a) \( x^2 - xy - y^2 = 1 \) at \((2, 1)\)

Solution.

- First we need to find the derivative \( \frac{dy}{dx} \). Using implicit differentiation, we have

\[
\frac{d}{dx} [x^2 - xy - y^2] = \frac{d}{dx} [x^2] - \frac{d}{dx} [xy] - \frac{d}{dx} [y^2] = 0 \leftrightarrow \\
2x - \left(y + x \frac{dy}{dx}\right) - 2y \frac{dy}{dx} = 0 \leftrightarrow 2x - y - x \frac{dy}{dx} - 2y \frac{dy}{dx} = 0 \leftrightarrow \\
(-x - 2y) \frac{dy}{dx} = -2x + y \leftrightarrow \frac{dy}{dx} = \frac{-2x + y}{-x - 2y}.
\]

- To get the slope at \((2, 1)\), we just have to substitute \(x\) by 2 and \(y\) by 1 in \( \frac{dy}{dx} \):

\[
m = \frac{-2(2) + 1}{(2) - 2(1)} = \frac{-4 + 1}{-2 - 2} = \frac{-3}{-4} = \frac{3}{4}.
\]

- Now we have the required equation: \( y - 1 = \frac{3}{4}(x - 2) \).

(b) \( (x^2 + y^2)^2 = 2(x^3 + y^2) \) at \((1, 1)\)

Solution.

- First we need to find the derivative \( \frac{dy}{dx} \). Using implicit differentiation, we have

\[
\frac{d}{dx} [(x^2 + y^2)^2] = \frac{d}{dx} [2(x^3 + y^2)] \leftrightarrow 2(x^2 + y^2) \frac{d}{dx} [x^2 + y^2] = 2 \frac{d}{dx} [x^3 + y^2] \leftrightarrow \\
(2x^2 + 2y^2) \left( \frac{d}{dx} [x^2] + \frac{d}{dx} [y^2] \right) = 2 \left( \frac{d}{dx} [x^3] + \frac{d}{dx} [y^2] \right) \leftrightarrow (2x^2 + 2y^2)(2x + 2y \frac{dy}{dx}) = \\
2(3x^2 + 2y \frac{dy}{dx}) \leftrightarrow 4x^3 + 4x^2 y \frac{dy}{dx} + 4y^2 x + 4y^3 \frac{dy}{dx} = 6x^2 + 4y \frac{dy}{dx} \leftrightarrow \\
4x^2 y \frac{dy}{dx} + 4y^3 \frac{dy}{dx} - 4y \frac{dy}{dx} = 6x^2 - 4x^3 - 4y^2 x \leftrightarrow (4x^2 + 4y^3 - 4y) \frac{dy}{dx} = 6x^2 - 4x^3 - 4y^2 x \leftrightarrow \\
\frac{dy}{dx} = \frac{6x^2 - 4x^3 - 4y^2 x}{4x^2 + 4y^3 - 4y} = \frac{3x^2 - 2x^3 - 2y^2 x}{2x^2 + 2y^3 - 2y}.
\]

- The slope at \((1, 1)\) is

\[
m = \frac{dy}{dx} = \frac{3(1)^2 - 2(1)^3 - 2(1)^2(1)}{2(1)^2 + 2(1)^3 - 2(1)} = \frac{3 - 2 - 2}{2 + 2 - 2} = -1.
\]
3. Find \( y'' = \frac{d^2y}{dx^2} = \frac{d}{dx} \left[ \frac{dy}{dx} \right] \) by implicit differentiation.

(a) \( x^2 - y^2 = 4 \)

**Solution.**

- First derivative:
  \[
  \frac{d}{dx}[x^2 - y^2] = \frac{d}{dx}[x^2] - \frac{d}{dx}[y^2] = 0 \iff 2x - 2y \frac{dy}{dx} = 0 \iff -2y \frac{dy}{dx} = -2x \iff \frac{dy}{dx} = \frac{x}{y}.
  \]

- Second derivative:
  \[
  y'' = \frac{d}{dx} \left[ \frac{dy}{dx} \right] = \frac{d}{dx} \left[ \frac{x}{y} \right] = \frac{\frac{d}{dx}[x]y - x \frac{d}{dx}[y]}{y^2} = \frac{y - x \frac{dy}{dx}}{y^2}.
  \]

We know that \( \frac{dy}{dx} = \frac{x}{y} \). So by substituting, we get

\[
  y'' = \frac{y - x \left( \frac{x}{y} \right)}{y^2} = \frac{y - \frac{x^2}{y}}{y^2} = y - \frac{x^2}{y^3}.
\]

(b) \( x^2 + xy + y^2 = 3 \)

**Solution.**

- First derivative:
  \[
  \frac{d}{dx}[x^2 + xy + y^2] = \frac{d}{dx}[3] \iff \frac{d}{dx}[x^2] + \frac{d}{dx}[xy] + \frac{d}{dx}[y^2] = 0 \iff 2x + x \frac{dy}{dx} + 2y \frac{dy}{dx} = 0 \iff (x + 2y) \frac{dy}{dx} = -2x - y \iff \frac{dy}{dx} = -\frac{2x + y}{x + 2y}.
  \]

- Second derivative:
  \[
  y'' = \frac{d}{dx} \left[ \frac{dy}{dx} \right] = \frac{d}{dx} \left[ -\frac{2x + y}{x + 2y} \right] = \frac{\frac{d}{dx}[-(2x + y)](x + 2y) - (-2x - y) \frac{d}{dx}[x + 2y]}{(x + 2y)^2} = \frac{\frac{d}{dx}(x + 2y) + (2x + y)(1 + 2 \frac{dy}{dx})}{(x + 2y)^2} = \frac{-2(x + 2y) - (x + 2y) \frac{dy}{dx} + 2x + y + 2(2x + y) \frac{dy}{dx}}{(x + 2y)^2} = \frac{-2x + 4y + (-x - 2y + 4x + 2y) \frac{dy}{dx} + 2x + y + 3y + 3x \frac{dy}{dx}}{(x + 2y)^2} = \frac{-3y + 3x \frac{dy}{dx}}{(x + 2y)^2} = \frac{-3y + 3x \frac{-2x - y}{x + 2y}}{(x + 2y)^2} = \frac{-3y + 3x \frac{3x(-2x - y)}{x + 2y}}{(x + 2y)^2} = \frac{-3xy - 6x^2 - 6x^2 - 3xy}{(x + 2y)^3} = \frac{-6x^2 - 6xy - 6y^2}{(x + 2y)^3}.
  \]
2.8 Solution to PB8

Section 2.8: Related Rates

1. If $V$ is the volume of a cube with edge length $x$ and the cube expands as time passes, find $\frac{dV}{dt}$ in terms of $\frac{dx}{dt}$.

**Solution.** Recall: if $y$ is a quantity that depends on time $t$, then by the General Power Rule, we have

$$\frac{d}{dt}[y^n] = ny^{n-1}\frac{dy}{dt}.$$  

We know that the volume of a cube with edge $x$ is given by the formula $V = x^3$. Differentiating both sides with respect to $t$, we get

$$\frac{dV}{dt} = \frac{d}{dt}[x^3] = 3x^2\frac{dx}{dt}.$$  

2. The radius of a sphere is increasing at a rate of 4 mm/s. How fast is the volume increasing when the diameter is 80 mm?

**Solution.** How to solve a related rates problem?

1. Read the problem carefully.
2. Draw a diagram if possible.
3. Introduce notation. Assign symbols to all quantities that are functions of time $t$.
4. Express the given information and the required rate in terms of derivatives.
5. Write an equation that relates the various quantities of the problem.
6. Use the Chain Rule to differentiate both sides of the equation with respect to $t$.
7. Substitute the given information into the resulting equation and solve for the unknown rate.

Here the quantities that are functions of time are the volume and the radius. Let $V$ denote the volume, and let $r$ denote the radius.

Given: $\frac{dr}{dt} = 4$ mm/s \hspace{1cm} Unknown: $\frac{dV}{dt}$ when $r = 40$ mm.

Equation that relates $V$ and $r$: we know that the volume of a sphere of radius $r$ is given by the formula: $V = \frac{4}{3}\pi r^3$. Differentiating both sides with respect to time $t$, we get

$$\frac{dV}{dt} = \frac{d}{dt}\left[\frac{4}{3}\pi r^3\right] = \frac{4}{3}\pi \frac{dr}{dt}[r^3] = \frac{4}{3}\pi (3r^2)\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$  

Substituting, we get

$$\frac{dV}{dt} = 4\pi (40)^2 4 = 16\pi 40 \times 40 = 16\pi \times 16 \times 100 = 256\pi \times 100 = 25600\pi \text{ mm}^3/s.$$  

3. The radius of a spherical ball is increasing at a rate of 2 cm/min. At what rate is the surface area of the ball increasing when the radius is 8 cm?

**Solution.** For this problem, the quantities that are functions of time are the radius $r$, and the surface $S$.

Given: $\frac{dr}{dt} = 2$ cm/min \hspace{1cm} Unknown: $\frac{dS}{dt}$ when $r = 8$ cm.
We know that the surface of a sphere with radius $r$ is given by the formula: $S = 4\pi r^2$. Differentiating both sides with respect to time $t$, we get

$$\frac{dS}{dt} = \frac{d}{dt}[4\pi r^2] = 4\pi \frac{d}{dt}[r^2] = 4\pi (2r) \frac{dr}{dt} = 8\pi r \frac{dr}{dt}.$$ 

Substituting, we get $\frac{dS}{dt} = 8\pi (8)(2) = 64\pi \times 2 = 128\pi \text{ cm}^2/\text{min}$.

4. A cylindrical tank with radius 5 m is being filled with water at a rate of 3 m$^3$/min. How fast is the height of the water increasing?

**Solution.** Here the quantities that depend on time are the volume $V$, and the height $h$. (Note that the radius is constant here: $r = 5$ m.)

| Given: $\frac{dV}{dt} = 3 \text{ m}^3/\text{min}$ | Unknown: $\frac{dh}{dt}$ |

We know that the volume of a cylinder with radius $r$ and height $h$ is given by the formula: $V = \pi r^2 h$. Differentiating both sides with respect to time $t$, we get

$$\frac{dV}{dt} = \frac{d}{dt}[\pi r^2 h] = \pi r^2 \frac{dh}{dt}.$$ 

So $\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$. Solving this latter equation for $\frac{dh}{dt}$, we get $\frac{dh}{dt} = \frac{1}{\pi r^2} \frac{dV}{dt}$. Now, by substituting, we get $\frac{dh}{dt} = \frac{1}{\pi (5)^2} (3) = \frac{3}{25\pi} \text{ m/min}$.

5. A water tank has the shape of an inverted circular cone with base radius 4 m and height 6 m. If water is being pumped into the tank at a rate of 8 m$^3$/min, find the rate at which the water level is rising when the water is 2 m deep.

**Solution.** Consider Figure 2.6.

![Figure 2.6: A water tank](image)

The quantities that depend on time and that we are interested in are the volume $V$ and the height $h$.

| Given: $\frac{dV}{dt} = 8 \text{ m}^3/\text{min}$ | Unknown: $\frac{dh}{dt}$ when $h = 2$ m. |

We know that the volume of a cone with radius $r$ and height $h$ is given by the formula: $V = \frac{1}{3} \pi r^2 h$. Since the radius $r$ depends also on time, it is useful to first eliminate $r$ in that equation before differentiating both sides as usual. To eliminate $r$, we can use similar triangles formula: $\frac{AD}{AB} = \frac{DE}{BC}$, which is equivalent to $\frac{h}{6} = \frac{r}{4}$. From this latter equation, we have $r = \frac{3}{2} h = \frac{2}{3} h$. Plug-in this in the formula of the volume, we have

$$V = \frac{1}{3} \pi \left(\frac{2}{3} h\right)^2 h = \frac{1}{3} \pi \left(\frac{4}{9} h^2\right) h = \frac{4\pi}{27} h^3.$$
2.8. SOLUTION TO PB8

Differentiating now both sides, we get

\[
\frac{dV}{dt} = \frac{d}{dt} \left[ \frac{4\pi}{27} h^3 \right] = \frac{4\pi}{27} \frac{d}{dt} \left[ h^3 \right] = \frac{4\pi}{27} (3h^2) \frac{dh}{dt} = \frac{4\pi h^2 dh}{9}.
\]

So \( \frac{dV}{dt} = \frac{4\pi h^2 \frac{dh}{dt}}{9} \). Solving this latter equation for \( \frac{dh}{dt} \), we get

\[
\frac{dh}{dt} = \frac{9}{4\pi h^2} \frac{dV}{dt}.
\]

Now, substituting \( \frac{dV}{dt} \) by 8 and \( h \) by 2, we get

\[
\frac{dh}{dt} = \frac{9}{4\pi (2)^2} (8) = \frac{9}{16\pi} (8) = 9 \frac{2}{\pi} \text{ m/min}.
\]

6. A ladder 10 m long is leaning against a vertical wall with its other end on the ground. The top end of the ladder is sliding down the wall. When the bottom of the ladder is 6 m from the wall, it is sliding at 1 m/s.

(a) How fast is the angle between the ladder and the ground changing at this instant?

(b) How fast is the top of the ladder sliding down at the same instant?

Solution.

(a) Consider Figure 2.7.

![Figure 2.7](image)

Given: \( \frac{dx}{dt} = 1 \text{ m/s} \)  
Unknown: \( \frac{d\theta}{dt} \) when \( x = 6 \text{ m} \).

Equation relating \( \theta \) and \( x \): since \( \cos \theta = \frac{x}{10} \), we have \( 10 \cos \theta = x \). Differentiating both sides, we get \( 10 \frac{d}{dt}[\cos \theta] = \frac{dx}{dt} \), which is equivalent to \( 10(-\sin \theta) \frac{d\theta}{dt} = \frac{dx}{dt} \). Solving this for \( \frac{d\theta}{dt} \), we get

\[
\frac{d\theta}{dt} = -\frac{1}{10 \sin \theta} \frac{dx}{dt}.
\]

Now we need to substitute. What is \( \sin \theta \) when \( x = 6 \)? By the Pythagorean theorem, we have \( x^2 + y^2 = 10^2 \), which is equivalent to \( x^2 + y^2 = 100 \) or \( y^2 = 100 - x^2 \). When \( x = 6 \), \( y^2 = 100 - 36 = 64 \), so that \( y = 8 \). Thus \( \sin \theta = \frac{y}{10} = \frac{8}{10} \). Substituting this in the equation above, we get

\[
\frac{d\theta}{dt} = -\frac{1}{10 \sin \theta} \frac{dx}{dt} = \frac{1}{10 \left( \frac{8}{10} \right)} (1) = \frac{1}{8} \text{ rad/s}.
\]

(b) For this part \( \frac{dx}{dt} = 1 \text{ m/s} \) is given and the unknown is \( \frac{dy}{dt} \) when \( x = 6 \). Again by the Pythagorean theorem, we have \( x^2 + y^2 = 100 \), which gives an equation relating \( x \) and \( y \). Differentiating both sides of this, we get \( 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \), which implies that

\[
\frac{dy}{dt} = \frac{-2x}{2y} \frac{dx}{dt} = -\frac{x}{y} \frac{dx}{dt} = \frac{-6}{8} (1) = \frac{-3}{4} \text{ m/s}.
\]
7. A plane flying horizontally at an altitude of 3 mi and a speed of 500 mi/h passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 5 mi away from the station.

**Solution.** Consider Figure 2.8.

\[ \text{Figure 2.8:} \]

\[
\text{Given: } \frac{dx}{dt} = 500 \text{ mi/h} \quad \text{Unknown: } \frac{dz}{dt} \text{ when } z = 5 \text{ mi.}
\]

Equation relating \( x \) and \( z \): \( x^2 + 9 = z^2 \). Differentiating both sides, we get \( 2x \frac{dx}{dt} + 0 = 2z \frac{dz}{dt} \), so that \( \frac{dz}{dt} = \frac{x \frac{dx}{dt}}{z} \). Using the Pythagorean theorem, one can see that \( x = 4 \) when \( z = 5 \). Thus

\[
\frac{dz}{dt} = \frac{4 \cdot 500}{5} = 400 \text{ mi/h}. 
\]

8. A hot air balloon rising vertically is viewed by an observer who is 2 km from the lift-off point. At a certain moment, the angle between the observer’s line of sight and the horizontal is \( \frac{\pi}{6} \) radians, and this angle is increasing at the rate of \( \frac{1}{5} \) radians per minute. How fast is the balloon rising (in km/min) at that moment?

**Solution.** Consider Figure 2.9.

\[ \text{Figure 2.9:} \]

\[
\text{Given: } \frac{d\theta}{dt} = \frac{1}{5} \text{ rad/min} \quad \text{Unknown: } \frac{dy}{dt} \text{ when } \theta = \frac{\pi}{6} \text{ rad.}
\]

Equation relating \( \theta \) and \( y \): \( \tan \theta = \frac{y}{2} \), which is equivalent to \( y = 2 \tan \theta \). Differentiating both sides with respect to \( t \), we get

\[
\frac{dy}{dt} = 2 \frac{d}{dt} [\tan \theta] = 2 \sec^2 \theta \frac{d\theta}{dt} = 2 \frac{1}{\cos^2 \theta} \frac{d\theta}{dt} = 2 \frac{1}{(\cos \frac{\pi}{6})^2} \left( \frac{1}{5} \right) = 
\]
2.8. Solution to PB8

\[ 2 - \frac{1}{(\sqrt[3]{3})^2} \left( \frac{1}{5} \right) = 2 \frac{1}{3} \left( \frac{1}{5} \right) = 2 \frac{4}{3} \left( \frac{1}{5} \right) = \frac{8}{15} \text{ km/min.} \]

Section 3.1: Maximum and Minimum Values

1. Find the critical numbers of each of the following functions.

(a) \( f(x) = -2x^2 + 8x - 3 \)

**Solution.** Recall that a number \( c \) in the domain of a function \( f \) is called a critical number if \( f'(c) = 0 \) or \( f'(c) \) is undefined. So to find critical numbers,

- we first find the derivative \( f'(x) \), and then we solve the equation \( f'(x) = 0 \), and
- we find all numbers \( c \) such that \( f'(c) \) is undefined and \( c \) lies in the domain of \( f \).

For \( f(x) = -2x^2 + 8x - 3 \), we have \( f'(x) = -4x + 8 \). Solving the equation \(-4x + 8 = 0\), we get \(-4x = -8 \) or \( x = \frac{8}{4} = 2 \). So 2 is the only number that makes the derivative equal to 0. Since \( f'(x) \) is defined everywhere (polynomial), it follows that there is no \( c \) such that \( f'(c) \) is undefined. Thus \( f \) has only one critical number: 2.

(b) \( f(x) = \sqrt{x} \)

**Solution.** First we have \( f(x) = x^{\frac{1}{2}} \). Now we have

\[ f'(x) = \frac{1}{3} x^{-\frac{1}{2}} = \frac{1}{3} \cdot \frac{1}{\sqrt{x}} = \frac{1}{3 \sqrt{x}}. \]

If \( \frac{1}{3 \sqrt{x}} = 0 \), then \( 1 = 0^2 \), which is impossible. So there is no number that makes the derivative equal to 0. However we can observe that \( f'(x) \) is undefined when \( x = 0 \). Since 0 is in the domain of \( f \), it follows that it is a critical number of \( f \).

(c) \( f(x) = x^3 + x^2 + 5 \)

**Solution.** We have \( f'(x) = 3x^2 + 2x = x(3x + 2) \) [Note: In order to find critical numbers, it is useful to factor the derivative whenever it is possible.]. \( f'(x) = 0 \) is equivalent to \( x = 0 \) or \( 3x + 2 = 0 \), so that \( x = 0 \) or \( x = -\frac{2}{3} \), which are the critical numbers.

(d) \( f(x) = x^3 + 6x^2 - 15x \)

**Solution.** We have

\[ f'(x) = 3x^2 + 12x - 15 = 3(x^2 + 4x - 5) = 3(x - 1)(x + 5). \]

So \( f'(x) = 0 \) is equivalent to \( x - 1 = 0 \) or \( x + 5 = 0 \), so that \( x = 1 \) or \( x = -5 \), which are the critical numbers of \( f \).

(e) \( f(x) = \sqrt{1 - x^2} \)

**Solution.** First we have \( f(x) = (1 - x^2)^{\frac{1}{2}} \). Now we have

\[ f'(x) = \frac{1}{2} (1 - x^2)^{-\frac{1}{2}} \frac{d}{dx} [1 - x^2] = \frac{1}{2} (1 - x^2)^{-\frac{1}{2}} (-2x) = -x \frac{1}{(1 - x^2)^{\frac{1}{2}}} = \frac{-x}{\sqrt{1 - x^2}}. \]

\( f'(x) = 0 \) is equivalent to \( -x = 0 \) or \( x = 0 \). So 0 is a critical number. Furthermore, we can observe that the derivative is undefined when its denominator is equal to zero: \( 1 - x^2 = 0 \), that is, \((1 - x)(1 + x) = 0 \) or \( x = 1 \) or \( x = -1 \). Since \(-1 \) and 1 belong to the domain of \( f \), it follows that they are also critical numbers of \( f \). Hence the critical numbers of \( f \) are: 0, -1, and 1.

\(^2\)If a fraction \( \frac{A}{B} = 0 \), then \( A = 0 \)
(f) \( f(x) = \frac{2x+1}{x-3} \)

**Solution.** Using the Quotient Rule, we get

\[
 f'(x) = \frac{2(x-3) - (2x+1)(1)}{(x-3)^2} = \frac{2x - 6 - 2x - 1}{(x-3)^2} = \frac{-7}{(x-3)^2}
\]

\( f'(x) = 0 \) is equivalent to \(-7 = 0\). This is impossible; so there is no number that makes the derivative equal to 0. However, one can see that \( f'(x) \) is undefined when \( x = 3 \). Since 3 does not lie in the domain of \( f \) (3 also makes the denominator of \( f(x) \) equal to 0), it follows 3 is not a critical number. Thus \( f \) has no critical number.

2. Find the absolute maximum and absolute minimum values of \( f \) on the given interval. Also state the locations of these absolute extrema.

(a) \( f(x) = 12 + 4x - x^2 \) on \([0, 5]\)

**Solution.** To find the absolute maximum and absolute minimum values of \( f \) on a closed interval, we can proceed as follows (this method is called the *Closed Interval Method*).

Step 1. Find the critical numbers of \( f \) on the open interval. First we need to find the derivative: \( f'(x) = 4 - 2x = 2(2-x) \). The equation \( f'(x) = 0 \) is equivalent to \( 2(2-x) = 0 \) or \( x = 2 \). So 2 is a critical number of \( f \). In fact it is the only one. Now we have to check whether that critical number lies in the open interval \((0, 5)\). Clearly 2 belongs to \((0, 5)\).

Step 2. Find the values of \( f \) at the critical numbers that lie in the open interval. We have \( f(2) = 12 + 4(2) - (2)^2 = 12 + 8 - 4 = 16 \).

Step 3. Find the values of \( f \) at the endpoints of the interval. (The endpoints of the interval here are 0 and 5.) We have \( f(0) = 12 \) and \( f(5) = 12 + 20 - 25 = 7 \).

Step 4. The absolute maximum (abbreviated abs max) value of \( f \) is the largest value from steps 2 and 3. So abs max = 16 located at 2 (it is located at 2 because \( f(2) = 16 \)). The absolute minimum (abbreviated abs min) is the smallest value from steps 2 and 3. So abs min = 7 located at 5.

(b) \( f(x) = -x^2 + 2x + 3 \) on \([0, 2]\)

**Solution.** We have \( f'(x) = -2x + 2 = -2(x-1) \). \( f'(x) = 0 \) is equivalent to \( x = 1 \). Does 1 lie in the open interval \((0, 2)\)? Yes. So 1 is the only critical number of \( f \) in \((0, 2)\). Now the value of \( f \) at that number is \( f(1) = -1 + 2 + 3 = 4 \), and the values of \( f \) at the endpoints of the interval are: \( f(0) = 3 \) and \( f(2) = -4 + 4 + 3 = 3 \). Thus abs max = 4 located at 1 and abs min = 3 located at 0 and 2.

(c) \( f(x) = x^3 - 3x + 5 \) on \([0, 3]\)

**Solution.** We have \( f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x-1)(x+1) \). \( f'(x) = 0 \) is equivalent to \( x = 1 \) or \( x = -1 \). Does 1 lie in \((0,3)\)? Yes. Does -1 lie in \((0,3)\)? No. So we throw away \(-1\). Now the value of \( f \) at 1 is \( f(1) = 1 - 3 + 5 = 3 \), and its values at the endpoints are: \( f(0) = 5 \) and \( f(3) = 27 - 9 + 5 = 23 \). Thus abs max = 23 located at 3, and abs min = 3 located at 1.

(d) \( f(x) = -x^3 + 3x^2 + 1 \) on \([-1, 2]\)

**Solution.** We have \( f'(x) = -3x^2 + 6x = -3x(x-2) \). The critical numbers are then 0 and 2. We can see that 0 is the only one that belongs to the open interval \((-1, 2)\). The value of \( f \) at that critical number is \( f(0) = 1 \), and its values at the endpoints are: \( f(-1) = -(-1)^3 + 3(-1)^2 + 1 = 1 + 3 + 1 = 5 \) and \( f(2) = -8 + 12 + 1 = 5 \). Thus abs max = 5 located at \(-1\) and abs min = 1 located at 0.

(e) \( f(x) = 2x^3 - 3x^2 - 12x + 1 \) on \([-2, 1]\)

**Solution.** The derivative is \( f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x+1)(x-2) \). This implies that \(-1\) and \(2\) are the critical numbers of \( f \). But \(2\) does not lie in \((-2, 1)\), so we throw
2.9 Solution to PB9

Section 3.2: The Mean Value Theorem

Verify that the following functions satisfy the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers \( c \) that satisfy the conclusion of the Mean Value Theorem.

1. \( f(x) = -3x^2 + x \) on \([0, 1]\)

   **Solution.** First let us recall the Mean Value Theorem: Let \( f \) be a function that satisfies the following hypotheses:
   
   1. \( f \) is continuous on \([a, b]\).
   2. \( f \) is differentiable on the open interval \((a, b)\).

   Then there is a number \( c \) in \((a, b)\) such that
   \[
   f'(c) = \frac{f(b) - f(a)}{b - a}.
   \]

   Clearly the function \( f(x) = -3x^2 + x \) is continuous and differentiable everywhere since it is a polynomial. So it is certainly continuous on \([0, 1]\) and differentiable on \((0, 1)\). Therefore, by the Mean Value Theorem, there is a number \( c \) in \((0, 1)\) such that
   \[
   f'(c) = \frac{f(1) - f(0)}{1 - 0}.
   \]

   However, \( f'(-x) = -6x + 1, \) and \( f'(x) = -6x + 1, \) so this equation becomes \(-6c + 1 = \frac{-2}{0}\), which gives \(-6c + 1 = -2, \) that is, \(-6c = -3 \) or \( c = \frac{1}{2}. \) Does \( \frac{1}{2} \) lie in \((0, 1)\)? Yes. So \( c = \frac{1}{2}. \)
2. \( f(x) = x^3 - 3x + 2 \) on \([-2, 2]\)

**Solution.** For the same reason as before, the function \( f \) is certainly continuous on \([-2, 2]\) and differentiable on \((-2, 2)\). Therefore, by the Mean Value Theorem, there is a number \( c \) in \((-2, 2)\) such that \( f'(c) = \frac{f(2) - f(-2)}{2 - (-2)} \). Now \( f(2) = 8 - 6 + 2 = 4, f(-2) = -8 + 6 + 2 = 0, f'(x) = 3x^2 - 3, \) and \( f'(c) = 3c^2 - 3 \), so this equation becomes \( 3c^2 - 3 = \frac{4 - 0}{2 - (-2)} \), that is, \( 3c^2 - 3 = 1 \) or \( 3c^2 = 4 \). This latter equation gives \( c^2 = \frac{4}{3} \), so that \( c = -\sqrt{\frac{4}{3}} \) or \( c = \sqrt{\frac{4}{3}} \). Certainly, those two numbers belong to the open interval \((-2, 2)\). Thus \( c = -\sqrt{\frac{4}{3}} = -\frac{2}{\sqrt{3}} \) or \( c = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}} \).

3. \( f(x) = x^3 - 2x \) on \([-1, 0]\)

**Solution.** Again as before, \( f \) is continuous on \([-1, 0]\) and differentiable on \((-1, 0)\). By the Mean Value Theorem, there is a number \( c \) in \((-1, 0)\) such that \( f'(c) = \frac{f(0) - f(-1)}{0 - (-1)} \). Now \( f(-1) = -1 + 2 = 1, f(0) = 0, f'(x) = 3x^2 - 2, \) and \( f'(c) = 3c^2 - 2 \), so this equation becomes \( 3c^2 - 2 = \frac{0 - 1}{-1} = -1 \), that is, \( 3c^2 = 1 \) or \( c^2 = \frac{1}{3} \). This latter equation gives \( c = \pm \sqrt{\frac{1}{3}} \). But \( c \) must lie in \((-1, 0)\), so \( c = -\sqrt{\frac{1}{3}} \).

4. \( f(x) = \frac{1}{x^2} \) on \([1, 3]\)

**Solution.** Here \( f \) is a rational function, which is then continuous and differentiable on its domain, which is \( \mathbb{R} \setminus \{0\} \) (all numbers except 0). So it is certainly continuous on \([1, 3]\) and differentiable on \((1, 3)\). By the Mean Value Theorem, there is \( c \) in \((1, 3)\) such that \( f'(c) = \frac{f(3) - f(1)}{3 - 1} \). Now \( f(3) = \frac{1}{3}, f(1) = 1, f'(x) = - \frac{1}{x^2}, \) and \( f'(c) = - \frac{1}{c^2} \). So this equation becomes

\[
- \frac{1}{c^2} = - \frac{1}{3} - 1 = - \frac{2}{3} = - \frac{2}{3} \times \frac{1}{2} = - \frac{1}{3}
\]

which gives \( - \frac{1}{c^2} = - \frac{1}{3} \) or \( \frac{1}{c^2} = \frac{1}{3} \), that is, \( c^2 = 3 \). This latter equation gives \( c = \pm \sqrt{3} \). But \( c \) must lie in \((1, 3)\), so \( c = \sqrt{3} \).

### Section 3.3: How Derivatives Affect the Shape of a Graph

For each of the following functions, find where it is increasing and where it is decreasing. Also find the local maximum and minimum values. Also find the intervals of concavity and the inflection points.

1. \( f(x) = -x^2 + 4x - 3 \)

**Solution.** We need to find the first derivative for increasing/decreasing, local max and min, and we need the second derivative for the concavity and inflection points. Recall the following.

**Increasing/Decreasing Test:**

(a) If \( f'(x) > 0 \) on an interval, then \( f \) is increasing on that interval.

(b) If \( f'(x) < 0 \) on an interval, then \( f \) is decreasing on that interval.

(c) If \( f'(x) = 0 \) on an interval, then \( f \) is constant on that interval.

**Local maximum and minimum values:** Suppose that \( c \) is a critical number of a continuous function \( f \).

(a) If \( f' \) changes from positive to negative at \( c \), then \( f \) has a local maximum at \( c \). In that case, \( f(c) \) is a local maximum value.

(b) If \( f' \) changes from negative to positive at \( c \), then \( f \) has a local minimum at \( c \). In that case, \( f(c) \) is a local minimum value.
(c) If $f'$ is positive to the left and right of $c$, or negative to the left and right of $c$, then $f$ has no local maximum or minimum.

**Concavity Test:**

(a) If $f''(x) > 0$ for all $x$ in an interval $I$, then the graph of $f$ is concave upward on $I$.

(b) If $f''(x) < 0$ for all $x$ in an interval $I$, then the graph of $f$ is concave downward on $I$.

So to find the intervals of concavity, we need to first find the second derivative $f''(x)$, then find the points that make $f''(x)$ equal to zero. And then deduce the sign of $f''(x)$.

**Inflection point:** An inflection point is a point where the concavity changes. That is, a point where the graph changes from concave upward to concave downward or from concave downward to concave upward.

We come back to the question.

- First derivative: we have $f'(x) = -2x + 4$.
- Critical numbers: $-2x + 4 = 0$ gives $-2x = -4$, that is, $x = -4 = 2$.
- Second derivative: $f''(x) = -2$.
- Table: see Figure 2.10. How to fill out that table? At the top row, we have the domain of $f$ (which is $\mathbb{R} = (-\infty, \infty)$) and the critical numbers (here we have only one critical number $= 2$). At the second row we have the sign of the first derivative. To get the sign on the interval $(-\infty, 2)$, pick any number in that interval and plug-in $f'(x)$. If the result is positive then $f'(x) > 0$. If the result is negative that $f'(x) < 0$. For example, if we pick 0, we get $f'(0) = 4 > 0$, so the sign is “+” on $(-\infty, 2)$. We use the same technique to get the sign of $f'(x)$ on the interval $(2, \infty)$: if for example, we pick 3, we get $f'(3) = -6 + 4 = -2 < 0$. So the sign is “−” on $(2, \infty)$. Thus $f$ is increasing on the interval $(-\infty, 2)$ and decreasing on $(2, \infty)$. The table shows that $f$ has a local maximum at 2; so $f(2) = -4 + 8 - 3 = 1$ is a local maximum value. However, there is no local minimum.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-\infty$</th>
<th>2</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sign of $f'(x)$</td>
<td>+</td>
<td>...</td>
<td>−</td>
</tr>
<tr>
<td>Increasing/ Decreasing</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

How about the concavity? Since $f''(x) = -2 < 0$ is negative everywhere, it follows that $f$ is concave downward on $\mathbb{R}$. So there is no inflection point.

2. $f(x) = x^3 - 3x^2 - 9x + 4$

**Solution.**
3. \( f(x) = -x^4 + 4x^3 + 1 \)

Solution.

- Critical numbers: The first derivative is \( f'(x) = -4x^3 + 12x^2 = 4x^2(-x + 3) \). \( f'(x) = 0 \) gives \( 4x^2 = 0 \) or \( -x + 3 = 0 \), that is, \( x = 0 \) or \( x = 3 \), which are the critical numbers. Now the value of \( f \) at 0 is: \( f(0) = 1 \). The value at 3 is: \( f(3) = -81 + 108 + 1 = 28 \).
- Second derivative: we have \( f''(x) = -12x^2 + 24x = 12x(-x + 2) \). Points that make \( f''(x) \) equal to zero: \( f''(x) = 0 \) gives \( x = 0 \) or \( x = 2 \). Values of \( f \) at those points: \( f(0) = 1 \) and \( f(2) = -16 + 32 + 1 = 17 \).
- Table: see Figure 2.12. From that table, one can see that \( f \) is increasing on the interval \((-\infty, 3)\) and decreasing on \((3, \infty)\). The number 28 is a local maximum value. There is no local minimum. For the concavity, one can again see from the same table that \( f \) is concave down on \((-\infty, 0)\), concave up on \((0, 2)\), and again concave down on \((2, \infty)\). So the concavity changes twice: at 0 and 2. So we have two inflection points: \((0, 1)\) and \((2, 17)\).

4. \( f(x) = \frac{x}{x^2 + 1} \)

Solution.
2.9. SOLUTION TO PB9

Figure 2.12:

- Critical numbers: using the Quotient Rule, we have
  \[ f'(x) = \frac{(1)(x^2 + 1) - x(2x)}{(x^2 + 1)^2} = \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2} = \frac{-x^2 + 1}{(x^2 + 1)^2}. \]
  
  \( f'(x) = 0 \) gives \(-x^2 + 1 = 0\), that is, \(-x^2 - 1 = 0\) or \(-(x-1)(x+1) = 0\), which gives \(x = 1\) or \(x = -1\). Values of \(f\) at those critical numbers: \(f(-1) = \frac{1}{1^2 + 1} = \frac{1}{2}\), and \(f(1) = \frac{1}{2}\).

- Second derivative: again by using the Quotient Rule, we have
  \[ f''(x) = \frac{-2x(x^2 + 1)^2 - (-x^2 + 1)2(x^2 + 1)(2x)}{(x^2 + 1)^4} = \frac{(x^2 + 1)[-2x(x^2 + 1) - 4x(-x^2 + 1)]}{(x^2 + 1)^4} = \frac{(x^2 + 1)(2x^2 - 3)}{(x^2 + 1)^4}. \]
  
  Points that make \(f''(x)\) equal to zero: \(f''(x) = 0\) gives \((x^2 + 1)2x(x^2 - 3) = 0\), that is, \(x^2 + 1 = 0\) or \(2x = 0\) or \(x^2 - 3 = 0\), that is, \(x = 0\) or \(x = \pm \sqrt{3}\). [Note that \(x^2 + 1 = 0\) has no solution at all as \(x^2 + 1\) is always positive, and then can not be equal to 0.] Values of \(f\) at those points:
  \(f(-\sqrt{3}) = -\frac{\sqrt{3}}{3+1} = -\frac{\sqrt{3}}{4}, f(0) = 0, f(\sqrt{3}) = \frac{\sqrt{3}}{4}\).

- Table: see Figure 2.13. That table shows that \(f\) is increasing on the interval \((-1, 1)\) and decreasing on \((-\infty, -1)\) and \((1, \infty)\). The number \(-\frac{1}{2}\) is a local minimum value, while \(\frac{1}{2}\) is a local maximum value. For the concavity, \(f\) is concave up on \((-\sqrt{3}, 0)\) and \((\sqrt{3}, +\infty)\), and concave down on \((-\infty, -\sqrt{3})\) and \((0, \sqrt{3})\). So we have two inflection points: \((-\sqrt{3}, -\frac{\sqrt{3}}{4})\) and \((\sqrt{3}, \frac{\sqrt{3}}{4})\).

Section 3.4: Limits at infinity; Horizontal Asymptotes

Find the following limits.

1. \(\lim_{x \to -\infty} \frac{1}{x^2}\)

   **Solution.** First recall the following basic limits at infinity:

   - If \(r\) is a positive real number, then \(\lim_{x \to \infty} \frac{1}{x^r} = 0\).
   - If \(n\) is a positive integer, then \(\lim_{x \to -\infty} \frac{1}{x^n} = 0\).
Now we have
\[
\lim_{x \to -\infty} \frac{1}{3x^4} = \frac{1}{3} \left( \lim_{x \to -\infty} \frac{1}{x^4} \right) = \frac{1}{3}(0) = 0.
\]

2. \(\lim_{x \to \infty} \frac{1}{\sqrt{x}}\)

Solution. We have \(\lim_{x \to \infty} \frac{1}{\sqrt{x}} = \lim_{x \to \infty} \frac{1}{x^{1/2}} = 0\). (Here \(r = \frac{1}{2}\).)

3. \(\lim_{x \to \infty} \frac{-x^2 + 1}{3x^2 + x + 1}\)

Solution. To find the limit of a rational function at infinity, we can proceed as follows:

Step 1 Divide every term by the largest power of \(x\) in the denominator. Here \(f(x) = \frac{-x^2 + 1}{3x^2 + x + 1}\), and the largest power of \(x\) in the denominator is \(x^2\). The terms of the function are: \(-x^2\), \(1\), \(3x^2\), \(x\), and \(1\). Dividing each term by \(x^2\), we get
\[
\lim_{x \to \infty} \frac{-x^2 + 1}{3x^2 + x + 1} = \lim_{x \to \infty} \frac{-\frac{x^2}{x^2} + \frac{1}{x^2}}{3 + \frac{1}{x} + \frac{1}{x^2}}.
\]

Step 2 Simplify each term whenever it is possible. For example, the term \(-\frac{x^2}{x^2}\) becomes \(-1\), the term \(\frac{1}{x^2}\) remains the same, etc. So we get
\[
\lim_{x \to \infty} \frac{-x^2 + 1}{3x^2 + x + 1} = \lim_{x \to \infty} \frac{-1 + \frac{1}{x}}{3 + \frac{1}{x} + \frac{1}{x^2}}.
\]

Step 3 Use basic limits above to get the final answer.
\[
\lim_{x \to \infty} \frac{-x^2 + 1}{3x^2 + x + 1} = \frac{-1 + 0}{3 + 0 + 0} = -\frac{1}{3}.
\]

4. \(\lim_{x \to \infty} \frac{7x^2}{x^2 - 4}\)

Solution. We have
\[
\lim_{x \to \infty} \frac{7x^2}{x^2 - 4} = \lim_{x \to \infty} \frac{7x^2}{x^2} = \lim_{x \to \infty} \frac{7}{1 - \frac{4}{x^2}} = \frac{7}{1 - 0} = 7.
\]

\(^3\text{A rational function is a function of the form}\ \frac{P(x)}{Q(x)}\text{, where } P(x) \text{ and } Q(x) \text{ are both polynomial.}\)
5. \( \lim_{x \to \infty} \frac{3x^2 - 2x}{-5x^3 + x^2 - x} \)

**Solution.** We have

\[
\lim_{x \to \infty} \frac{3x^2 - 2x}{-5x^3 + x^2 - x} = \lim_{x \to \infty} \frac{3x^2 - 2x}{-5x^3 + x^2 - x} = \\
\lim_{x \to \infty} \frac{\frac{3}{x} - \frac{2}{x^2}}{-5 + \frac{1}{x} - \frac{1}{x^2}} = \frac{0 - 0}{-5 + 0 + 0} = \frac{0}{-5} = 0.
\]

6. \( \lim_{x \to \infty} \sqrt{x^2 + 3 - x} \)

**Solution.** First we rationalize.

\[
\lim_{x \to \infty} \sqrt{x^2 + 3 - x} = \lim_{x \to \infty} \frac{(\sqrt{x^2 + 3} - x)(\sqrt{x^2 + 3} + x)}{\sqrt{x^2 + 3} + x} = \lim_{x \to \infty} \frac{(x^2 + 3) - x^2}{\sqrt{x^2 + 3} + x} = \lim_{x \to \infty} \frac{3}{\sqrt{x^2 + 3} + x}.
\]

From this point, we can use the same method as before (that is, divide each term by the largest power of \( x \) in the denominator). It turns out that that method does not work very well when dealing with roots and limits as \( x \) goes to \(-\infty\). Fortunately, there is another technique, that consists of pulling out the largest power of \( x \) in the numerator and the denominator, and then simplify. For example, \( \sqrt{x^2 + 3} \) becomes

\[
\sqrt{x^2(1 + \frac{3}{x^2})} = \sqrt{x^2} \sqrt{1 + \frac{3}{x^2}} = |x| \sqrt{1 + \frac{3}{x^2}} = x \sqrt{1 + \frac{3}{x^2}}.
\]

We used the fact that \( \sqrt{x^2} = |x| \). The last equality follows from the fact that when \( x \) goes to positive infinity, \( x > 0 \), and therefore \( |x| = x \). (If it was \( x \to -\infty \), then \( |x| = -x \).) So the limit above becomes

\[
\lim_{x \to \infty} \frac{3}{\sqrt{1 + \frac{3}{x^2} + 1}} = \frac{0}{\sqrt{1 + 0} + 1} = \frac{0}{2} = 0.
\]

**Note.** \( \sqrt{x^2} \) is not always equal to \( x \) as \( x \) could be a negative number. For example, if \( x = -2 \), \( \sqrt{(-2)^2} = \sqrt{4} = 2 \). So \( \sqrt{(-2)^2} \) is not equal to \(-2\), but \( 2 \) or \(-(-2) = -x \). However, if \( x \) is positive, then \( \sqrt{x^2} \) is definitely equal to \( x \). Hence

\[
\sqrt{x^2} = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x < 0
\end{cases}
\]

How about \( \sqrt{x^2} \) for all \( x \)? Since \( x^2 \) is always a positive number, we have \( \sqrt{x^2} = x^2 \) for all \( x \).

7. \( \lim_{x \to \infty} \sqrt{x^2 + x + 1} - x \)

**Solution.** We have

\[
\lim_{x \to \infty} \sqrt{x^2 + x + 1} - x = \lim_{x \to \infty} \frac{(\sqrt{x^2 + x + 1} - x)(\sqrt{x^2 + x + 1} + x)}{\sqrt{x^2 + x + 1} + x} = \lim_{x \to \infty} \frac{x^2 + x + 1 - x^2}{\sqrt{x^2 + x + 1} + x} = \\
\lim_{x \to \infty} \frac{x + 1}{\sqrt{x^2 + x + 1} + x} = \lim_{x \to \infty} \frac{x(1 + \frac{1}{x})}{\sqrt{x^2(1 + \frac{x}{x^2} + \frac{1}{x^2}) + x} = \\
\lim_{x \to \infty} \frac{x(1 + \frac{1}{x})}{\sqrt{x^2} \sqrt{1 + \frac{x}{x^2} + \frac{1}{x^2} + x}} = \lim_{x \to \infty} \frac{x(1 + \frac{1}{x})}{x \sqrt{1 + \frac{x}{x^2} + \frac{1}{x^2} + x}} = \\
\lim_{x \to \infty} \frac{x(1 + \frac{1}{x})}{x \left( \sqrt{1 + \frac{x}{x^2} + \frac{1}{x^2} + 1} \right)} = \lim_{x \to \infty} \frac{1 + \frac{1}{x}}{1 + \frac{1}{x^2} + \frac{1}{x^2} + 1} = \frac{1 + 0}{\sqrt{1 + 0 + 0 + 1}} = \frac{1}{2}
\]


8. \( \lim_{x \to \infty} \frac{\sqrt{x} + x^2}{2x - x^2} \)

**Solution.** Here we can use the first method.

\[
\lim_{x \to \infty} \frac{\sqrt{x} + x^2}{2x - x^2} = \lim_{x \to \infty} \frac{\frac{\sqrt{x}}{x} + \frac{x^2}{x^2}}{\frac{2x}{x^2} - \frac{x^2}{x^2}} = \lim_{x \to \infty} \frac{x^{\frac{1}{2}} + 1}{2x^{-1} - 1} = \lim_{x \to \infty} \frac{\frac{1}{x^{\frac{1}{2}}} + 1}{0 - 1} = -1.
\]

9. Find the horizontal asymptotes of \( f(x) = \frac{x^2}{\sqrt{x^4 + 1}} \)

**Solution.** Recall. The line \( y = L \) is called a horizontal asymptote of the curve \( y = f(x) \) if either \( \lim_{x \to \infty} f(x) = L \) or \( \lim_{x \to -\infty} f(x) = L \). So to find horizontal asymptotes, we need to find two limits: \( \lim_{x \to \infty} f(x) \) and \( \lim_{x \to -\infty} f(x) \). Let us begin with the first limit.

\[
\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x^2}{\sqrt{x^4 + 1}} = \lim_{x \to \infty} \frac{x^2}{\sqrt{x^4(1 + \frac{1}{x^4})}} = \lim_{x \to \infty} \frac{x^2}{x^2 \sqrt{1 + \frac{1}{x^4}}} = \lim_{x \to \infty} \frac{1}{\sqrt{1 + \frac{1}{x^4}}} = 1.
\]

The limit as \( x \to -\infty \) is the same because \( \sqrt{x^4} = x^2 \) for all \( x \). So \( y = 1 \) is a horizontal asymptote.

10. Find the horizontal asymptotes of \( f(x) = \frac{\sqrt{x^4 + 1}}{2x - 1} \)

**Solution.** We have

\[
\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{\sqrt{x^4 + 1}}{2x - 1} = \lim_{x \to \infty} \frac{x^2(1 + \frac{1}{x^2})}{x(2 - \frac{1}{x})} = \lim_{x \to \infty} \frac{x^2(1 + \frac{1}{x^2})}{x(2 - \frac{1}{x})} = \lim_{x \to \infty} \frac{\sqrt{x^2(1 + \frac{1}{x^2})}}{x(2 - \frac{1}{x})}.
\]

\[
\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{\sqrt{x^4 + 1}}{2x - 1} = \lim_{x \to -\infty} \frac{x^2(1 + \frac{1}{x^2})}{x(2 - \frac{1}{x})} = \lim_{x \to -\infty} \frac{\sqrt{x^2(1 + \frac{1}{x^2})}}{x(2 - \frac{1}{x})} = \lim_{x \to -\infty} \frac{-\sqrt{1 + \frac{1}{x^2}}}{2 - \frac{1}{x}} = \frac{-\sqrt{1 + 0}}{2 - 0} = -\frac{1}{2}.
\]

So \( y = \frac{1}{2} \) is a horizontal asymptote. Now we find the limit at \( -\infty \).

Here we used the fact that \( \sqrt{x^2} = -x \) (this is because as \( x \) goes to \( -\infty \), \( x \) is negative, and therefore \( \sqrt{x^2} = -x \); see the note above). So \( y = -\frac{1}{2} \) is another horizontal asymptote.

### 2.10 Solution to PB10

Section 3.4: Limits at infinity; Horizontal Asymptotes (continued)

1. \( \lim_{x \to \infty} -2x^{10} \)

**Solution.** First we recall the following basic limits.
2.10. SOLUTION TO PB10

- If \( r > 0 \) is a positive real number, then \( \lim_{x \to \infty} x^r = \infty \).

- Let \( n \) be a positive integer. Then

\[
\lim_{x \to -\infty} x^n = \begin{cases} 
\infty & \text{if } n \text{ is even} \\
-\infty & \text{if } n \text{ is odd}
\end{cases}
\]

Also recall the following properties. If \( k > 0 \), then \( k \times \infty = \infty \), and \( k \times (-\infty) = -\infty \). If \( k < 0 \), then \( k \times \infty = -\infty \), and \( k \times (-\infty) = \infty \).

We come back to the question. We have

\[
\lim_{x \to \infty} -\frac{2}{x^{10}} = -2 \lim_{x \to \infty} x^{10} = -2(\infty) = -\infty.
\]

2. \( \lim_{x \to -\infty} -5x^{10} \)

**Solution.** We have

\[
\lim_{x \to -\infty} -5x^{10} = -5 \lim_{x \to -\infty} x^{10} = -5(\infty) = -\infty.
\]

3. \( \lim_{x \to -\infty} 5x^9 \)

**Solution.** We have

\[
\lim_{x \to -\infty} 5x^9 = 5 \lim_{x \to -\infty} x^9 = 5(-\infty) = -\infty.
\]

4. \( \lim_{x \to \infty} 3x^2 \)

**Solution.** We have \( \lim_{x \to \infty} 3x^2 = \infty \).

5. \( \lim_{x \to \infty} -4x^3 + x - 1 \)

**Solution.** Limits of polynomials will follow the behavior of the term with respect to the highest power. So \( \lim_{x \to \infty} -4x^3 + x - 1 = \lim_{x \to \infty} (-4x^3) = -\infty \).

6. \( \lim_{x \to -\infty} 1 - 8x^3 + 2x^2 \)

**Solution.** We have \( \lim_{x \to -\infty} 1 - 8x^3 + 2x^2 = \lim_{x \to -\infty} (-8x^3) = -8(-\infty) = \infty \).

7. \( \lim_{x \to \infty} 2x - x^3 + 7x^5 - 1 \)

**Solution.** We have \( \lim_{x \to \infty} 2x - x^3 + 7x^5 - 1 = \lim_{x \to \infty} (7x^5) = \infty \).

8. \( \lim_{x \to -\infty} \frac{3x^2 - x^3}{1 + x^2} \)

**Solution.** We have

\[
\lim_{x \to -\infty} \frac{3x^2 - x^3}{1 + x^2} = \lim_{x \to -\infty} \frac{3x^2 - x^3}{x^2} = \lim_{x \to -\infty} \frac{3 - x}{\frac{1}{x^2} + 1} = \frac{3 - (-\infty)}{0 + 1} = \frac{3 + \infty}{1} = 3 + \infty = \infty.
\]

Note: If \( k \) is a real number, then \( \infty + k = \infty \), and \( -\infty + k = -\infty \).

9. \( \lim_{x \to \infty} \frac{1 - x + 2x^3}{2 - 4x^2 - 6x} \)

**Solution.** We have

\[
\lim_{x \to \infty} \frac{1 - x + 2x^3}{2 - 4x^2 - 6x} = \lim_{x \to \infty} \frac{\frac{1}{x^3} - \frac{1}{x} + \frac{2}{x^2}}{\frac{2}{x^3} - \frac{4}{x^2} - \frac{6}{x}} = \lim_{x \to \infty} \frac{\frac{1}{x^3} - \frac{1}{x} + \frac{2}{x^2}}{\frac{2}{x^3} - 4 - \frac{6}{x}} = \frac{0 - 0 + \infty}{-4} = \frac{\infty}{-4} = -\infty.
\]

10. \( \lim_{x \to \infty} \sqrt{x} \)

**Solution.** We have \( \lim_{x \to \infty} \sqrt{x} = \lim_{x \to \infty} x^{\frac{1}{2}} = \infty \).
11. \( \lim_{x \to \infty} \sqrt{x^2 - x + 1} \)

**Solution.** Since \( \lim_{x \to \infty} (x^2 - x + 1) = \lim_{x \to \infty} (x^2) = \infty \), we have \( \lim_{x \to \infty} \sqrt{x^2 - x + 1} = \infty \).

12. \( \lim_{x \to \infty} \frac{1}{x} \sin x \)

**Solution.** For \( x \neq 0 \), we have \(-1 \leq \sin x \leq 1\), which implies \(-\frac{1}{x} \leq \frac{1}{x} \sin x \leq \frac{1}{x}\). Since \( \lim_{x \to \infty} \left(-\frac{1}{x}\right) = 0 \) and \( \lim_{x \to \infty} \frac{1}{x} = 0 \), it follows by the Squeeze Theorem that \( \lim_{x \to \infty} \frac{1}{x} \sin x = 0 \).

**Section 3.5: Summary of curve sketching**

1. Sketch the graph of the following functions.

   (a) \( f(x) = -x^3 + 3x^2 - 1 \)

   **Solution.** Guidelines for sketching.

   i. **Domain.** The domain, \( D \), is the set of values of \( x \) for which \( f(x) \) is defined. Here \( f \) is defined everywhere since it is a polynomial. So \( D = \mathbb{R} = (-\infty, \infty) \).

   ii. **Intercepts.** We have two types of intercepts. • The \( y \)-intercept is the the point where the graph intersects the \( y \)-axis. So the \( y \)-intercept is \( f(0) = 0^3 + 3(0)^2 = 0 \). • The \( x \)-intercepts are the points where the graph intersects the \( x \)-axis. So to find the \( x \)-intercepts, we set \( f(x) = 0 \) and solve for \( x \). (You can omit this step if the equation is difficult to solve.) Set \(-x^3 + 3x^2 - 1 = 0\). This equation is not easy to solve, so we will skip this step.

   iii. **Asymptotes.** • Horizontal asymptotes. If \( \lim_{x \to \infty} f(x) = L \) or \( \lim_{x \to -\infty} f(x) = L \), then the line \( y = L \) is a horizontal asymptote. Here we have \( \lim_{x \to \infty} f(x) = \lim_{x \to \infty} (x^3 + 3x^2 - 1) = \lim_{x \to \infty} (-x^3) = -\infty \) and \( \lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} (x^3) = \infty \). So there is no horizontal asymptote.

   • Vertical asymptote. If there exists a number \( a \) such that

     \[
     \lim_{x \to a^+} f(x) = \infty, \text{ or } \lim_{x \to a^-} f(x) = -\infty, \text{ or } \lim_{x \to a} f(x) = \infty, \text{ or } \lim_{x \to a} f(x) = -\infty
     \]

   then the line \( x = a \) is a vertical asymptote. For rational functions, you can locate the vertical asymptote by equating the denominator to 0 after canceling any common factor. For the function \( f(x) = -x^3 + 3x^2 - 1 \), there is no vertical asymptote. In fact, when dealing with polynomials, there is no horizontal asymptote, no vertical asymptote.

   iv. **Intervals of increase or decrease.** Compute \( f'(x) \), find critical numbers, and the sign of \( f'(x) \). For \( f(x) = -x^3 + 3x^2 - 1 \), we have \( f'(x) = -3x^2 + 6x \). The equation \(-3x^2 + 6x = 0\) gives \( 3x(-x + 2) = 0 \), that is, \( 3x = 0 \) or \(-x + 2 = 0\). The former equation gives \( x = 0 \) and the latter gives \(-x = -2\) or \( x = 2 \). So we have two critical numbers: 0 and 2. For the sign of \( f'(x) \), see the table from Figure 2.14. (To learn how to fill out that table, we refer the reader to the Solution to Practice Problems 9 - Section 3.3.)

   v. **Local maximum and minimum values.** Local max and min occur at the critical numbers. If the sign of the first derivative \( f'(x) \) changes from + to − at some point \( c \), then there is a local max at \( c \). If the sign of \( f'(x) \) changes from − to + at some point \( c \), then there is a local min at \( c \). If the sign of \( f'(x) \) is the same at the left and right of \( c \), then there is no local max, no local min. From Figure 2.14, we can see that the sign of \( f'(x) \) changes from − to + at the critical number 0. So there is a local min at 0, and the local min value at that point is \( f(0) = -1 \). Again from the same table, we can see that the sign of \( f'(x) \) changes from + to − at the critical number 2. So there is a local max at 2, and the local max value is \( f(2) = -8 + 12 - 1 = 3 \).
vi. Concavity and inflection points. Compute the second derivative $f''(x)$, find points that make $f''(x) = 0$, and determine the sign of $f''(x)$. For $f(x) = -x^3 + 3x^2 - 1$, we have $f'(x) = -3x^2 + 6x$ and $f''(x) = -6x + 6$. Setting $-6x + 6 = 0$, we get $-6x = -6$, that is, $x = \frac{6}{6} = 1$. The sign of $f''(x)$, and the concavity of $f$ are provided by Figure 2.14. Since the sign of $f''(x)$ changes at 1 and since 1 lies in the domain, we have an inflection point at 1. The value of $f$ at 1 is $f(1) = -1 + 3 - 1 = 1$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-\infty$</th>
<th>0</th>
<th>2</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sign of $f'(x)$</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>Increasing/Decreasing</td>
<td>$\infty$</td>
<td>$3$</td>
<td>$-\infty$</td>
<td></td>
</tr>
<tr>
<td>$x$</td>
<td>$-\infty$</td>
<td>1</td>
<td>$\infty$</td>
<td></td>
</tr>
<tr>
<td>sign of $f''(x)$</td>
<td>+</td>
<td>-</td>
<td></td>
<td></td>
</tr>
<tr>
<td>concavity</td>
<td>concave up</td>
<td></td>
<td>concave down</td>
<td></td>
</tr>
</tbody>
</table>

Figure 2.14:

vii. Sketch the graph. See Figure 2.15.

![Graph of $f(x) = -x^3 + 3x^2 - 1$](image)

Figure 2.15: Graph of $f(x) = -x^3 + 3x^2 - 1$

(b) $f(x) = x^4 - 4x^3 + 21$

**Solution.** The domain is $\mathbb{R} = (-\infty, \infty)$ since $f$ is a polynomial. The $y$-intercept is $f(0) = 21$. For the $x$-intercepts, we need to solve the equation $x^4 - 4x^3 + 21 = 0$ for $x$. Since this is difficult to solve, we omit this step. Since $f$ is a polynomial, there is no horizontal asymptote, no vertical asymptote. Let’s find the limits at infinity anyway (this is still useful information for sketching the graph). We have $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} (x^4) = \infty$ and $\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} (x^4) = \infty$. For the intervals
of increase and decrease, we need to find the critical numbers. We have \( f'(x) = 4x^3 - 12x^2 \). Setting \( f'(x) = 0 \), we get \( 4x^2(x - 3) = 0 \), that is, \( 4x^2 = 0 \) or \( x - 3 = 0 \). The former equation gives \( x = 0 \), and the latter gives \( x = 3 \). So there are two critical numbers: 0 and 3. Value of \( f \) at 0: \( f(0) = 21 \). Value of \( f \) at 3: \( f(3) = 81 - 108 + 21 = -6 \). There is no local min, no local max at 0 since the sign of \( f'(x) \) is the same at the left and right of 0 (see Figure 2.16). But \( f \) has a local min at 3 since the sign of \( f'(x) \) changes from − to + at that point. For the concavity, we need to find the second derivative, which is \( f''(x) = 12x - 24x = 12x(x - 2) \). Setting \( f''(x) = 0 \), we get \( x = 0 \) or \( x = 2 \). Value of \( f \) at 2: \( f(2) = 16 - 32 + 21 = 5 \). All those information are summarized in the table Figure 2.16. From that table we can see that \( f \) has two inflection points: (0, 21) and (2, 5). For the graph of \( f \), see Figure 2.17.

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-\infty)</th>
<th>0</th>
<th>3</th>
<th>( \infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sign of ( f'(x) )</td>
<td>−</td>
<td>−</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Increasing/ Decreasing</td>
<td>( \infty )</td>
<td>21</td>
<td>5</td>
<td>−6</td>
</tr>
<tr>
<td>( x )</td>
<td>(-\infty)</td>
<td>0</td>
<td>2</td>
<td>( \infty )</td>
</tr>
<tr>
<td>sign of ( f''(x) )</td>
<td>+</td>
<td>−</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>concavity</td>
<td>concave up</td>
<td>concave down</td>
<td>concave up</td>
<td>concave up</td>
</tr>
</tbody>
</table>

Figure 2.16:

Figure 2.17: Graph of \( f(x) = x^4 - 4x^3 + 21 \)

2. Consider the following functions.
In each case, determine the following features of $f$: domain, even, odd, neither, intercepts, asymptotes (show the relevant limits), intervals of increase and decrease, local maxima and minima, intervals of upward and downwards concavity, points of inflections. Use all those features to sketch the graph of $f$.

1. $f(x) = \frac{-2x^2}{x^2-1}$, $f'(x) = \frac{4x}{(x^2-1)^2}$, and $f''(x) = \frac{-(12x^2+4)}{(x^2-1)^3}$.

**Solution.**

(a) *Domain.* The denominator is equal to zero when $x^2 - 1 = 0$, that is, $(x + 1)(x - 1) = 0$, that is, $x = -1$ or $x = 1$. So the function is undefined when $x = -1$ or $x = 1$. Thus $D = \mathbb{R}\setminus\{-1, 1\}$ (all numbers except $-1$ and $1$).

(b) • A function $f$ is said to be **even** if $f(-x) = f(x)$ for all $x$ in $D$. That is, the graph of $f$ is symmetric about the $y$-axis. For $f(x) = \frac{-2x^2}{x^2-1}$, we have $f(-x) = \frac{-2(-x)^2}{(-x)^2-1} = \frac{-2x^2}{x^2-1} = f(x)$. So $f$ is even. • A function $f$ is said to be **odd** if $f(-x) = -f(x)$ for all $x$ in $D$. That is, the graph of $f$ is symmetric about the origin. For $f(x) = \frac{-2x^2}{x^2-1}$, we have $f(-x) = f(x) \neq -f(x)$. So $f$ is not odd.

(c) *Intercepts.* The $y$-intercept is $f(0) = \frac{-2(0)^2}{0^2-1} = 0$. For the $x$-intercepts, we need to solve the equation $f(x) = 0$, which is equivalent to $-2x^2 = 0$. That is, $-2x^2 = 0$ or $x = 0$. So the graph intersects the $x$-axis at $x = 0$.

(d) *Asymptotes.* Let us begin with horizontal asymptotes (H.A). We have

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{-2x^2}{x^2-1} = \lim_{x \to \pm\infty} \frac{-2x^2}{x^2} \frac{1}{1-\frac{1}{x^2}} = \lim_{x \to \pm\infty} \frac{-2}{1} = -2.$$

Similarly, we have $\lim_{x \to -\infty} f(x) = -2$. So the line $y = -2$ is a horizontal asymptote. Now we find vertical asymptotes (V.A). An equation $x = a$ is a vertical asymptote if the factor $(x - a)$ appears in the denominator and it is not possible to cross it out. Here $f(x) = \frac{-2x^2}{x^2-1}$. We can see that the factor $(x + 1)$ appears in the denominator, and it is not possible to cross it out, so the line $x = -1$ is a vertical asymptote. Likewise, the line $x = 1$ is another vertical asymptote. To be more precise, we have to find the one-sided limits as $x \to -1$ and $x \to 1$. If we plug-in $x = -1$ or $1$ in $f(x)$, we get $\frac{-2}{0}$, which is a problem of the form $\frac{\pm}{0}$. So we need the sign analysis test (if you need to refresh about the **sign analysis test**, see the solution to Practice Problems 3 – Section 1.6 – Questions 1, 3, and 5), which is given by Figure 2.18. From Figure 2.18, we have that

![Figure 2.18: Sign analysis test for $f(x) = \frac{-2x^2}{x^2-1}$](image)

$$\lim_{x \to -1^+} f(x) = -\infty, \quad \lim_{x \to -1^-} f(x) = \infty, \quad \lim_{x \to 1^+} f(x) = \infty, \quad \lim_{x \to 1^-} f(x) = -\infty.$$  

This proves that $x = -1$ and $x = 1$ are vertical asymptotes.

(e) *Intervals of increase and decrease.* The first derivative is given: $f'(x) = \frac{4x}{(x^2-1)^2}$. Setting $f'(x) = 0$, we get $4x = 0$, that is, $x = 0$. So 0 is a critical number of $f$. For the sign of $f'(x)$, and the intervals of increase and decrease, see Figure 2.19.
(f) **Local max and local min.** From Figure 2.19, one can see that $f$ has a local min at 0, and the local min value is $f(0) = 0$. There is no local max.

(g) **Concavity Inflection points.** For the concavity we need the second derivative, which is also given: $f''(x) = \frac{-12x^2 + 4}{(x^2 - 1)^3}$. Setting $f''(x) = 0$, we get $12x^2 + 4 = 0$, that is, $x^2 = -\frac{4}{12}$. This is impossible since $x^2$ is always greater than or equal to zero and $-\frac{4}{12}$ is negative. So there is no number that makes the second derivative equal to zero. For the sign of $f''(x)$, see Figure 2.19. Though the sign of $f''(x)$ changes at $-1$ and $1$, there is no inflection point. This is due to the fact that $-1$ and $1$ do not lie in the domain.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-\infty$</th>
<th>$-1$</th>
<th>$0$</th>
<th>$1$</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Sign of $f'(x)$</strong></td>
<td>$-$</td>
<td>$-$</td>
<td>$+$</td>
<td>$+$</td>
<td>$+$</td>
</tr>
<tr>
<td><strong>Increasing/Decreasing</strong></td>
<td>$-\infty$</td>
<td>$-\infty$</td>
<td>$0$</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
<tr>
<td><strong>Sign of $f''(x)$</strong></td>
<td>$-$</td>
<td>$-$</td>
<td>$+$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td><strong>Concavity</strong></td>
<td>concave down</td>
<td>concave up</td>
<td>concave down</td>
<td>concave up</td>
<td>concave down</td>
</tr>
</tbody>
</table>

Figure 2.19:

(h) **Graph of $f$.** See Figure 2.20.

Figure 2.20: Graph of $f(x) = \frac{-2x^2}{x^2 - 4}$

2. $f(x) = \frac{x^2 + 3x - 4}{x^2 - 4}$, $f'(x) = \frac{-3(x^2 + 4)}{(x^2 - 4)^2}$, and $f''(x) = \frac{6x(x^2 + 12)}{(x^2 - 4)^3}$.

**Solution.**
2.10. SOLUTION TO PB10

(a) Domain. \( D = \mathbb{R} \setminus \{-2, 2\} \).

(b) Even, odd, neither. For all \( x \) in \( D \), we have \( f(-x) = \frac{(-x)^2 + 3(-x) - 4}{(-x)^2 - 4} = \frac{x^2 - 3x - 4}{x^2 - 4} \). Since \( f(-x) \neq \frac{1}{f(x)} \), and since \( f(-x) \neq -f(x) \), we have that \( f \) is neither even nor odd.

(c) Intercepts. The \( y \)-intercept is \( f(0) = \frac{9 + 0 - 4}{0 - 4} = \frac{5}{-4} = 1 \). For the \( x \)-intercepts, we need to solve the equation \( f(x) = 0 \), which is equivalent to \( x^2 + 3x - 4 = 0 \), that is, \( (x + 4)(x - 1) = 0 \), that is, \( x = -4 \) or \( x = 1 \).

(d) Asymptotes. Let us begin with the horizontal asymptote. We have

\[
\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} \frac{x^2 + 3x - 4}{x^2 - 4} = \lim_{x \to \infty} \frac{x^2 + 3x - 4}{x^2 - 4} = \lim_{x \to -\infty} \frac{1 + \frac{3}{x} - \frac{4}{x^2}}{1 - \frac{4}{x^2}} = \frac{1 + 0 - 0}{1 - 0} = 1.
\]

Similarly, we have \( \lim_{x \to -\infty} f(x) = -\infty \). So the line \( y = 1 \) is a horizontal asymptote. For vertical asymptotes, we need to make the sign analysis test, which is given by Figure 2.21. From that figure, we have the following limits.

\[
\lim_{x \to -2^+} f(x) = -\infty, \quad \lim_{x \to -2^-} f(x) = \infty, \quad \lim_{x \to 2^+} f(x) = -\infty, \quad \lim_{x \to 2^-} f(x) = \infty.
\]

This shows that the lines \( x = -2 \) and \( x = 2 \) are both vertical asymptotes.

![Figure 2.21: Sign analysis test for \( f(x) = \frac{x^2 + 3x - 4}{x^2 - 4} \)](image)

(e) Intervals of increase or decrease. Setting \( f'(x) = 0 \), we get \(-3(x^2 + 4) = 0\), that is, \( x^2 + 4 = 0 \) or \( x^2 = -4 \), which is impossible. So there is no number that makes \( f'(x) \) equal to zero. Looking at \( f'(x) \), we can see that it is not defined when \( x = -2 \) or \( x = 2 \). But, since \(-2 \) and \( 2 \) do not lie in the domain, there is no critical number. It is easy to get the sign of \( f'(x) \). Since \( x^2 + 4 \) is always positive, it follows that \(-3(x^2 + 4)\) is always negative. Therefore, since the bottom \((x^2 - 4)^2\) is always positive, we have that \( f'(x) \) is always negative. So \( f \) is decreasing in the intervals \((-\infty, -2), (-2, 2)\), and \((2, \infty)\) (see Figure 2.22).

(f) Local max and local min. We know that local max and min occur only at critical numbers. Since there is no critical number, there is no local max and min.

(g) Concavity and inflection points. Setting \( f''(x) = 0 \), we get \( 6x(x^2 + 12) = 0 \), that is, \( x = 0 \) or \( x^2 = -12 \). The latter equation is impossible. So the second derivative vanishes when \( x = 0 \). The sign of \( f''(x) \) is given by Figure 2.22. This sign shows that the concavity changes at \(-2, 0\), and \(2\). Since \(-2 \) and \( 2 \) do not lie in the domain, we have only one inflection point: \((0, 1)\).

(h) Graph of \( f \). See Figure 2.23.

3. \( f(x) = \frac{x}{x^2 + 1}, \quad f'(x) = \frac{1 - x^2}{(1 + x^2)^2}, \quad \text{and} \quad f''(x) = \frac{2x(x^2 - 3)}{(x^2 + 1)^3} \).

**Solution.** For the domain, let’s try to find numbers that make the denominator equal to zero. Setting \( x^2 + 1 = 0 \), we get \( x^2 = -1 \), which is impossible. So there is no number that makes the bottom equal to
0. This means that $f$ is defined everywhere, so that $D = \mathbb{R}$. Since $f(-x) = \frac{-x}{(-x)^2+1} = -\frac{x}{x^2+1} = -f(x)$, it follows that $f$ is odd. Clearly $f$ is not even. For the asymptotes, we have that the line $y = 0$ is a horizontal asymptote since $\lim_{x \to \infty} f(x) = 0$ and $\lim_{x \to -\infty} f(x) = 0$. There is no vertical asymptote since $f$ is defined everywhere. For the intervals of increase, decrease, for the local max and local min values, and for the concavity and inflection points, see the solution to Practice Problems 9 – Section 3.3 – Question 4. All those information are summarized in Figure 2.24. The graph of $f$ is given by Figure 2.25.

### 2.11 Solution to PB11

**Section 3.7: Optimization Problems**
1. A farmer has 2000 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

Solution. We are going to solve this step by step.

Step 1 Draw a picture (see Figure 2.26).

Step 2 Introduce notation. Assign a symbol to the quantity that is to be maximized or minimized. Also select symbols for other unknown quantities. Here we want to maximize the area that we denote $A$. The other unknown are $x$ and $y$ as shown Figure 2.26.

Step 3 Express the quantity you want to maximize (or minimize) in terms of the other variables from step 2. Here we have $A = xy$, which is the formula for the area of a rectangle of sides $x$ and $y$. 
Step 4 If there is more than one variable, in step 3, use the given information to find relationships among these variables (called constraint), and try to write \( A \) as a function of one variable. Here the constraint is that the total length of the fencing is 2000 ft, that is, \( 2x + y = 2000 \). (Note that from the problem no fence is needed along the river. This is why the variable \( y \) appears only once in the constraint.) From the constraint we have \( y = 2000 - 2x \). Substituting this into the expression for \( A \), we get \( A = x(2000 - 2x) = 2000x - 2x^2 \).

Step 5 Find the interval and use the methods of Section 3.1 (the Closed Interval Method) or Section 3.3 (The First Derivative Test or the Second Derivative Test) to find absolute max (or absolute min). To get the interval, the following two questions might be helpful. Question 1: What is the least value you could make \( x \)? Answer: 0 (this is because \( x \) can’t be negative). Question 2: What is the maximum value you could make \( x \)? Answer: 1000 (this uses all the fence for the depth and none for the width). So the interval is \([0, 1000]\). Now the problem is to find the absolute max value of the function \( A(x) = 2000x - 2x^2 \) on the interval \([0, 1000]\). Since the interval is closed, we will use the Closed Interval Method (this was introduced in the Solution to Practice Problems 8 – Section 3.1 – Question 2).

- Critical numbers: We have \( A'(x) = 2000 - 4x \). Setting this to 0, we get \(-4x = -2000\), that is, \( x = 500 \).
- Values of \( A \) at the critical numbers: \( A(500) = 500(2000 - 1000) = 500000 \).
- Values of \( A \) at the endpoints: \( A(0) = 0 \) and \( A(1000) = 1000(2000 - 2000) = 0 \).
- The absolute max is then 500000 located at \( x = 500 \). The corresponding value for \( y \) is \( y = 2000 - 2(500) = 1000 \).

Conclusion: The maximum area is 500000 ft\(^2\) when \( x = 500 \) ft and \( y = 1000 \) ft.

2. A farmer wants to fence in a rectangular field beside a river. No fencing is required along the river and the farmer’s neighbour will pay half of the cost of one of the sides perpendicular to the river. If fencing costs 20 per linear meter and the field must have an area of 600 m\(^2\), what are the dimensions of the field that will minimize the cost to the farmer?

Solution. Consider the same picture as before (see Figure 2.26). Let \( C \) denote the cost, which is the quantity we want to minimize.

- Equation. We have \( C = 20x + 20y + \frac{20y}{x} = 20x + 20y + 10x = 30x + 20y \). This equation has two variables, so we have to eliminate one of them by using the constraint, which states the area must be equal to 600, that is, \( xy = 600 \) or \( y = \frac{600}{x} \). Substituting this into the formula for the cost, we get

\[
C = 30x + 20 \times \frac{600}{x} = 30 \left( x + 20 \times \frac{20}{x} \right) = 30 \left( x + \frac{400}{x} \right).
\]

So the function we want to minimize is \( C(x) = 30(x + \frac{400}{x}) \).

- Interval. For the interval, we can’t use the argument of Question 1 as there is no constraint here about the total length of the fencing. Since \( x \) can’t be 0 here (otherwise the area would be 0, which would contradict the fact that the area is 600), and since there is no maximum value for \( x \), the physical domain for \( x \) is the open interval \((0, \infty)\).

- Critical numbers. The first derivative is \( C'(x) = 30(1 - \frac{400}{x^2}) = 30 \left( \frac{x^2 - 400}{x^2} \right) \). Setting \( C'(x) = 0 \), we get \( x^2 - 400 = 0 \), that is, \( x^2 = 400 \). This latter equation gives \( x = -20 \) or \( x = 20 \). Rejecting a negative length leaves \( x = 20 \). So the only critical number is \( x = 20 \).

- Absolute min. To find the absolute min, we can’t use the Closed Interval Method here since the interval is open. We are going to use the Second Derivative Test, which states that if \( f'(c) = 0 \) and \( f''(c) > 0 \), then \( f \) has a local min at \( c \). (The second part states that if \( f'(c) = 0 \) and \( f''(c) < 0 \),
2.11. SOLUTION TO PB11

then $f$ has a local max at $c$.) The second derivative is $C''(x) = 30 \left( \frac{800x}{x^4} \right)$. Since $C'(20) = 0$ and $C''(20) > 0$, the function $C$ has a local min at $20$. That local min is actually an absolute local min since $C''(x) > 0$ on the interval $(0, \infty)$ (which means that $C$ is concave up on $(0, \infty)$). So the absolute min cost occurs when $x = 20$. The value of $y$ corresponding to $x = 20$ is $y = \frac{600}{20} = 30$.

Hence the dimensions of the field that will minimize the cost to the farmer are $x = 20$ m and $y = 30$ m.

3. A piece of wire 1 m long is cut into two pieces. One piece is bent in a square and the other is bent into an equilateral triangle. How much wire should be used for the square in order to minimize the sum of the areas of the square and triangle? [Recall that the area of an equilateral triangle with side $s$ is $A = \frac{\sqrt{3}}{4} s^2$.]

**Solution.** Let $x$ be the side of the square and let $y$ be the side of the triangle. The quantity we want to minimize is the sum of the areas of the square and triangle, which we denote $A$. The formula for $A$ is $A = x^2 + \frac{\sqrt{3}}{4} \left( 1 - 4x \right)^2$. The constraint is that the sum of the perimeters of the square and triangle must be equal to 1. That is, $4x + 3y = 1$ or $y = \frac{1}{3}(1 - 4x)$. Substituting this into the expression for $A$, we get

$$A(x) = x^2 + \frac{\sqrt{3}}{4} \left( 1 - 4x \right)^2 = x^2 + \frac{\sqrt{3}}{9} \left( 1 - 4x \right)^2.$$

Since the least $x$ is 0, and the largest $x$ is $\frac{1}{4}$, the interval for $x$ is $[0, \frac{1}{4}]$. Now we find the critical numbers. The first derivative is

$$A'(x) = 2x + \frac{\sqrt{3}}{36} (2)(1 - 4x)(-4) = 2x + \frac{\sqrt{3}}{36} (-8)(1 - 4x) = 2x - \frac{2\sqrt{3}}{9} (1 - 4x) = 2x - \frac{2\sqrt{3}}{9} + \frac{8\sqrt{3}}{9} x = \left( 2 + \frac{8\sqrt{3}}{9} \right) x - \frac{2\sqrt{3}}{9} = \left( \frac{18 + 8\sqrt{3}}{9} \right) x - \frac{2\sqrt{3}}{9}.$$

Setting $A'(x) = 0$, we get $\left( \frac{18 + 8\sqrt{3}}{9} \right) x = \frac{2\sqrt{3}}{9}$, that is, $(18 + 8\sqrt{3})x = 2\sqrt{3}$, or $x = \frac{2\sqrt{3}}{18 + 8\sqrt{3}} = \frac{\sqrt{3}}{9 + 4\sqrt{3}}$.

Using the Closed Interval Method, one can verify that $A$ has an absolute minimum value at $x = \frac{\sqrt{3}}{9 + 4\sqrt{3}}$.

4. A rectangular picture frame (see Figure 2.27) encloses an area of 600 cm$^2$. The top edge of the frame is constructed out of heavier material than the other three sides. If the material for the top edge weighs 200 gram/cm and the other three sides are made from material weighing 100 gram/cm, find the dimensions of the frame that would minimize the total weight of the material used.

**Figure 2.27:**

**Solution.** This question is similar to Question 2. Let $W$ denote the total weight of the material, which is the quantity we want to minimize. From the problem, we have $W = 100x + 100y + 200x + 100y =$
CHAPTER 2. SOLUTION TO PRACTICE PROBLEMS

5. The volume $V$ of a cylinder of height $h$ and radius $r$ is $V = \pi r^2 h$, whereas the area of the cylinder's surface, including top and bottom, is $A = 2\pi r^2 + 2\pi rh$. Of all cylinders of volume $V = 1$, determine the height and radius of the cylinder that has minimal surface area.

**Solution.** The quantity we want to minimize is the surface area $A = 2\pi r^2 + 2\pi rh$, and the constraint states that the volume must be equal to 1, that is, $\pi r^2 h = 1$, or $h = \frac{1}{\pi r^2}$. Substituting this into the expression for $A$, we get

$$A = 2\pi r^2 + 2\pi r \left(\frac{1}{\pi r^2}\right) = 2\pi r^2 + \frac{2}{r}.$$ 

Since the radius $r$ can’t be negative, and since $r \neq 0$ (otherwise, the volume would be 0, and this would contradict the fact that the volume is 1), we have $r > 0$. So the interval for $r$ is $(0, \infty)$. Now, we find the critical numbers. The first derivative is

$$A'(r) = 4\pi r - \frac{2}{r^2} = \frac{4\pi r^3 - 2}{r^2}.$$ 

Setting this to 0, we get $4\pi r^3 - 2 = 0$, that is, $r^3 = \frac{1}{2\pi}$. Taking the cube root, we have $r = \sqrt[3]{\frac{1}{2\pi}}$, which is the only critical number. To see that $A$ has an absolute min at that number, we can use the Second Derivative Test by arguing in the same way as we did in Question 2. Namely, one can easily see that the second derivative, $A''(r) = 4\pi + \frac{6\pi}{r^2}$ is positive on $(0, \infty)$. The value of $h$ corresponding to $r = \sqrt[3]{\frac{1}{2\pi}}$ is

$$h = \frac{1}{\pi r^2} = \frac{1}{\pi \left(\sqrt[3]{\frac{1}{2\pi}}\right)^2}.$$ 

Thus to minimize the surface area, the radius should be $r = \sqrt[3]{\frac{1}{2\pi}}$ and the height should be $h = \frac{1}{\pi (\sqrt[3]{\frac{1}{2\pi}})^2}$.

6. The entrance to a tent is in the shape of an isosceles triangle as shown Figure 2.28. Zipper run vertically along the middle of the triangle and horizontally along the bottom of it. If the designers of the tent want to have a total zipper length of 5 metres, find the lengths $x$ and $y$ (see diagram) that will maximize the area of the entrance. Also find this maximum area. (Note the area of a triangle equals $(\frac{1}{2})(\text{base})(\text{height}).$)

**Solution.** Here we want to maximize the area of the entrance, which is given by the formula $A = \frac{1}{2} (2x) y = xy$. From the problem, the constraint is $x + x + y = 5$, that is, $2x + y = 5$. Solving this for $y$, we get $y = 5 - 2x$. Substituting into $A$, we get $A = x(5 - 2x) = 5x - 2x^2$. Since the least $x$ is 0 and since the largest $x$ is obtained when $2x = 5$, that is, $x = \frac{5}{2}$, we have the interval $[0, \frac{5}{2}]$. Now we find the critical numbers. The first derivative is $A'(x) = 5 - 4x$. This is equal to zero when $4x = 5$, that is, $x = \frac{5}{4}$. So the only critical number is $x = \frac{5}{4}$. To get the absolute max value, we use the Closed Interval Method (since the interval is closed).
2.12 Solution to PB12

Indefinite Integrals

Integration table. Evaluate the following indefinite integrals.

1. $\int 5 dx$

Solution. Let’s first recall the Table of Indefinite integrals.
CHAPTER 2. SOLUTION TO PRACTICE PROBLEMS

\[
\begin{align*}
\int k \, dx &= kx + C \\
\int x^r \, dx &= \frac{1}{r+1} x^{r+1} + C \quad (r \neq -1) \\
\int \cos x \, dx &= \sin x + C \\
\int \sin x \, dx &= -\cos x + C \\
\int \sec^2 x \, dx &= \tan x + C \\
\int \csc^2 x \, dx &= -\cot x + C \\
\int \sec x \, \tan x \, dx &= \sec x + C \\
\int \csc x \, \cot x \, dx &= -\csc x + C
\end{align*}
\]

The integral \( \int 5 \, dx \) fits to table. In fact it is the top left formula with \( k = 5 \). So \( \int 5 \, dx = 5x + C \).

2. \( \int x^8 \, dx \)

**Solution.** The integral \( \int x^8 \, dx \) also fits to table. (This is the top middle formula with \( r = 8 \).) So \( \int x^8 \, dx = \frac{1}{8+1} x^{8+1} + C = \frac{1}{9} x^9 + C \).

3. \( \int \frac{1}{x^3} \, dx \)

**Solution.** This integral does not fit to table, but by using very basic algebra, we can make it easier. Recall the formula \( \frac{1}{x^m} = x^{-m} \).

Using that formula, we get \( \int \frac{1}{x^3} \, dx = \int x^{-3} \, dx \). This latter integral fits to table (in fact, it corresponds to \( \int x^r \, dx \) with \( r = -3 \)). Therefore

\[
\int \frac{1}{x^3} \, dx = \int x^{-3} \, dx = \frac{1}{-3+1} x^{-3+1} + C = \frac{1}{-2} x^{-2} + C = -\frac{1}{2x^2} + C.
\]

**Note.** In order to do integral, it must fit into the integration table. So the idea is to try to manipulate and make it fit somehow.

4. \( \int \sqrt{x} \, dx \)

**Solution.** First recall the formula \( \sqrt{x} = x^{\frac{1}{2}} \).

Using that formula, we get \( \int \sqrt{x} \, dx = \int x^{\frac{1}{2}} \, dx \), which fits to table (here \( r = \frac{1}{2} \)). So

\[
\int \sqrt{x} \, dx = \int x^{\frac{1}{2}} \, dx = \frac{1}{\frac{1}{2}+1} x^{\frac{1}{2}+1} + C = \frac{2}{3} x^{\frac{3}{2}} + C = \frac{3}{4} x^{\frac{3}{2}} + C.
\]

5. \( \int \frac{1}{\sqrt{x}} \, dx \)

**Solution.** We have

\[
\int \frac{1}{\sqrt{x}} \, dx = \int \frac{1}{x^{\frac{1}{2}}} \, dx = \int x^{-\frac{1}{2}} \, dx = \frac{1}{-\frac{1}{2}+1} x^{-\frac{1}{2}+1} + C = \frac{2}{1} x^{\frac{1}{2}} + C = \frac{6}{5} x^{\frac{1}{2}} + C.
\]

6. \( \int \frac{x}{\sqrt{x}} \, dx \)

**Solution.** First recall the formula \( \frac{x^n}{x^m} = x^{n-m} \).

We have

\[
\int \frac{x}{\sqrt{x}} \, dx = \int \frac{x}{x^{\frac{1}{2}}} \, dx = \int x^{1-\frac{1}{2}} \, dx = \int x^{\frac{1}{2}} \, dx = \frac{3}{2} x^{\frac{3}{2}} + C = \frac{3}{5} x^{\frac{3}{2}} + C.
\]

7. \( \int \sqrt{x}^2 \, dx \)

**Solution.** First recall the formula \( \sqrt{x} = x^{\frac{1}{2}} \).

Now we have

\[
\int \sqrt{x}^2 \, dx = \int x^{\frac{2}{2}} \, dx = \frac{1}{\frac{3}{2}} x^{\frac{3}{2}} + C = \frac{4}{9} x^{\frac{3}{2}} + C.
\]
8. \( \int \frac{\sqrt{x^2}}{x^4} \, dx \)

**Solution.** We have
\[
\int \frac{\sqrt{x^2}}{x^4} \, dx = \int \frac{x^2}{x^4} \, dx = \int x^{-4} \, dx = \int x^{-\frac{10}{3}} \, dx = -\frac{10}{3} + 1 \times x^{-\frac{7}{3}} + C = -\frac{3}{3} \times \frac{1}{x^\frac{7}{3}} + C.
\]

Integration Table + Properties. Evaluate the following indefinite integrals.

9. \( \int 4x \, dx \)

**Solution.** First we recall the basic properties of integrals.

(a) \( \int kf(x) \, dx = k \int f(x) \, dx \). In words, this property says that if we have the integral of a constant times a function, we can pull the constant outside.

(b) \( \int [f(x) + g(x)] \, dx = \int f(x) \, dx + \int g(x) \, dx \). The integral of a sum is the sum of integrals.

(c) \( \int [f(x) - g(x)] \, dx = \int f(x) \, dx - \int g(x) \, dx \). The integral of a difference is the difference of integrals.

(d) WARNING! The equation \( \int f(x)g(x) \, dx = \left( \int f(x) \, dx \right) \left( \int g(x) \, dx \right) \) does not work. In other words, the integral of a product in NOT equal to the product of integrals.

Applying the first property, we get
\[
\int 4x \, dx = 4 \int x \, dx = 4 \times \frac{x^2}{2} + C = 2x^2 + C.
\]

10. \( \int \sqrt{3x} \, dx \)

**Solution.** We have
\[
\int \sqrt{3x} \, dx = \sqrt{3} \int \sqrt{x} \, dx = \sqrt{3} \int x^\frac{1}{2} \, dx = \sqrt{3} \left( \frac{x^\frac{3}{2}}{3} \right) + C = \frac{2}{3} \sqrt{3}x^\frac{3}{2} + C.
\]

11. \( \int (-3x^2 + 5x - 1) \, dx \)

**Solution.** We have
\[
\int (-3x^2 + 5x - 1) \, dx = \int -3x^2 \, dx + \int 5x \, dx - \int 1 \, dx = -3 \int x^2 \, dx + 5 \int x \, dx - \int 1 \, dx =
\]
\[
-3 \times \frac{x^3}{3} + 5 \times \frac{x^2}{2} - x + C = -x^3 + \frac{5}{2} x^2 - x + C.
\]

**Note:** When dealing with the sum or difference of integrals, add the constant \( C \) at the end. (Don’t add one constant for each integral.) This is because a constant times a number is still a constant, and the sum of constants is still a constant.

12. \( \int (x^2 + \sqrt{x} - 1) \, dx \)

**Solution.** We have
\[
\int (x^2 + \sqrt{x} - 1) \, dx = \int x^2 \, dx + \int \sqrt{x} \, dx - \int 1 \, dx = \int x^2 \, dx + \int x^\frac{1}{2} \, dx - \int 1 \, dx =
\]
\[
\frac{x^3}{3} + \frac{x^\frac{3}{2}}{\frac{3}{2}} - x + C = \frac{1}{3} x^3 + \frac{2}{3} x^\frac{3}{2} - x + C.
\]
13. \( \int (-2\sqrt{x} + \cos x)dx \)

**Solution.** We have
\[
\int (-2\sqrt{x} + \cos x)dx = -2 \int \sqrt{x}dx + \int \cos xdx = -2 \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \sin x + C =
-2(\frac{2}{3})x^{\frac{3}{2}} + \sin x = -\frac{4}{3}x^{\frac{3}{2}} + \sin x + C.
\]

14. \( \int (3\sec^2 x - 4 \sin x - \csc^2 x)dx \)

**Solution.** We have
\[
\int (3\sec^2 x - 4 \sin x - \csc^2 x)dx = 3 \int \sec^2 xdx - 4 \int \sin xdx - \int \csc^2 xdx =
3 \tan x - 4(-\cos x) - (-\cot x) + C = 3 \tan x + 4 \cos x + \cot x + C.
\]

15. \( \int (7 \sec x \tan x + 11 \csc x \cot x + 1)dx \)

**Solution.** We have
\[
\int (7 \sec x \tan x + 11 \csc x \cot x + 1)dx =
7 \int \sec x \tan x dx + 11 \int \csc x \cot x dx + \int 1 dx =
7 \sec x + 11(- \csc x) + x = 7 \sec x - 11 \csc x + x + C.
\]

16. \( \int \frac{4x - 3}{\sqrt{x}} dx \)

**Solution.** We have
\[
\int \frac{4x - 3}{\sqrt{x}} dx = \int \left( \frac{4x}{\sqrt{x}} - \frac{3}{\sqrt{x}} \right) dx = \int \frac{4x}{\sqrt{x}} dx - \int \frac{3}{\sqrt{x}} dx =
4 \int x^{\frac{1}{2}} dx - 3 \int x^{-\frac{1}{2}} dx = 4 \int x^{\frac{3}{2}} dx - 3 \int x^{-\frac{3}{2}} dx =
4 \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 3 \frac{x^{-\frac{3}{2}}}{-\frac{3}{2}} + C = 4(\frac{2}{3})x^{\frac{3}{2}} - 3(\frac{2}{1})x^{-\frac{3}{2}} + C = \frac{8}{3}x^{\frac{3}{2}} - 6x^{-\frac{3}{2}} + C.
\]

17. \( \int \frac{2 - \sqrt{2t} + \sqrt{7t^2}}{\sqrt{t}} dt \)

**Solution.** We have
\[
\int \frac{2 - \sqrt{2t} + \sqrt{7t^2}}{\sqrt{t}} dt = \int \frac{2}{\sqrt{t}} dt - \int \sqrt{2t} \sqrt{t} dt + \int \sqrt{7t^2} \sqrt{t} dt =
\int \frac{2}{t^{\frac{1}{2}}} dt - \int \sqrt{2} t^{\frac{1}{2}} dt + \int \sqrt{7} t^{\frac{3}{2}} dt = 2t^{\frac{1}{2}} - \sqrt{2} \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + \frac{\sqrt{7} t^{\frac{5}{2}}}{\frac{5}{2}} + C = 4t^{\frac{1}{2}} - \sqrt{2} \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + \frac{\sqrt{7} t^{\frac{5}{2}}}{\frac{5}{2}} + C.
\]

18. \( \int 2x(3 - x^{-3})dx \)

**Solution.** We first need to distribute before taking the integral.
\[
\int 2x(3 - x^{-3})dx = \int (6x - 2x^{-2})dx = 6 \int xdx - 2 \int x^{-2}dx =
6 \frac{x^2}{2} - 2 \frac{x^{-1}}{-1} + C = 3x^2 + 2x^{-1} + C.
\]
2.12. **SOLUTION TO PB12**

19. $\int (t+4)(2t+1)dt$

**Solution.** Again, we first need to distribute.

$$\int (t+4)(2t+1)dt = \int (2t^2 + t + 8t + 4)dt = \int (2t^2 + 9t + 4)dt = \frac{2}{3}t^3 + \frac{9}{2}t^2 + 4t + C.$$

**Integration by Substitution.** Evaluate the following indefinite integrals.

20. $\int (1-2x)^9 dx$

**Solution.** We will do this integral step by step.

Step 1 Pick $u$ so the integral is easier. Usually, $u$ is the “inside” of something. So $u = 1 - 2x$.

Step 2 Express $dx$ in terms of $du$. We have that the derivative of $u$ with respect to $x$ is $\frac{du}{dx} = -2$, so that $dx = \frac{du}{-2}$.

Step 3 Rewrite the original integral only in terms of $u$ and do it.

$$\int (1-2x)^9dx = \int (u)^9 \frac{du}{-2} = -\frac{1}{2} \int u^9 du = -\frac{1}{2} \frac{u^{10}}{10} + C = -\frac{1}{20}u^{10} + C.$$

**Note:** All $x$’s must be eliminated before you integrate.

Step 4 Translate back to $x$.

$$\int (1-2x)^9dx = -\frac{1}{20}(1-2x)^{10} + C.$$

21. $\int 2x(x^2 + 3)^{15}dx$

**Solution.** Let $u = x^2 + 3$. Then $\frac{du}{dx} = 2x$, so $dx = \frac{du}{2x}$. Therefore

$$\int 2x(x^2 + 3)^{15}dx = \int 2xu^{15} \frac{du}{2x} = \int u^{15} du = \frac{u^{16}}{16} + C = \frac{1}{16}(x^2 + 3)^{16} + C.$$

22. $\int x^2(x^3 - 7)^6 dx$

**Solution.** Let $u = x^3 - 7$. Then $\frac{du}{dx} = 3x^2$, so $dx = \frac{du}{3x^2}$. Therefore

$$\int x^2(x^3 - 7)^6 dx = \int x^2 u^6 \frac{du}{3x^2} = \frac{1}{3} \int u^6 du = \frac{1}{3} \frac{u^7}{7} + C = \frac{1}{21}(x^3 - 7)^7 + C.$$

23. $\int \sqrt{1-x^2} dx$

**Solution.** Let $u = 1 - x^2$. Then $\frac{du}{dx} = -2x$, so $dx = \frac{du}{-2x}$. Therefore

$$\int \sqrt{1-x^2} dx = \int \sqrt{u} \frac{du}{-2x} = -\frac{1}{2} \int u^{\frac{1}{2}} du = -\frac{1}{2} \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = -\frac{1}{2} \cdot \frac{2}{3} u^{\frac{3}{2}} + C = -\frac{1}{3} u^{\frac{3}{2}} + C = -\frac{1}{3} (1-x^2)^{\frac{3}{2}} + C.$$

24. $\int \sin(3x) dx$

**Solution.** Let $u = 3x$. Then $\frac{du}{dx} = 3$, so $dx = \frac{du}{3}$. Therefore

$$\int \sin(3x)dx = \int \sin u \frac{du}{3} = \frac{1}{3} \int \sin u du = \frac{1}{3} (-\cos u) + C = -\frac{1}{3} \cos(3x) + C.$$
25. \[ \int -5x^2 \cos(x^3) \, dx \]

**Solution.** Let \( u = x^3 \). Then \( \frac{du}{dx} = 3x^2 \), so \( dx = \frac{du}{3x^2} \). Therefore

\[
\int -5x^2 \cos(x^3) \, dx = \int -5x^2 \cos(u) \frac{du}{3x^2} = \frac{5}{3} \int \cos u \, du = -\frac{5}{3} \sin u + C = -\frac{5}{3} \sin(x^3) + C.
\]

26. \[ \int 7 \tan^2 x \sec^2 x \, dx \]

**Solution.** Let \( u = \tan x \). Then \( \frac{du}{dx} = \sec^2 x \), so \( dx = \frac{du}{\sec^2 x} \). Therefore

\[
\int 7 \tan^2 x \sec^2 x \, dx = \int 7u^2 \sec^2 x \frac{du}{\sec^2 x} = 7 \int u^2 \, du = \frac{7}{3} u^3 + C = \frac{7}{3} (\tan x)^3 + C.
\]

27. \[ \int \frac{\sec^2 x}{(3 + \tan x)^5} \, dx \]

**Solution.** First we can rewrite the original integral as \( \int \frac{\sec^2 x}{(3 + \tan x)^5} \, dx = \int \sec^2 x (3 + \tan x)^{-5} \, dx \). Now we make the substitution \( u = 3 + \tan x \). Then \( \frac{du}{dx} = \sec^2 x \), so \( dx = \frac{du}{\sec^2 x} \). Therefore

\[
\int \frac{\sec^2 x}{(3 + \tan x)^5} \, dx = \int \sec^2 x (3 + \tan x)^{-5} \, dx = \int \sec^2 x u^{-5} \, du = \frac{1}{4} (3 + \tan x)^{-4} + C = \frac{1}{4} (3 + \tan x)^{-4} + C.
\]

28. \[ \int \sin \theta \cos \theta \sqrt{\sin^2 \theta + 1} \, d\theta \]

**Solution.** Let \( u = \sin \theta \). Then by using the Chain Rule, we get

\[
\frac{du}{d\theta} = \cos \theta \] and \[ \frac{d\theta}{du} = \frac{1}{\cos \theta}. \] Therefore

\[
\frac{d\theta}{du} = \frac{d}{du} [\sin \theta] = \frac{d}{du} [\sin \theta] = 2 \sin \theta \cos \theta.
\]

So \( \frac{du}{d\theta} = 2 \sin \theta \cos \theta \). This implies that \( \frac{d\theta}{du} = \frac{1}{2 \sin \theta \cos \theta} \). Therefore

\[
\frac{1}{2} \int u^2 \, du = \frac{1}{2} \left( \frac{u^3}{3} \right) + C = \frac{1}{2} \left( \frac{2}{3} \right) u^3 + C = \frac{1}{3} u^3 + C = \frac{1}{3} (\sin^2 \theta + 1)^{\frac{3}{2}} + C.
\]

29. \[ \int (x^2 + 1) \cos(x^3 + 3x) \, dx \]

**Solution.** Let \( u = x^3 + 3x \). Then \( \frac{du}{dx} = 3x^2 + 3 \), so \( dx = \frac{du}{3x^2 + 3} \). Therefore

\[
\int (x^2 + 1) \cos(x^3 + 3x) \, dx = \int (x^2 + 1) \cos(u) \frac{du}{3x^2 + 3} = \int (x^2 + 1) \cos(u) \frac{du}{3(x^2 + 1)} = \int \cos u \frac{du}{3} = \frac{1}{3} \int \cos u \, du = \frac{1}{3} \sin u + C = \frac{1}{3} \sin(x^3 + 3x) + C.
\]

30. \[ \int \frac{-3x^4}{(x^5 + 1)^7} \, dx \]

**Solution.** First we have \( \int \frac{-3x^4}{(x^5 + 1)^7} \, dx = \int -3x^4(x^5 + 1)^{-7} \, dx \). Let \( u = x^5 + 1 \). Then \( \frac{du}{dx} = 5x^4 \), so \( dx = \frac{du}{5x^4} \). Therefore

\[
\int \frac{-3x^4}{(x^5 + 1)^7} \, dx = \int -3x^4(u)^{-7} \frac{du}{5x^4} = -\frac{3}{5} \int u^{-7} \, du = -\frac{3}{5} \left( \frac{u^{-6}}{-6} \right) + C = \frac{1}{10} u^{-6} + C = \frac{1}{10} (x^5 + 1)^{-6} + C = \frac{1}{10} (x^5 + 1)^{-6} + C.
\]
31. \[ \int \frac{-4x + 4}{\sqrt{(x^2 - 2x + 1)^2}} \, dx \]

**Solution.** First we have

\[ \int \frac{-4x + 4}{\sqrt{(x^2 - 2x + 1)^2}} \, dx = \int \frac{-4x + 4}{(x^2 - 2x + 1)^2} \, dx = \int (4x + 4)(x^2 - 2x + 1)^{-\frac{1}{2}} \, dx. \]

Let \( u = x^2 - 2x + 1 \). Then \( \frac{du}{dx} = 2x - 2 \), so \( dx = \frac{du}{2x - 2} \). Therefore

\[ \int \frac{-4x + 4}{\sqrt{(x^2 - 2x + 1)^2}} \, dx = \int (4x + 4)u^{-\frac{1}{2}} \, \frac{du}{2x - 2} = -2 \int (2x - 2)u^{-\frac{1}{2}} \, du = -2 \int u^{-\frac{1}{2}} \, du = -2 \sqrt{u} + C = -2 \left( \frac{3}{1} \right) u^{\frac{1}{2}} + C = -6u^{\frac{1}{2}} + C = -6(x^2 - 2x + 1)^{\frac{1}{2}} + C. \]

32. \( \int x\sqrt{x + 7} \, dx \)

**Solution.** First we have \( \int x\sqrt{x + 7} \, dx = \int x(x + 7)^{\frac{1}{2}} \, dx \). Let \( u = x + 7 \). Then \( \frac{du}{dx} = 1 \), so \( dx = \frac{du}{1} = du \). Therefore \( \int x\sqrt{x + 7} \, dx = \int xu^{\frac{1}{2}} \, du \). It looks like it is not possible to eliminate all \( x \)'s. Since we want to write the original integral only in terms of \( u \), we can use the substitution formula to express \( x \) as a function of \( u \). This gives \( x = u - 7 \). Substituting \( x = u - 7 \) into \( \int xu^{\frac{1}{2}} \, du \), we get

\[ \int x\sqrt{x + 7} \, dx = \int xu^{\frac{1}{2}} \, du = \int (u - 7)u^{\frac{1}{2}} \, du = \int (u^{\frac{3}{2}} - 7u^{\frac{1}{2}}) \, du = \]

\[ u^{\frac{3}{2}} - 14u^{\frac{1}{2}} + C = \frac{2}{5}(x + 7)^{\frac{3}{2}} - \frac{14}{3}(x + 7)^{\frac{1}{2}} + C. \]

33. \( \int \frac{x}{\sqrt{1 + 2x}} \, dx \)

**Solution.** We have \( \int \frac{x}{\sqrt{1 + 2x}} \, dx = \int \frac{x}{(1 + 2x)^{\frac{1}{2}}} \, dx = \int x(1 + 2x)^{-\frac{1}{2}} \, dx \). Let \( u = 1 + 2x \). Then \( \frac{du}{dx} = 2 \), so \( dx = \frac{du}{2} \). Therefore \( \int \frac{x}{\sqrt{1 + 2x}} \, dx = \frac{1}{2} \int xu^{-\frac{1}{2}} \, du \). Again it is not possible to cross out \( x \). From the substitution formula, \( u = 1 + 2x \), we have \( 2x = u - 1 \), that is, \( x = \frac{u - 1}{2} \). Substituting this into \( \frac{1}{2} \int xu^{-\frac{1}{2}} \, du \), we get

\[ \int \frac{x}{\sqrt{1 + 2x}} \, dx = \frac{1}{2} \int u - 1 \cdot u^{-\frac{1}{2}} \, du = \frac{1}{4} \int (u - 1)u^{-\frac{1}{2}} \, du = \frac{1}{4} \int (u^{\frac{1}{2}} - u^{-\frac{1}{2}}) \, du =
\]

\[ \frac{1}{4} \left( \int u^{\frac{1}{2}} \, du - \int u^{-\frac{1}{2}} \, du \right) = \frac{1}{4} \int u^{\frac{3}{2}} - \frac{1}{4} \int u^{-\frac{1}{2}} \, du =
\]

\[ \frac{1}{4} \left( \frac{u^{\frac{3}{2}}}{\frac{3}{2}} - \frac{1}{\frac{1}{2}} \right) + C = \frac{1}{4} \left( \frac{2}{3}u^{\frac{3}{2}} - 2 \right) + C =
\]

\[ \frac{1}{6}u^{\frac{3}{2}} - \frac{1}{2}u^{\frac{1}{2}} + C = \frac{1}{6}(1 + 2x)^{\frac{3}{2}} - \frac{1}{2}(1 + 2x)^{\frac{1}{2}} + C. \]

34. \( \int -3\cot^2 x \csc^2 x \, dx \)

**Solution.** Let \( u = \cot x \). Then \( \frac{du}{dx} = -\csc^2 x \), so \( dx = \frac{du}{-\csc^2 x} \). Therefore

\[ \int -3\cot^2 x \csc^2 x \, dx = \int -3u^2 \csc^2 x \, \frac{du}{-\csc^2 x} = 3 \int u^2 \, du =
\]

\[ \frac{3}{3}u^3 + C = u^3 + C = (\cot x)^3 + C. \]
2.13 Solution to PB13

Definite Integrals and Area Between Two Curves

1. The Fundamental Theorem of Calculus, Part 2. Evaluate the following definite integrals.

(a) \( \int_{-3}^{1} 5 \, dx \)

Solution. A definite integral is an integral of the form \( \int_{a}^{b} f(x) \, dx \). The numbers \( a \) and \( b \) are called bounds of the integral. We can refer to \( a \) as the bottom bound, and to \( b \) as the top bound. Note that the result of a definite integral is a number, while that of an indefinite integral (that is, an integral of the form \( \int f(x) \, dx \) without bounds) is a function or a family of functions. The Fundamental Theorem of Calculus, Part 2 (FTC2), gives a nice way to find a definite integral. It states that if \( f \) is a continuous on \([a,b]\), then

\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a),
\]

where \( F \) is any antiderivative of \( f \), that is, a function \( F \) such that \( F' = f \). We will write \( [F(x)]_{a}^{b} \) for \( F(b) - F(a) \). Note that when evaluating \( [F(x)]_{a}^{b} \), we first plug-in the top bound, and then the bottom bound. (Always respect that order!)

We come back to the integral \( \int_{-3}^{1} 5 \, dx \). The function is \( f(x) = 5 \). Clearly an antiderivative of \( f \) is \( F(x) = 5x + C \), where \( C \) is a constant. So by the FTC2, we have

\[
\int_{-3}^{1} 5 \, dx = [F(x)]_{-3}^{1} = F(1) - F(-3) =
\]

\[
(5(1) + C) - (5(-3) + C) = 5 + C - (-15 + C) = 5 + 15 - C = 20.
\]

Note. For Definite Integrals, \( C \) will be always cancelled out. So we don’t need to write it. But for indefinite integrals, we have to write \( C \) (see Practice Problems 12). The following shows how we will take definite integrals from now on.

\[
\int_{-3}^{1} 5 \, dx = [5x]_{-3}^{1} = 5(1) - 5(-3) = 5 + 15 = 20.
\]

When evaluating a definite integral, be careful with the signs! Add parentheses or brackets if necessary.

(b) \( \int_{4}^{1} 3x \, dx \)

Solution. We have

\[
\int_{4}^{1} 3x \, dx = 3 \int_{4}^{1} x \, dx = 3 \left[ \frac{x^2}{2} \right]_{4}^{1} =
\]

\[
3 \left( \frac{2}{2} - \frac{1}{2} \right) = 3 \left( \frac{16}{2} - \frac{1}{2} \right) = 3 \left( \frac{15}{2} \right) = \frac{45}{2}.
\]

(c) \( \int_{4}^{9} \frac{1}{\sqrt{x}} \, dx \)

Solution. We have

\[
\int_{4}^{9} \frac{1}{\sqrt{x}} \, dx = \int_{4}^{9} \frac{1}{x^{1/2}} \, dx = \int_{4}^{9} x^{-1/2} \, dx = \left[ \frac{x^{1/2}}{1/2} \right]_{4}^{9} =
\]

\[
\left[ 2\sqrt{x} \right]_{4}^{9} = 2\sqrt{9} - 2\sqrt{4} = 2(3) - 2(2) = 6 - 4 = 2.
\]
2.13. **SOLUTION TO PB13**

(d) $\int_1^2 (-x^2 + 3x - 1) \, dx$

**Solution.** We have

\[
\int_1^2 (-x^2 + 3x - 1) \, dx = \int_1^2 -x^2 \, dx + \int_1^2 3x \, dx - \int_1^2 1 \, dx = -\int_1^2 x^2 \, dx + 3 \int_1^2 x \, dx - \int_1^2 1 \, dx = \\
- \left[ \frac{x^3}{3} \right]_1^2 + 3 \left[ \frac{x^2}{2} \right]_1^2 - [x]_1^2 = - \left( \frac{8}{3} - \frac{1}{3} \right) + 3 \left( \frac{4}{2} - \frac{1}{2} \right) - (2 - 1) = \\
- \left( \frac{7}{3} \right) + 3 \left( \frac{3}{2} \right) - (1) = - \frac{7}{3} + \frac{9}{2} - 1 = - \frac{14 + 27}{6} - 1 = \frac{13}{6} - 1 = \frac{13 - 6}{6} = \frac{7}{6}.
\]

(e) $\int_1^2 \left( \frac{1}{x^2} - \frac{4}{x^3} \right) \, dx$

**Solution.** We have

\[
\int_1^2 \left( \frac{1}{x^2} - \frac{4}{x^3} \right) \, dx = \int_1^2 \left( x^{-2} - 4x^{-3} \right) \, dx = \int_1^2 x^{-2} \, dx - 4 \int_1^2 x^{-3} \, dx = \\
\left[ \frac{x^{-1}}{-1} \right]_1^2 - 4 \left[ \frac{x^{-2}}{-2} \right]_1^2 = \left[ -\frac{1}{x} \right]_1^2 - 4 \left[ \frac{1}{2x^2} \right]_1^2 = \\
- \left( \frac{1}{2} - \frac{1}{1} \right) + 2 \left( \frac{1}{4} - \frac{1}{1} \right) = - \left( \frac{3}{2} \right) + 2 \left( \frac{3}{4} \right) = \frac{1}{2} - \frac{3}{2} = - \frac{2}{2} = -1.
\]

(f) $\int_{-2}^{3} (x^2 - 3) \, dx$

**Solution.** We have

\[
\int_{-2}^{3} (x^2 - 3) \, dx = \left[ \frac{x^3}{3} - 3x \right]_{-2}^{3} = \left( \frac{27}{3} - 9 \right) - \left( \frac{-8}{3} + 6 \right) = 9 - 9 - \left( \frac{-8 + 18}{3} \right) = 0 - \left( \frac{10}{3} \right) = -\frac{10}{3}.
\]

(g) $\int_1^{4} \frac{4 + 6x}{\sqrt{x}} \, dx$

**Solution.** We have

\[
\int_1^{4} \frac{4 + 6x}{\sqrt{x}} \, dx = \int_1^{4} \left( \frac{4}{x^{\frac{1}{2}}} + \frac{6x}{x^{\frac{1}{2}}} \right) \, dx = \int_1^{4} \left( 4x^{-\frac{1}{2}} + 6x^{\frac{1}{2}} \right) \, dx = 4 \int_1^{4} x^{-\frac{1}{2}} \, dx + 6 \int_1^{4} x^{\frac{1}{2}} \, dx = \\
4 \left[ \frac{x^{\frac{1}{2}}}{\frac{1}{2}} \right]_1^{4} + 6 \left[ \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_1^{4} = 4 \left( \frac{2}{1} \right) \left[ \frac{4}{2} \right]_1^{4} + 6 \left( \frac{2}{3} \right) \left[ \frac{4^2}{3} \right]_1^{4} = 8 \left[ \frac{4}{2} \right]_1^{4} + 4 \left( \frac{4^3}{3} \right) = \\
8 \left[ \sqrt{4} \right]_1^{4} + 4 \left( \sqrt[3]{4} \right)_1^{4} = 8 \left( \sqrt[3]{4} - \sqrt[3]{1} \right) + 4 \left( \frac{\sqrt[3]{4}}{3} - \frac{\sqrt[3]{1}}{3} \right) = \\
8(2 - 1) + 4(2^3 - 1^3) = 8(1) + 4(8 - 1) = 8 + 4(7) = 8 + 28 = 36.
\]

(h) $\int_1^{8} \frac{2 + t}{\sqrt{t^2}} \, dt$

**Solution.** We have

\[
\int_1^{8} \frac{2 + t}{\sqrt{t^2}} \, dt = \int_1^{8} \left( \frac{2}{t^{\frac{1}{2}}} + \frac{t}{t^{\frac{1}{2}}} \right) \, dt = \int_1^{8} \left( 2t^{-\frac{1}{2}} + t^{\frac{1}{2}} \right) \, dt = \\
2 \int_1^{8} t^{-\frac{1}{2}} \, dt + \int_1^{8} t^{\frac{1}{2}} \, dt = 2 \left[ \frac{t^{\frac{1}{2}}}{\frac{1}{2}} \right]_1^{8} + \left[ \frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right]_1^{8} = \\
2 \left[ \frac{8^{\frac{1}{2}}}{\frac{1}{2}} \right] + \left[ \frac{8^{\frac{3}{2}}}{\frac{3}{2}} \right] =
\]
\[
2 \left( \frac{3}{4} \right) \left[ t^4 \right]_1^8 + \frac{3}{4} \left[ t^4 \right]_1^8 = 6(8^4 - 1^4) + \frac{3}{4} \left( 8^4 - 1^4 \right) = \\
6 \left( \sqrt{3} - 1 \right) + \frac{3}{4} \left( (8^4 - 1) \right) = 6(2 - 1) + \frac{3}{4} \left( (2)^4 - 1 \right) = \\
6(1) + \frac{3}{4} (16 - 1) = 6 + \frac{3}{4} (15) = 6 + \frac{45}{4} = \frac{24 + 45}{4} = \frac{69}{4}.
\]

(i) \( \int_0^2 (2x - 3)(4x^2 + 1) \, dx \)

**Solution.** Here we distribute first.
\[
\int_0^2 (2x - 3)(4x^2 + 1) \, dx = \int_0^2 (8x^3 + 2x - 12x^2 - 3) \, dx = \\
8 \int_0^2 x^3 \, dx + 2 \int_0^2 x \, dx - 12 \int_0^2 x^2 \, dx - \int_0^2 3 \, dx = \\
8 \left[ \frac{x^4}{4} \right]_0^2 + 2 \left[ \frac{x^2}{2} \right]_0^2 - 12 \left[ \frac{x^3}{3} \right]_0^2 - 3(x)_0^2 = \\
8 \left( \frac{16}{4} - 0 \right) + 2 \left( \frac{4}{2} - 0 \right) - 12 \left( \frac{8}{3} - 0 \right) - (6 - 0) = \\
8(4 + 2) - 4(8) - 6 = 32 + 4 - 32 - 6 = -2.
\]

(j) \( \int_0^4 (4-x)\sqrt{x} \, dx \)

**Solution.** Again we distribute first.
\[
\int_0^4 (4-x)\sqrt{x} \, dx = \int_0^4 (4\sqrt{x} - x\sqrt{x}) \, dx = \int_0^4 (4x^{1/2} - xx^{1/2}) \, dx = 4 \int_0^4 x^{1/2} \, dx - \int_0^4 x^{3/2} \, dx = \\
4 \left[ \frac{x^{3/2}}{3/2} \right]_0^4 - \left[ \frac{x^{3/2}}{3/2} \right]_0^4 = 4 \left( \frac{2}{3} \right) \left[ \frac{x^{3/2}}{3/2} \right]_0^4 - 2 \left[ \frac{x^{3/2}}{3/2} \right]_0^4 = \\
8 \left( \frac{4^{3/2}}{3} - 0 \right) - 2 \left( \frac{4^{3/2}}{3} - 0 \right) = \frac{8}{3} \left( \frac{4^{3/2} - 2^{3/2}}{5} \right) = \frac{8}{3} \left( \frac{2^{3} - 2^{5}}{5} \right) = \frac{8}{3} \left( \frac{2^{5} - 2^{5}}{5} \right) = \frac{8}{3} \left( \frac{5 - 3}{15} \right) = \frac{128}{15}.
\]

(k) \( \int_0^\pi (2 - 3 \sin \theta) \, d\theta \)

**Solution.** We have
\[
\int_0^\pi (2 - 3 \sin \theta) \, d\theta = \int_0^\pi 2 \, d\theta - 3 \int_0^\pi \sin \theta \, d\theta = [2\theta]_0^\pi - 3[- \cos \theta]_0^\pi = \\
[2\theta]_0^\pi + 3[\cos \theta]_0^\pi = (2\pi - 2(0)) + 3(\cos \pi - \cos 0) = 2\pi - 0 + 3(-1 - 1) = 2\pi - 6.
\]

Recall that \( \cos 0 = 1, \sin 0 = 0, \cos \pi = -1, \) and \( \sin \pi = 0. \)

(l) \( \int_0^\pi \csc^2 x \, dx \)

**Solution.** We have
\[
\int_0^\pi \csc^2 x \, dx = [- \cot x]_0^\pi = [- \cot x]_0^\pi = - \left( \cot \frac{\pi}{4} - \cot \frac{\pi}{6} \right) = \\
- \left( \frac{1}{\tan \frac{\pi}{4}} - \frac{1}{\tan \frac{\pi}{6}} \right) = - \left( 1 - \frac{1}{\sqrt{3}} \right) = - \left( 1 - \frac{3}{\sqrt{3}} \right) = - \left( \frac{\sqrt{3} - 3}{\sqrt{3}} \right) = \frac{-\sqrt{3} + 3}{\sqrt{3}}.
\]

Recall the cotangent function: \( \cot x = \frac{1}{\tan x} \). Also recall that \( \tan 0 = 0, \tan \frac{\pi}{6} = \frac{\sqrt{3}}{3}, \tan \frac{\pi}{4} = 1, \tan \frac{\pi}{3} = \sqrt{3}, \tan \frac{\pi}{2} \) Does Not Exist, and \( \tan \pi = 0. \)
2.13. SOLUTION TO PB13

(m) \( \int_0^\pi (\sec x - \tan x) \sec x \, dx \)

**Solution.** We have

\[
\int_0^\pi (\sec x - \tan x) \sec x \, dx = \int_0^\pi (\sec^2 x - \sec x \tan x) \, dx = \int_0^\pi \sec^2 x \, dx - \int_0^\pi \sec x \tan x \, dx = [\tan x]_0^\pi - [\sec x]_0^\pi = (\tan \frac{\pi}{4} - \tan 0) - (\sec \frac{\pi}{4} - \sec 0) = (1 - 0) - \left( \frac{1}{\cos \frac{\pi}{4}} - \frac{1}{\cos 0} \right) = 1 - \left( \frac{1}{\sqrt{2}} - 1 \right) = 1 - \left( \frac{2}{\sqrt{2}} - 1 \right) = 1 - \frac{2}{\sqrt{2}} + 1 = 2 - \frac{2}{\sqrt{2}}.
\]

Recall the secant function: \( \sec x = \frac{1}{\cos x} \). Also recall that \( \cos 0 = 1, \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}, \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}, \cos \frac{\pi}{3} = \frac{1}{2}, \cos \frac{\pi}{2} = 0, \cos \pi = -1. \)

2. Definite integrals with respect to Substitution. Evaluate the following definite integrals.

(a) \( \int_0^2 2x(x^2 - 2)^3 \, dx \)

**Solution.** When evaluating a definite integral by substitution, two methods are possible.

**Method 1.** Take the indefinite integral first and then evaluate it. Let us explain that with the integral of the question. We make the substitution \( u = x^2 - 2 \). Then \( \frac{du}{dx} = 2x, \) so \( dx = \frac{du}{2x} \). Therefore the indefinite integral is

\[
\int 2x(x^2 - 2)^3 \, dx = \int 2xu^3 \frac{du}{2x} = \int u^3 \, du = \frac{u^4}{4} = \frac{(x^2 - 2)^4}{4}.
\]

Evaluating this from 0 to 2, we get

\[
\int_0^2 2x(x^2 - 2)^3 \, dx = \left[ \frac{(x^2 - 2)^4}{4} \right]_0^2 = \left( \frac{((2)^2 - 2)^4}{4} \right) - \left( \frac{((0)^2 - 2)^4}{4} \right) = \frac{(2)^4}{4} - \frac{(-2)^4}{4} = \frac{16}{4} - \frac{16}{4} = 0.
\]

**Method 2 (Usually Preferable).** Change the bounds during substitution. Let us explain that with the same integral. First we make the (same) substitution \( u = x^2 - 2 \). Then \( \frac{du}{dx} = 2x, \) so \( dx = \frac{du}{2x} \). Then we change the bounds.

When \( x = 0, \) \( u = (0)^2 - 2 = -2 \) and when \( x = 2, \) \( u = (2)^2 - 2 = 2. \)

(The new bounds are \( -2 \) and 2). Therefore

\[
\int_0^2 2x(x^2 - 2)^3 \, dx = \int_{-2}^2 u^3 \, du = \left[ \frac{u^4}{4} \right]_{-2}^2 = \left( \frac{2^4}{4} \right) - \left( \frac{(-2)^4}{4} \right) = \frac{16}{4} - \frac{16}{4} = 0.
\]

For the following integrals, we will use the second method.

(b) \( \int_0^1 (4t - 1)^{50} \, dt \)

**Solution.** We make the substitution \( u = 4t - 1 \). Then \( \frac{du}{dt} = 4, \) so \( dt = \frac{du}{4} \). Now we change the bounds.

When \( t = 0, \) \( u = -1 \) and when \( t = 1, \) \( u = 4(1) - 1 = 3. \)

So the new bounds are \( -1 \) and 3. Therefore

\[
\int_0^1 (4t - 1)^{50} \, dt = \int_{-1}^3 u^{50} \frac{du}{4} = \frac{1}{4} \int_{-1}^3 u^{50} \, du = \frac{1}{4} \left[ \frac{u^{51}}{51} \right]_{-1}^{3} = \frac{1}{204} \left[ 3^{51} - (-1)^{51} \right] = \frac{1}{204} \left( 3^{51} + 1 \right).
\]
(c) \( \int_0^1 \sqrt{26x + 1} \, dx \)

**Solution.** We can rewrite that integral as \( \int_0^1 \sqrt{26x + 1} \, dx = \int_0^1 (26x + 1)^{\frac{1}{2}} \, dx \). Let \( u = 26x + 1 \).

Then \( \frac{du}{dx} = 26 \), so \( dx = \frac{du}{26} \). Now we change the bounds.

When \( x = 0 \), \( u = 1 \) and when \( x = 1 \), \( u = 27 \).

The integral becomes

\[
\int_0^1 (26x + 1)^{\frac{1}{2}} \, dx = \int_1^{27} u^{\frac{1}{2}} \frac{du}{26} = \frac{1}{26} \int_1^{27} u^{\frac{1}{2}} du = \frac{1}{26} \left[ \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right]_1^{27} = \frac{1}{26} \left( \frac{3}{4} \right) \left[ \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right]_1^{27} = \frac{3}{104} \left( (27^{\frac{3}{2}}) - 1 \right) = \frac{3}{104} \left( (3^3) - 1 \right) = \frac{3}{104} (81 - 1) = \frac{3}{104} (80) = \frac{30}{13}.
\]

(d) \( \int_0^\pi \cos x \sin^4 x \, dx \)

**Solution.** Let \( u = \sin x \). Then \( \frac{du}{dx} = \cos x \), so \( dx = \frac{du}{\cos x} \).

When \( x = 0 \), \( u = \sin(0) = 0 \) and when \( x = \frac{\pi}{2} \), \( u = \sin(\frac{\pi}{2}) = 1 \).

Therefore

\[
\int_0^\pi \cos x \sin^4 x \, dx = \int_0^1 \cos x \, u^4 \frac{du}{\cos x} = \int_0^1 u^4 \frac{du}{5} = \left[ \frac{u^5}{5} \right]_0^1 = \frac{(1)^5}{5} - \frac{(0)^5}{5} = \frac{1}{5}.
\]

(e) \( \int_0^\pi \frac{\sin t}{\cos^2 t} \, dt \)

**Solution.** First we rewrite the integral as \( \int_0^\pi \frac{\sin t}{\cos^2 t} \, dt = \int_0^\pi \sin t (\cos t)^{-2} \, dt \). Let \( u = \cos t \). Then \( \frac{du}{dt} = -\sin t \), so \( dt = -\frac{du}{\sin t} \).

When \( t = 0 \), \( u = \cos(0) = 1 \) and when \( t = \frac{\pi}{6} \), \( u = \cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2} \).

Therefore

\[
\int_0^\pi \frac{\sin t}{\cos^2 t} \, dt = \int_1^{\frac{\sqrt{3}}{2}} \sin t \, u^{-2} \frac{du}{-\sin t} = - \int_1^{\frac{\sqrt{3}}{2}} u^{-2} \frac{du}{1} = - \left[ \left( \frac{u^{-1}}{1} \right) \right]_1^{\frac{\sqrt{3}}{2}} = \left[ \frac{1}{\sqrt{3}} \right]_1^{\frac{\sqrt{3}}{2}} = \left( \frac{1}{\sqrt{3}} - 1 \right) = \frac{2\sqrt{3} - 3}{3}.
\]

(f) \( \int_0^8 \frac{1}{\sqrt{1 + 2x}} \, dx \)

**Solution.** First we rewrite the integral as \( \int_0^8 \frac{1}{\sqrt{1 + 2x}} \, dx = \int_0^8 (1 + 2x)^{-\frac{1}{2}} \, dx \). Let \( u = 1 + 2x \). Then \( \frac{du}{dx} = 2 \), so \( dx = \frac{du}{2} \).

When \( x = 0 \), \( u = 1 \) and when \( x = 8 \), \( u = 17 \).

Therefore

\[
\int_0^8 \frac{1}{\sqrt{1 + 2x}} \, dx = \int_1^{17} u^{-\frac{1}{2}} \frac{du}{2} = \frac{1}{2} \int_1^{17} u^{-\frac{1}{2}} \, du = \frac{1}{2} \left[ \frac{u^{\frac{1}{2}}}{\frac{1}{2}} \right]_1^{17} = \frac{1}{2} \left( \frac{2}{1} \right) \left[ u^{\frac{1}{2}} \right]_1^{17} = \left[ u^{\frac{1}{2}} \right]_1^{17} = \sqrt{17} - 1 = 17^{\frac{1}{2}} - 1 = \sqrt{17} - 1.
\]
2.13. SOLUTION TO PB13

(g) \( \int_0^2 t^2 \sqrt{1 + t^3} \, dt \)

**Solution.** First we have \( \int_0^2 t^2 \sqrt{1 + t^3} \, dt = \int_0^2 t^2 (1 + t^3)^{\frac{1}{2}} \, dt \). We make the substitution \( u = 1 + t^3 \). Then \( \frac{du}{dt} = 3t^2 \), so \( dt = \frac{du}{3t^2} \).

Therefore

\[
\int_0^2 t^2 \sqrt{1 + t^3} \, dt = \int_1^9 t^2 u^{\frac{1}{2}} \, \frac{du}{3t^2} = \frac{1}{3} \int_1^9 u^{\frac{1}{2}} \, du = \frac{1}{3} \left[ \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right]_1^9 = \frac{1}{3} \left( \frac{2}{3} \right) \left[ u^{\frac{3}{2}} \right]_1^9 = \frac{2}{9} \left( g^{\frac{3}{2}} - 1 \right) = \frac{2}{9} \left( \left( g^{\frac{3}{2}} \right)^3 - 1 \right) = \frac{2}{9} \left( 3^3 - 1 \right) = \frac{2}{9} (27 - 1) = \frac{2}{9} (26) = \frac{52}{9}.
\]

(h) \( \int_0^\pi \cos(3x) \, dx \)

**Solution.** Let \( u = 3x \). Then \( \frac{du}{dx} = 3 \), so \( dx = \frac{du}{3} \).

When \( x = 0 \), \( u = 0 \) and when \( x = \frac{\pi}{6} \), \( u = \frac{3\pi}{6} = \frac{\pi}{2} \).

Therefore

\[
\int_0^\pi \cos(3x) \, dx = \int_0^\pi \cos u \, \frac{du}{3} = \frac{1}{3} \int_0^\pi \cos u \, du = \frac{1}{3} \left[ \sin u \right]_0^\pi = \frac{1}{3} \left( \sin \frac{\pi}{2} - \sin 0 \right) = \frac{1}{3} (1 - 0) = \frac{1}{3}.
\]

(i) \( \int_1^2 \frac{x+1}{\sqrt{x^2+2x-1}} \, dx \)

**Solution.** First we have \( \int_1^2 \frac{x+1}{\sqrt{x^2+2x-1}} \, dx = \int_1^2 (x+1)(x^2+2x-1)^{-\frac{1}{2}} \, dx \). Let \( u = x^2 + 2x - 1 \). Then \( \frac{du}{dx} = 2x + 2 \), so \( dx = \frac{du}{2x+2} \).

When \( x = 1 \), \( u = 2 \) and when \( x = 2 \), \( u = 4 + 4 - 1 = 7 \).

Therefore

\[
\int_1^2 \frac{x+1}{\sqrt{x^2+2x-1}} \, dx = \int_2^7 (x+1)u^{-\frac{1}{2}} \, \frac{du}{2x+2} = \int_2^7 (x+1)u^{-\frac{1}{2}} \, \frac{du}{2(x+1)} = \int_2^7 u^{-\frac{1}{2}} \, \frac{du}{2} \]

\[
= \frac{1}{2} \int_2^7 u^{-\frac{1}{2}} \, du = \frac{1}{2} \left[ \frac{u^{\frac{1}{2}}}{\frac{1}{2}} \right]_2^7 = \frac{1}{2} \left( \frac{7}{2} \right) \left[ u^{\frac{1}{2}} \right]_2^7 = \frac{7^2}{2} - 2^2 = \sqrt{7} - \sqrt{2}.
\]

(j) \( \int_\frac{\pi}{2}^\pi \csc^2 \left( \frac{1}{2} t \right) \, dt \)

**Solution.** Let \( u = \frac{1}{2} t \). Then \( \frac{du}{dt} = \frac{1}{2} \), so \( dt = \frac{2 \, du}{\pi} \).

When \( t = \frac{\pi}{3} \), \( u = \frac{\pi}{6} \) and when \( t = \frac{\pi}{2} \), \( u = \frac{\pi}{4} \).

So

\[
\int_\frac{\pi}{2}^\pi \csc^2 \left( \frac{1}{2} t \right) \, dt = \int_\frac{\pi}{2}^\pi \csc^2 u \, (2 \, du) = 2 \int_\frac{\pi}{2}^\pi \csc^2 u \, du = \]

\[
= -2 \left[ \cot u \right]_\frac{\pi}{2}^\pi = -2 \left[ \cot \left( \frac{\pi}{4} \right) - \cot \left( \frac{\pi}{6} \right) \right] = -2 \left( \frac{1}{\tan \frac{\pi}{4}} - \frac{1}{\tan \frac{\pi}{6}} \right) =
\]

\[
-2 \left( \frac{1}{1} - \frac{1}{\sqrt{3}} \right) = -2 \left( \frac{\sqrt{3} - 3}{\sqrt{3}} \right) = \frac{-2\sqrt{3} + 6}{\sqrt{3}}.
\]
(k) \[ \int_0^4 \frac{x}{\sqrt{1+2x}} \, dx \]

**Solution.** First we have \( \int_0^4 \frac{x}{\sqrt{1+2x}} \, dx = \int_0^4 x(1+2x)^{-\frac{1}{2}} \, dx \). Let \( u = 1 + 2x \). Then \( \frac{du}{dx} = 2 \), so \( dx = \frac{du}{2} \).

When \( x = 0, \ u = 1 \) and when \( x = 4, \ u = 9 \).

Therefore \( \int_0^4 \frac{x}{\sqrt{1+2x}} \, dx = \int_1^9 xu^{-\frac{1}{2}} \frac{du}{2} = \frac{1}{4} \int_1^9 (u-1)u^{-\frac{1}{2}} \, du = \frac{1}{4} \left[ \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right]_1^9 - \frac{1}{4} \left[ \frac{u^{\frac{1}{2}}}{\frac{1}{2}} \right]_1^9 = \frac{1}{6} \left[ 27 - 1 \right] = 13 - 1 = 10/3 \).

(l) \[ \int_0^\pi x \sin(x^2) \, dx \]

**Solution.** Let \( u = x^2 \). Then \( \frac{du}{dx} = 2x \), so \( dx = \frac{du}{2x} \).

When \( x = 0, \ u = 0 \) and when \( x = \pi, \ u = (\pi)^2 = \pi \).

Therefore \( \int_0^\pi x \sin(x^2) \, dx = \int_0^\pi x \sin u \frac{du}{2x} = \frac{1}{2} \int_0^\pi \sin u \, du = \frac{1}{2} \left[ -\cos u \right]_0^\pi = -\frac{1}{2} \left[ \cos \pi - \cos 0 \right] = -\frac{1}{2} (-1 - 1) = -\frac{1}{2} (-2) = 1 \).

3. **The Fundamental Theorem of Calculus, Part 1.** Find the derivative of each of the following functions.

(a) \( g(x) = \int_1^x \cos(t^2) \, dt \)

**Solution.** The Fundamental Theorem of Calculus, Part 1, (FTC1) states that if \( f \) is continuous on \([a, b]\), then the function \( g \) defined by \( g(x) = \int_a^x f(t) \, dt \), \( a \leq x \leq b \), is continuous on \([a, b]\) and differentiable on \((a, b)\), and \( g'(x) = f(x) \). **Note** that in the FTC1, the variable is the top bound and the number is the bottom bound.

Here \( f(t) = \cos(t^2) \), which is continuous and differentiable everywhere. So by the FTC1, \( g'(x) = f(x) = \cos(x^2) \).

(b) \( g(x) = \int_a^x t^3 \sin t \, dt \)

**Solution.** Recall the following property (saying that if we switch the bounds, then we must add the sign “−” in front of the integral).

\[ \int_a^b f(x) \, dx = -\int_b^a f(x) \, dx. \]

In order to apply the FTC1, we need to get the variable on top. Applying the property, we get \( g(x) = -\int_a^x t^3 \sin t \, dt \). Let \( h(x) = \int_a^x t^3 \sin t \, dt \). Then \( g(x) = -h(x) \), and therefore \( g'(x) = -h'(x) \). Now we find \( h'(x) \). Let \( f(t) = t^3 \sin t \). Then by the FTC1, we have that \( h'(x) = f(x) = \)
2.13. SOLUTION TO PB13

\(x^3 \sin x\). Thus \(g'(x) = -h'(x) = -x^3 \sin x\). We can rewrite all this as follows (by using the “\(\frac{d}{dx}\)” notation).

\[
g'(x) = \frac{d}{dx} \left[ \int_x^2 t^3 \sin t \, dt \right] = \frac{d}{dx} \left[ - \int_2^x t^3 \sin t \, dt \right] = - \frac{d}{dx} \left[ \int_2^x t^3 \sin t \, dt \right] = -x^3 \sin x.
\]

(c) \(g(x) = \int_1^{3x+2} \frac{t}{1+t^2} \, dt\)

**Solution.** Let \(u = 3x + 2\), and let \(h(u) = \int_1^{u} \frac{t}{1+t^2} \, dt\). Then \(g'(x) = \frac{d}{dx}[h(u)]\). By the Chain Rule, we have \(\frac{d}{dx}[h(u)] = \frac{d}{du}[h(u)] \frac{du}{dx}\). By the FTC1, we have \(\frac{d}{du}[h(u)] = \frac{u}{1+u^2}\). Putting all these together, we get \(g'(x) = \frac{u}{1+u^2} (3) = \frac{3(3x+2)}{1+(3x+2)^2}\). One can rewrite all this as follows.

\[
g'(x) = \frac{d}{dx} \left[ \int_1^{u} \frac{t}{1+t^2} \, dt \right] = \frac{d}{du} \left[ \int_1^{u} \frac{t}{1+t^2} \, dt \right] \frac{du}{dx} = \frac{u}{1+u^2} (3) = \frac{3(3x+2)}{1+(3x+2)^2}.
\]

(d) \(g(x) = \int_{x^4}^{3} \cos^2 t \, dt\)

**Solution.** Let \(u = x^4\). Then

\[
g'(x) = \frac{d}{dx} \left[ \int_3^{x^4} \cos^2 t \, dt \right] = \frac{d}{dx} \left[ \int_3^{x^4} \cos^2 t \, dt \right] = -\cos^2(u)(4x^3) = -4x^3 \cos^2(x^4).
\]

(e) \(g(x) = \int_{x^2}^{\sqrt{x}} (t^2 + 1) \, dt\)

**Solution.** First recall the following property.

\[
\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx.
\]

By using that property, we can split the integral into two parts:

\[
g(x) = \int_{x^2}^{0} (t^2 + 1) \, dt + \int_0^{\sqrt{x}} (t^2 + 1) \, dt = -\int_0^{x^2} (t^2 + 1) \, dt + \int_0^{\sqrt{x}} (t^2 + 1) \, dt.
\]

Let’s take the derivative of the first part. Let \(u = x^2\). Then

\[
\frac{d}{dx} \left[ -\int_0^{x^2} (t^2 + 1) \, dt \right] = -\frac{d}{dx} \left[ \int_0^{x^2} (t^2 + 1) \, dt \right] = -\frac{d}{du} \left[ \int_0^{u} (t^2 + 1) \, dt \right] \frac{du}{dx} =
\]

\[
-(u^2 + 1) \frac{d}{dx} [u^2] = -(x^2)^2 + 1)(2x) = -2x(x^4 + 1).
\]

Now we take the derivative of the second part. Let \(v = \sqrt{x}\). Then

\[
\frac{d}{dx} \left[ \int_0^{\sqrt{x}} (t^2 + 1) \, dt \right] = \frac{d}{dv} \left[ \int_0^{v} (t^2 + 1) \, dt \right] \frac{dv}{dx} =
\]

\[
(v^2 + 1) \frac{d}{dx} [\sqrt{x}] = ((\sqrt{x})^2 + 1) \frac{1}{2\sqrt{x}} = \frac{x+1}{2\sqrt{x}}.
\]

Adding up the two derivatives, we get

\[
g'(x) = -2x(x^4 + 1) + \frac{x+1}{2\sqrt{x}}.
\]
4. (Section 5.1) Area Between Two Curves.

(a) Find the area bounded above by \( y = 2x + 5 \) and below by \( y = x^2 \) on \([0, 2]\).

**Solution.** Recall the formula for the area between two curves. If \( f \) and \( g \) are two continuous functions on \([a, b]\) such that \( f(x) \) is above \( g(x) \) on \([a, b]\). That is, \( f(x) \geq g(x) \) for all \( x \) in \([a, b]\).

Then the area between the graph of \( f \) and that of \( g \) is given by the formula

\[
A = \int_a^b [f(x) - g(x)] \, dx.
\]

**Note.** In order to find the area, we need to determine which function is on top. Then the area is the integral of the function on top minus the function on bottom. Here we know from the problem that \( y = 2x + 5 \) is on top. Again from the problem, we know that the interval is \([0, 2]\). So

\[
A = \int_0^2 (2x + 5 - x^2) \, dx = 2 \int_0^2 x \, dx + \int_0^2 5 \, dx - \int_0^2 x^2 \, dx =
\]

\[
2 \left[ \frac{x^2}{2} \right]_0^2 + [5x]_0^2 - \left[ \frac{x^4}{4} \right]_0^2 = (2^2 - 0) + (5(2) - 0) - \left( \frac{2^4}{4} - 0 \right) = 4 + 10 - 4 = 10.
\]

(b) Find the area of the region enclosed by the curves \( y = x^2 \) and \( y = x + 2 \).

**Solution.** Let us solve this problem step by step.

Step 1. Find the \( x \)-coordinates of the intersection of the curves (set \( f(x) = g(x) \), and solve for \( x \)). This gives the bounds of integration. Let \( f(x) = x^2 \) and \( g(x) = x + 2 \). Setting \( f(x) = g(x) \), we get \( x^2 = x + 2 \), that is, \( x^2 - x - 2 = 0 \) or \((x + 1)(x - 2) = 0\). So \( x = -1 \) or \( x = 2 \), and therefore the interval is \([-1, 2]\).

Step 2. Which function is on Top? To answer this question, we can pick a number, say \( c \), in the interval and plug-in into both \( f(x) \) and \( g(x) \). If \( f(c) > g(c) \) then \( f \) is on top. If \( g(c) > f(c) \) then \( g \) is on top. Here we can pick for example \( 0 \), which is clearly between \(-1 \) and \( 2 \). Plug-in: \( f(0) = 0 \) and \( g(0) = 2 \). Since \( g(0) > f(0) \), it follows that \( g \) is on top.

Step 3. Set-up and solve. Since \( g \) is on top, the required area is the integral of \( g \) minus \( f \) on the interval \([-1, 2]\). Specifically,

\[
A = \int_{-1}^2 x + 2 - x^2 \, dx = \int_{-1}^2 x \, dx + \int_{-1}^2 2 \, dx - \int_{-1}^2 x^2 \, dx = \left[ \frac{x^2}{2} \right]_{-1}^2 + [2x]_{-1}^2 - \left[ \frac{x^3}{3} \right]_{-1}^2 =
\]

\[
\left\{ \frac{2^2}{2} - \frac{(-1)^2}{2} \right\} + (2(2) - 2((-1))) - \left( \frac{2^3}{3} - \frac{(-1)^3}{3} \right) =
\]

\[
\left( \frac{4}{2} - \frac{1}{2} \right) + (4 + 2) - \left( \frac{8}{3} + \frac{1}{3} \right) = \frac{3}{2} + 6 - 3 = \frac{3}{2} + 3 = \frac{3 + 6}{2} = \frac{9}{2}.
\]

(c) Find the area of the region enclosed by the curves \( y = x^2 - 8 \) and \( y = -x^2 + 10 \). Include a sketch of the relevant region as part of your solution.

**Solution.** Let \( f(x) = x^2 - 8 \) and \( g(x) = -x^2 + 10 \). Setting \( f(x) = g(x) \), we get \( x^2 - 8 = -x^2 + 10 \), that is, \( 2x^2 - 18 = 0 \) or \( 2(x^2 - 9) = 0 \), that is, \( 2(x + 3)(x - 3) = 0 \). So \( x = -3 \) or \( x = 3 \), and therefore the interval is \([-3, 3]\). To see which function is on top, pick \( 0 \), which is clearly a number between \(-3 \) and \( 3 \). We have \( f(0) = -8 \) and \( g(0) = 10 \). So \( g \) is on top since \( g(0) > f(0) \). Therefore the area is

\[
A = \int_{-3}^3 [(x^2 - 8) - (-x^2 + 8)] \, dx = \int_{-3}^3 (-2x^2 + 18) \, dx = \left[ -\frac{2x^3}{3} + 18x \right]_{-3}^3 =
\]

\[
\left( -\frac{54}{3} + 54 \right) - \left( \frac{54}{3} - 54 \right) = -108 + 108 = 108 \left( \frac{1}{3} + 1 \right) = 108 \left( \frac{2}{3} \right) = (36)(2) = 72.
\]

For the sketch, see Figure 2.29.
2.13. SOLUTION TO PB13

Figure 2.29: Graphs of $f(x) = x^2 - 8$ and $g(x) = -x^2 + 10$

(d) Find the area of the region enclosed by the curves $y = x^2$ and $y = 4x - x^2$. Include a sketch of the relevant region as part of your solution.

**Solution.** Let $f(x) = x^2$ and $g(x) = 4x - x^2$. Setting $f(x) = g(x)$, we get $x^2 = 4x - x^2$, that is, $2x^2 - 4x = 0$, or $2x(x - 2) = 0$. So $x = 0$ or $x = 2$, and the interval is then $[0, 2]$. To see which function is on top, we pick $1$, which is clearly in the interval. We have $f(1) = 1$ and $g(1) = 3$. Since $g(1) > f(1)$, the graph of $g$ is on top. Therefore,

$$A = \int_0^2 4x - x^2 - x^2 \, dx = \int_0^2 4x - 2x^2 \, dx = \left[ \frac{4x^2}{2} - \frac{2x^3}{3} \right]_0^2 =$$

$$\frac{16}{2} - \frac{16}{3} = 16 \left( \frac{1}{2} - \frac{1}{3} \right) = 16 \left( \frac{1}{6} \right) = \frac{8}{3}.$$

For the sketch, see Figure 2.30.

Figure 2.30: Graphs of $f(x) = x^2$ and $g(x) = 4x - x^2$

(e) Find the area of the region enclosed by the curves $y = x^3$ and $y = x$. Include a sketch of the relevant region as part of your solution.

**Solution.** Let $f(x) = x^3$ and $g(x) = x$. Setting $f(x) = g(x)$, we get $x^3 = x$, that is, $x^3 - x = 0$, or $x(x^2 - 1) = 0$, that is, $x(x + 1)(x - 1) = 0$. So $x = -1$ or $x = 0$ or $x = 1$. Here we have
three solutions. This gives rise to two intervals: $[-1,0]$ and $[0,1]$. Pick $-\frac{1}{2}$ in the first interval. We have $f(-\frac{1}{2}) = -\frac{1}{8}$ and $g(-\frac{1}{2}) = -\frac{1}{2}$. Since $f(-\frac{1}{2}) > g(-\frac{1}{2})$, the function $f$ is on top in the interval $[-1,0]$. Picking $\frac{1}{2}$ in the second interval, we get $f(\frac{1}{2}) = \frac{1}{2}$ and $g(\frac{1}{2}) = \frac{1}{2}$. So $g$ is on top in the interval $[0,1]$. Now the required area is equal to the area in the interval $[-1,0]$ plus the area in the interval $[0,1]$. Specifically, we have

$$A = \int_{-1}^{0} [f(x) - g(x)] \, dx + \int_{0}^{1} [g(x) - f(x)] \, dx =$$

$$\int_{-1}^{0} (x^3 - x) \, dx + \int_{0}^{1} (x - x^3) \, dx = \left[ \frac{x^4}{4} - \frac{x^2}{2} \right]_{-1}^{0} + \left[ \frac{x^2}{2} - \frac{x^4}{4} \right]_{0}^{1} =$$

$$\left( 0 - \left( \frac{1}{4} - \frac{1}{2} \right) \right) + \left( \left( \frac{1}{2} - \frac{1}{4} \right) - 0 \right) = -\frac{1}{4} + \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = 1 - \frac{1}{2} = \frac{1}{2}.$$ 

For the sketch, see Figure 2.31.

Figure 2.31: Graphs of $f(x) = x^3$ and $g(x) = x$