Covering Arrays and Covering Arrays on Graphs

Karen Meagher

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University of Ottawa

Covering Arrays

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- r rows of length n (n is the size)
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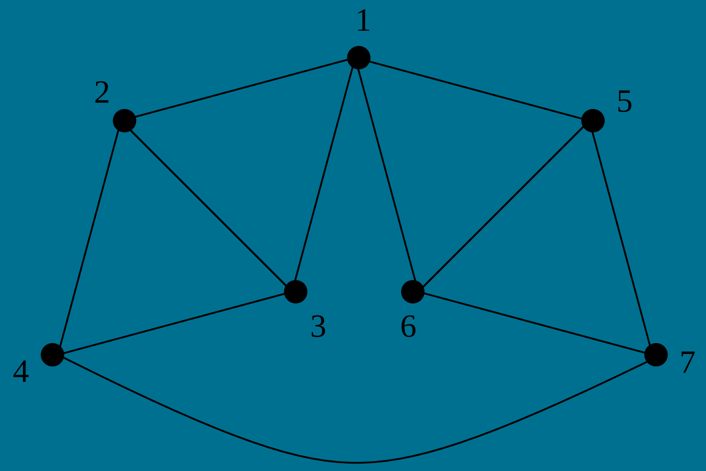

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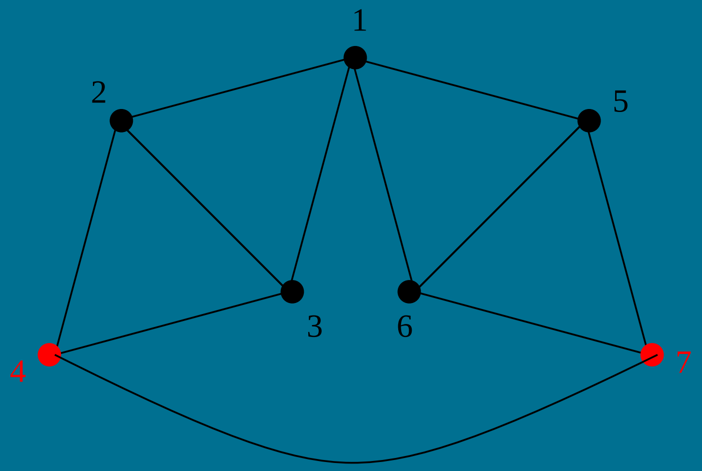

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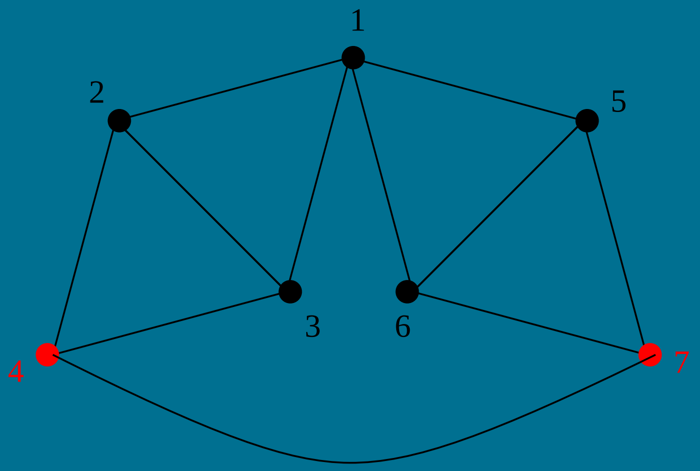

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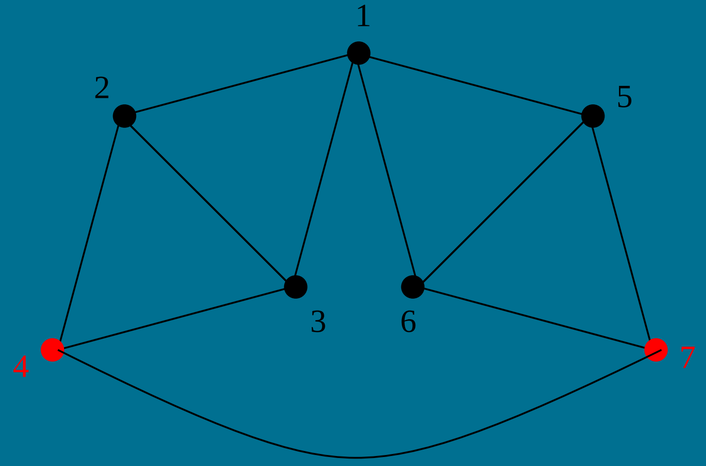

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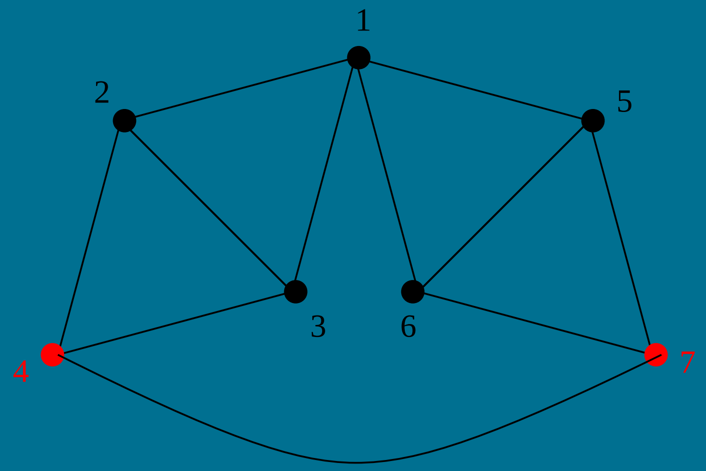

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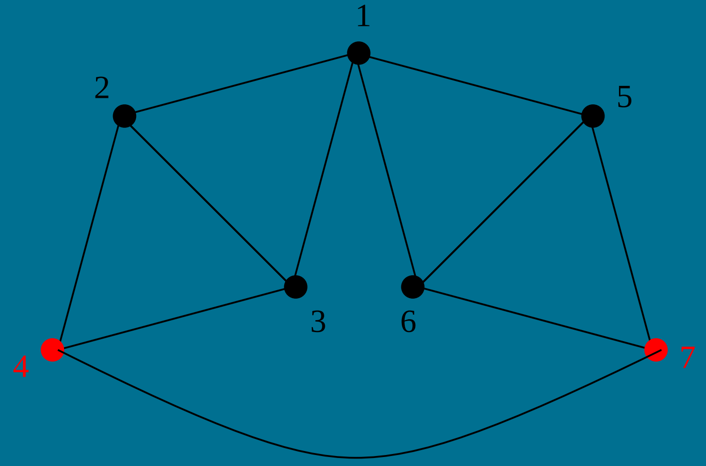

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- Covering arrays correspond to covering arrays on complete graphs

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$$CAN(\omega(G), k) \leq CAN(G, k) \leq CAN(\chi(G), k).$$

- ⋆ and a bound from the chromatic number.

Qualitative Independence Graph

Define graph $QI(n, k)$ by

- Vertices are all vectors which could go into a row of a covering array of size n on $\mathbb{Z}_k$.
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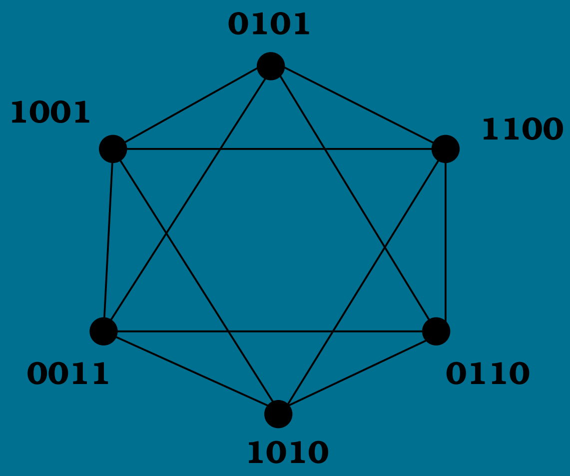

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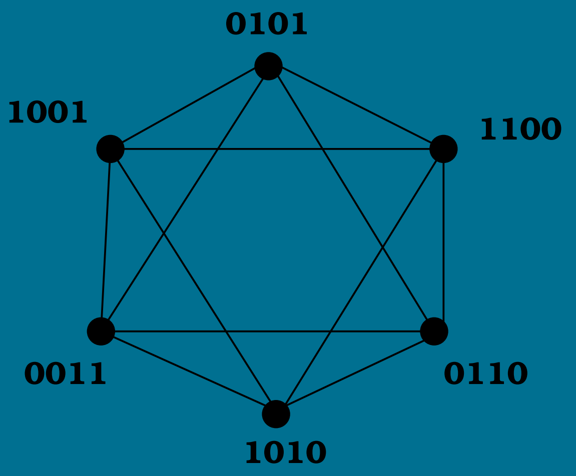

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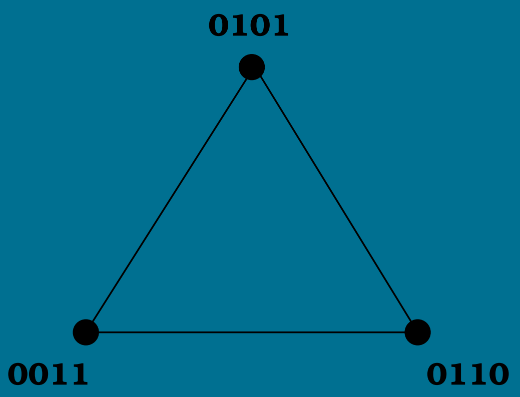

The graph $QI(4, 2)$.

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Why is $QI(n, g)$ Interesting?

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Theorem (M. and Stevens, 2002)

$$CAN(G, k) = \min\{n : G \to QI(n, k)\}$$

Facts for $QI(n,2)$

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An **almost-uniform k-partition** is a k-partition in which all classes have size $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$.

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- $\omega(QI(k^2, k)) \leq k + 1$, equality holds if k is a prime power

Better results for $QI(k^2, k)$

- $\alpha(QI(k^2,k)) = \binom{k^2-2}{k-2}\binom{k^2-k}{k}\cdots\binom{k}{k}$. The set of all partitions with 1 and 2 in the same class is an independent set.

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General $QI(n, k)$

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- ⋆ $QI(n,k)$ is difficult

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How can we find the eigenvalues?

Equitable Partitions

Equitable partition for a graph G:

- $\star$ partition π of $V(G)$ with cells $C_1, C_2, \ldots, C_r$,
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Quotient graph of G over π, G/π is the directed mulitgraph with

- ⋆ the r cells C_i as its vertices,
- ⋆ b_{ij} arcs between the $i^{th}$and j^{th} cells.

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Finding Eigenvalues

For a graph G with an equitable partition π, the adjacency matrix of the quotient graph of G over π is: $A(G/\pi) = [b_{i,j}]$.

If λ is an eignevalue of $A(G/\pi) = [b_{i,j}]$ then λ is an eigenvalue of $A(G)$.

If π has a cell with exactly 1 vertex then the eignevalues $A(G/\pi)$ are exactly the eigenvalues of $A(G)$.

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The adjacency matrix of the quotient graph $QI(k^2, k)/\pi$ is

$$\begin{pmatrix} 0 & k!^{k-1} \\ \frac{k!^{k-1}}{k} & k!^{k-1} - \frac{k!^{k-1}}{k} \end{pmatrix}$$

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The eigenvalues are d and $a-b$.
How to count a, b and d is not so easy!

Meet Tables

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For $P = 123|456|789$ and $Q = 147|258|369$,

$$M_{P,Q} = \begin{array}{c|ccc|} & Q_1 & Q_2 & Q_3 \\ \hline P_1 & 1 & 1 & 1 \\ P_2 & 1 & 1 & 1 \\ P_3 & 1 & 1 & 1 \\ \hline \end{array}$$

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Two meet tables are **isomorphic** if there is some permutation of the rows and columns of one array that produces the other array.

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Build a computer program that builds the adjacent matrix of $UQI(ck,k)/\pi$.

Spectrums of Some $UQI(ck, k)$

Graph	Eigenvalues and corresponding multiplicities
$9, 3$	(-4, 2, 8, -12, 36) (84, 120, 48, 27, 1)
$12, 3$	(0, 8, -12, 18, -27, 48, 108, -252, 1728) (275,2673,462,616,1408,132,154,54,1)
$15, 3$	(4, 8, -10, -22, 29, 34, -76, 218, -226, 284, 1628, -5060, 62000) (1638, 21450, 910, 25025, 32032, 22113, 11583, 1925, 7007, 2002, 350, 90,1)
$18, 3$	(8, 15, 18, -60, 60, -102, -120, 120, 368, 648, -655, -2115, 2370, -2115, 2370, 2460, -4140, 24900, -89550, 1876500, $954 \pm 18\sqrt{10209}$) (787644, 678912, 136136, 87516, 331500, 259896, 102102, 219912, 99144, 11934, 88128, 22848, 4641, 5508, 2244, 663, 135, 1, 9991)
$16, 4$	(-72,$-56 \pm 8\sqrt{193}$, $-96 \pm 96\sqrt{37}$, $24 \pm 24\sqrt{97}$, -96, 96, -288, 8, -144, 24, 192, 32, 1728, -64, -16, 432, 48, 1296,-48, -576, 128, -3456, 576, 13824, -1152, 144) (266240, 137280, 7280, 76440, 69888, 91520, 24960, 262080, 73920, 24024, 65520,150150,440,51480,753324,20020,420420,1260,23100,10752,60060,104, 4070,1,1260,32032)

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Is this an association scheme or some related structure?