Two Approaches in the Study of Covering Arrays

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A *covering array* $CA(n, r, k)$ is an $r \times n$ array with:

- ⋆ entries from $\mathbb{Z}_k$ (k is the alphabet),
- ⋆ and between any two rows all pairs from $\mathbb{Z}_k$ occur. (This property is called qualitative independence.)
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This is a $CA(5, 4, 2)$

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Two Areas of Covering Arrays

- Extremal Set Theory
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- Extremal Set Theory
  - The Erdős-Ko-Rado Theorem and Sperner's Theorem apply to binary covering arrays.
  - Extend such results to partition systems.
- Graph Theory
  - Add a graph structure to covering arrays.
  - Use methods from algebraic graph theory.

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- ⋆ A set system is a collection of subsets of an n-set.
- ⋆ A k-set system is a set system with subsets of size k.
- ⋆ The rows of a binary covering array correspond to a set system.

Qualitatively Independent Sets

A set system $\mathcal{F}$ is qualitatively independent if for distinct sets $A, B \in \mathcal{F}$,

$$A \cap B \neq \emptyset \quad \overline{A} \cap B \neq \emptyset \quad A \cap \overline{B} \neq \emptyset \quad \overline{A} \cap \overline{B} \neq \emptyset.$$

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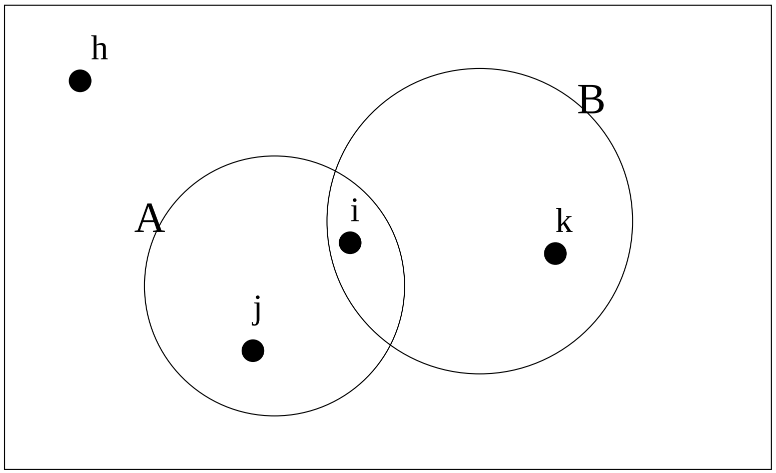

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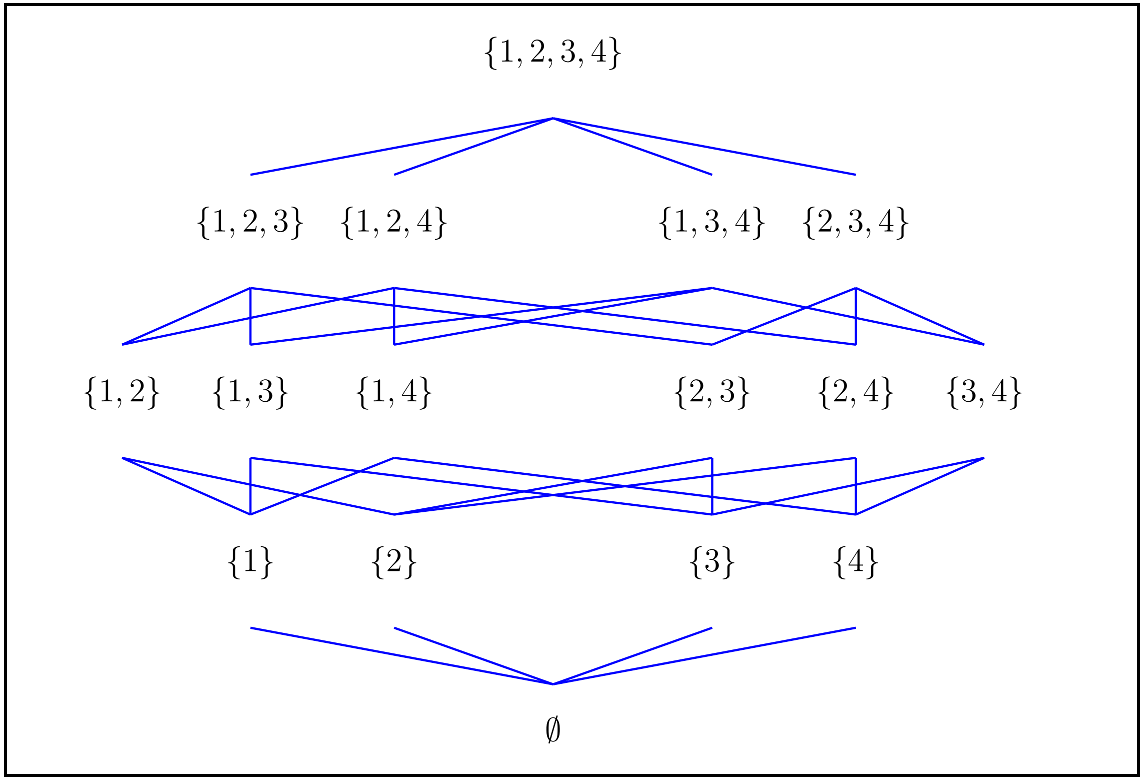

Sperner's Theorem

Theorem (Sperner - 1928) Let $\mathcal{F}$ be a Sperner set system over an n-set. Then

1. $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$.
2. Equality holds if and only if $\mathcal{F}$ is the system of all sets of size $\lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$.

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If $\mathcal{F}$ is a qualitatively independent set system on an n-set, then

$$\mathcal{F}^* = \{A, \overline{A} : A \in \mathcal{F}\}$$

is a Sperner set system. In particular,

$$|\mathcal{F}| \leq \tfrac{1}{2}\binom{n}{\lfloor n/2 \rfloor}.$$

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A trivially t-intersecting k-set system on an n-set has cardinality

$$\binom{n-t}{k-t}.$$

Erdős-Ko-Rado Theorem

Theorem (Erdős, Ko and Rado - 1961) For n **sufficiently large,** if $\mathcal{F}$ is a t-intersecting k-set system on an n-set then

1. $|\mathcal{F}| \leq \binom{n-t}{k-t}$,
2. and $\mathcal{F}$ meets this bound only if it is a trivially t-intersecting set system.

Qualitatively Independent Sets

Theorem (Kleitman and Spencer, Katona - 1973)

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Assume n is even.

- ⋆ By Sperner's Theorem $|\mathcal{F}| \leq \frac{1}{2}\binom{n}{\frac{n}{2}} = \binom{n-1}{\frac{n}{2}-1}$.
- ⋆ The set of all $\frac{n}{2}$-sets that contain 1 meets this bound.

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Assume n is odd.

- ⋆ If a set in $\mathcal{F}$ is larger than $\frac{n-1}{2}$, replace it with its complement (this makes n sufficiently large).
- ⋆ $\mathcal{F}$ is intersecting, by EKR, $|\mathcal{F}| \leq \binom{n-1}{\frac{n-1}{2}}$.
- ⋆ The set of all $\frac{n-1}{2}$-sets that contain 1 meets this bound.

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1 2 3 | 4 5 6 | 7 8 9	1 4 7 | 2 5 8 | 3 6 9

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⋆ A k-partition of an n-set is a set of k disjoint non-empty subsets (called classes) of the n-set whose union is the n-set.

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- ⋆ A k-partition of n-set is uniform if each class is size n/k.

Qualitative Independence

Partitions $P = \{P_1, \ldots, P_k\}$ and $Q = \{Q_1, \ldots, Q_k\}$ are qualitatively independent if

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Can we extend Sperner's Theorem and the Erdős-Ko-Rado Theorem to partitions?

Sperner Partition Systems

A k-partition system $\mathcal{P}$ is a Sperner partition system if for all distinct $P, Q \in \mathcal{P}$, with $P = \{P_1, \ldots, P_k\}$ and $Q = \{Q_1, \ldots, Q_k\}$,

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- $\star$ A partition system that is qualitatively independent is a Sperner partition system.

Sperner's Theorem for Partitions

Theorem (Meagher, Moura and Stevens - 2005) Let $\mathcal{F}$ be a Sperner k-partition system on an n-set.

- ⋆ If $n = ck$, then
  1. $|\mathcal{F}| \leq \frac{1}{k}\binom{n}{c} = \binom{ck-1}{c-1}$.
  2. Only uniform systems meet this bound.
- ⋆ If $n = ck + r$ with $0 \leq r < k$, then

$$|\mathcal{F}| \leq \frac{1}{(k-r) + \frac{r(c+1)}{n-c}}\binom{n}{c}.$$

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- ⋆ A trivially intersecting partition system is a partition system with all the partitions that contain a given class.
- ⋆ If $n = ck$, a trivially intersecting uniform k-partition system on an n-set has size

$$\frac{1}{(k-1)!}\binom{ck-c}{c}\binom{ck-2c}{c}\cdots\binom{c}{c}.$$

Erdős-Ko-Rado for Partitions

Theorem (Meagher and Moura - 2004)
Let $k, c \geq 1$ and $n = kc$.
Let $\mathcal{F}$ be an intersecting uniform k-partition system on an n-set. Then,

1. $|\mathcal{F}| \leq \frac{1}{(k-1)!}\binom{ck-c}{c}\binom{ck-2c}{c}\cdots\binom{c}{c}$.
2. If $\mathcal{F}$ meets this bound, then $\mathcal{F}$ is a trivially intersecting uniform partition system.

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Covering Arrays on Graphs

A covering array on a graph G, denoted $CA(n, G, k)$, is:

- $\star$ an $r \times n$ array where $r = |V(G)|$,
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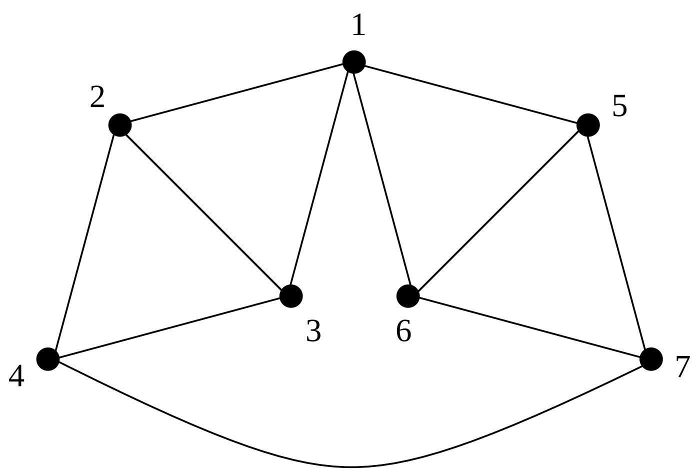

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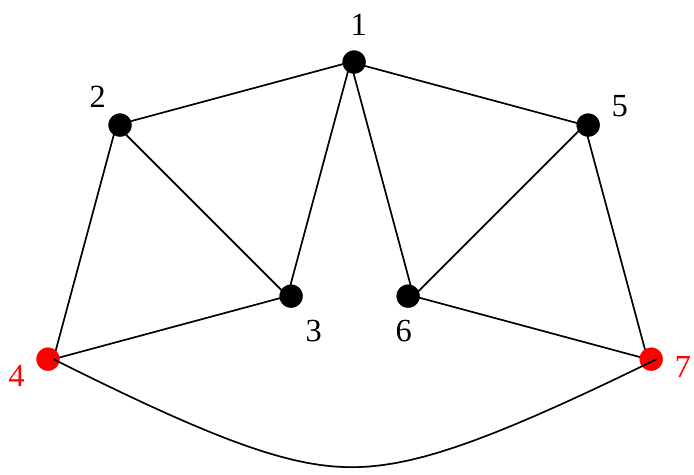

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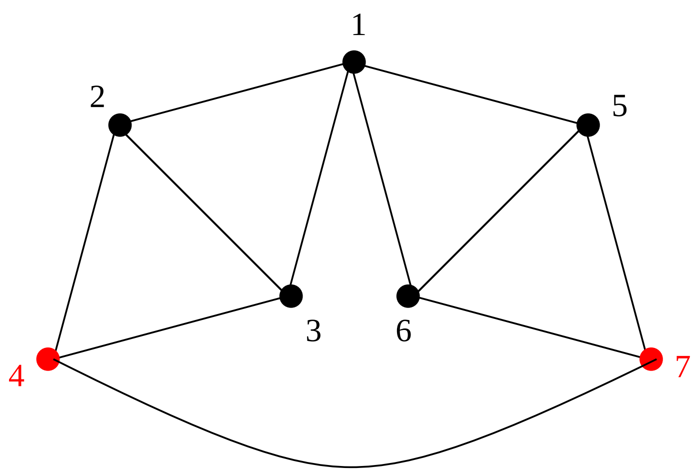

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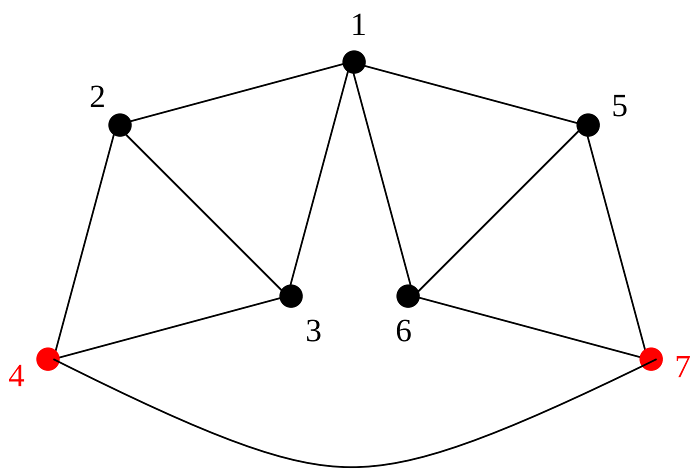

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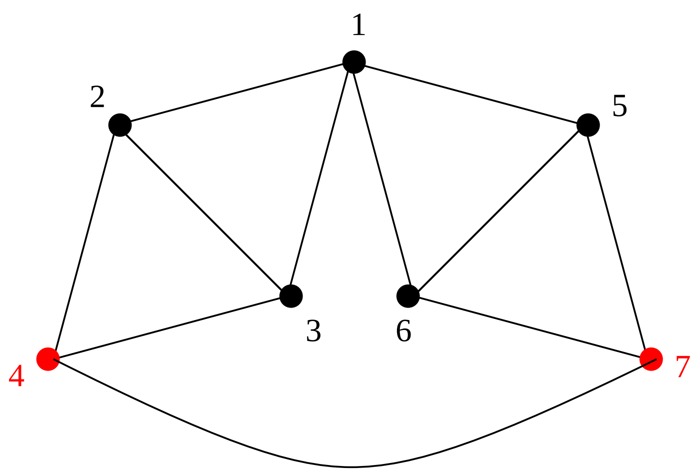

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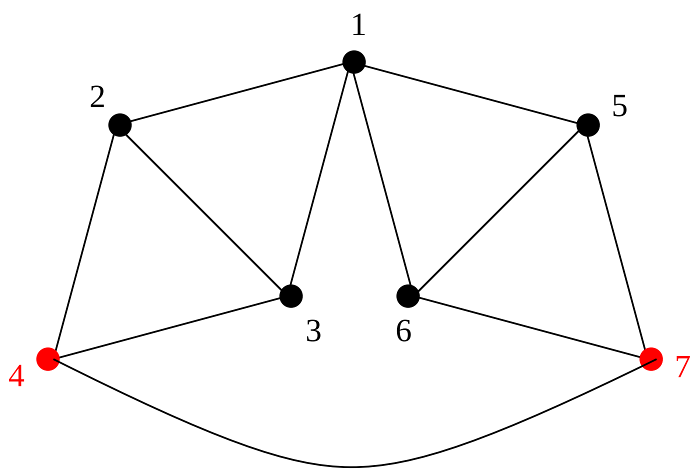

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Qualitative Independence Graph

Define the **qualitative independence graph** $QI(n,k)$ as follows:

- the vertex set is the set of all k-partitions of an n-set with every class of size at least k,
- and vertices are connected if and only if the partitions are qualitatively independent.

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The graph $QI(4, 2)$:

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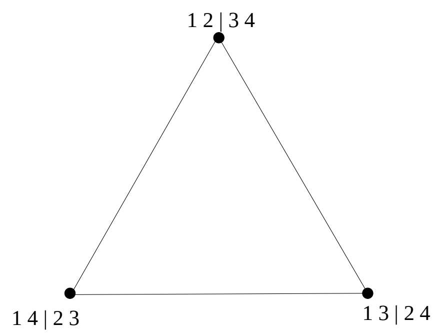
1	4	|	2	3



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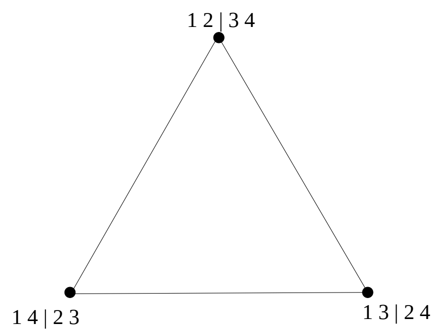

By construction, $CAN(QI(n,k),k) \leq n$.

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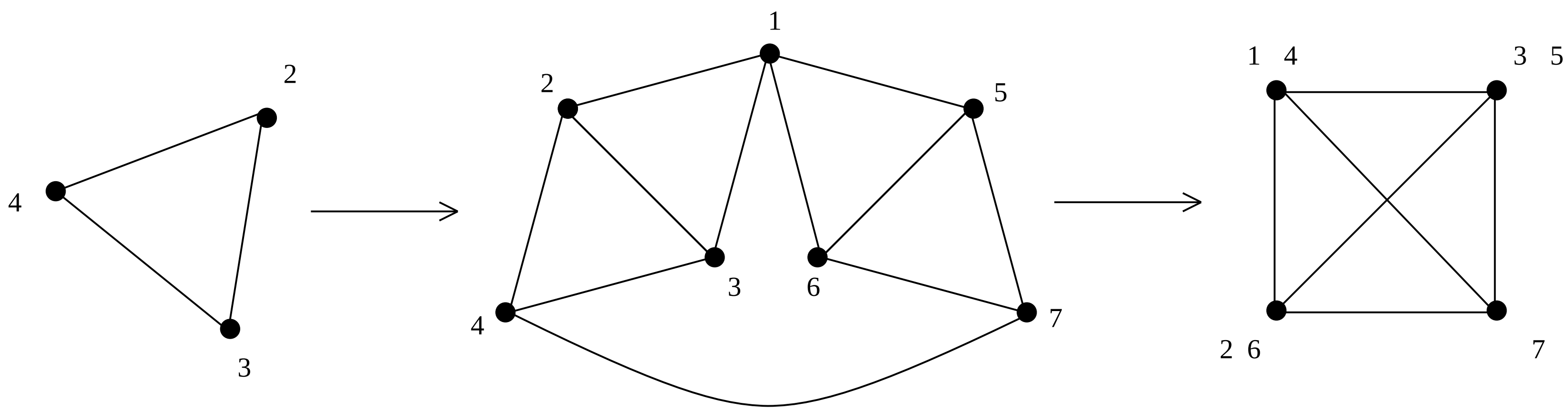

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Theorem (Meagher and Stevens - 2002) A covering array on a graph G with n columns and alphabet k exists if and only if there is a graph homomorphism

$$G \to QI(n,k).$$

Facts for $QI(n,2)$

Theorem (Kleitman and Spencer, Katona - 1973)

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Theorem (Meagher and Stevens - 2002)

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Theorem (Meagher and Stevens - 2002) If $CAN(G,2) \leq n$, then there exists a uniform binary covering array on G with n columns. (Each row has $\lceil \frac{n}{2} \rceil$ 0's and $\lfloor \frac{n}{2} \rfloor$ 1's.)

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$1\ 2\ 3 \mid 4\ 5\ 6 \mid 7\ 8\ 9 \qquad 1\ 2\ 7 \mid 4\ 5\ 8 \mid 3\ 6\ 9$

Is this set the largest independent set in $QI(k^2, k)$?