Covering Arrays on Graphs:

Extremal Partition Theory and Qualitative Independence Graphs

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Testing Systems

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A *covering array* $CA(n, r, k)$ is an $r \times n$ array with:

- entries from $\mathbb{Z}_k$ (k is the alphabet),
- and between any two rows all pairs from $\mathbb{Z}_k$ occur. (This property is called qualitative independence.)
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This is a $CA(5, 4, 2)$

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Two Areas of Covering Arrays

- Extremal Set Theory
  - ⋆ Sperner's Theorem and the Erdős-Ko-Rado Theorem can be used for binary covering arrays.
  - ⋆ Extend such results to partition systems.

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- Extremal Set Theory
  - ⋆ Sperner's Theorem and the Erdős-Ko-Rado Theorem can be used for binary covering arrays.
  - ⋆ Extend such results to partition systems.
- Graph Theory
  - ⋆ Add a graph structure to covering arrays.
  - ⋆ Use methods from algebraic graph theory.

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0 0 1 1 1	and	0 1 0 1 1	and	0 0 1 1 0
1 2 3 4 5		1 2 3 4 5		1 2 3 4 5
{3, 4, 5}	and	{2, 4, 5}	and	{3, 4}

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- A set system is a collection of subsets of an n-set.
- The rows of a binary covering array correspond to a set system.

Qualitatively Independent Sets

A set system $\mathcal{F}$ is qualitatively independent if for distinct sets $A, B \in \mathcal{F}$,

$$A \cap B \neq \emptyset \quad \overline{A} \cap B \neq \emptyset \quad A \cap \overline{B} \neq \emptyset \quad \overline{A} \cap \overline{B} \neq \emptyset.$$

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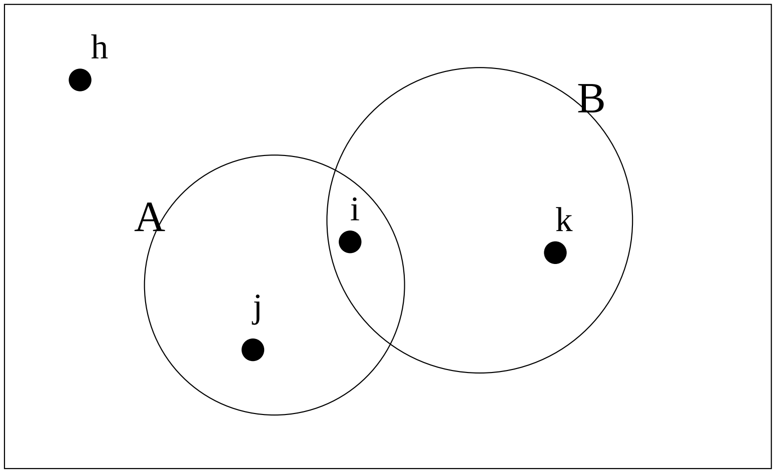

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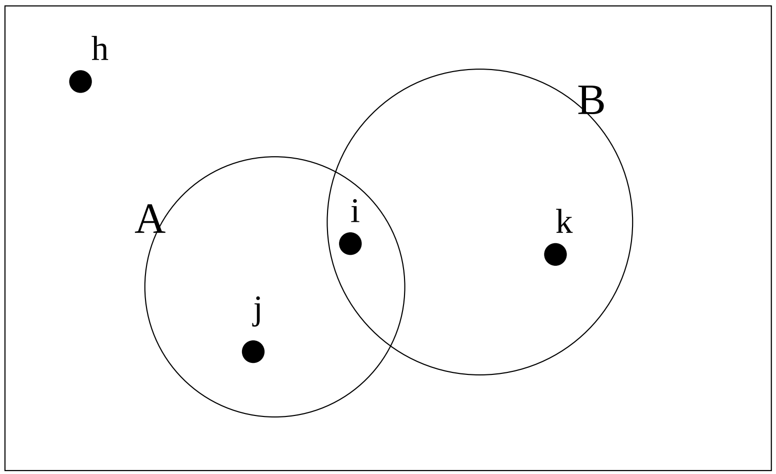

If sets A and B are qualitatively independent, knowing $i \in A$ gives no information if $i \in B$.

Sperner Set Systems

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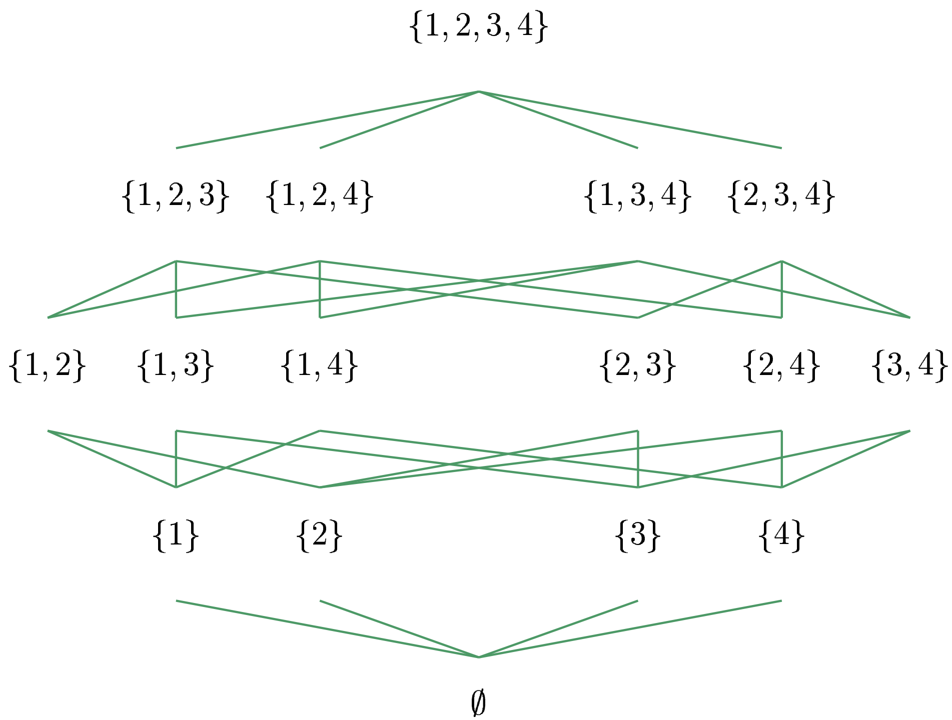

Sperner's Theorem

Theorem (Sperner - 1928) Let $\mathcal{F}$ be a Sperner set system over an n-set. Then

1. $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$.
2. Equality holds if and only if $\mathcal{F}$ is the system of all sets of size $\lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$.

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If $\mathcal{F}$ is a qualitatively independent set system on an n-set,

$$\mathcal{F}^* = \{A, \overline{A} : A \in \mathcal{F}\}$$

is a Sperner set system. In particular,

$$|\mathcal{F}| \leq \frac{1}{2}\binom{n}{\lfloor n/2 \rfloor}.$$

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A trivially t-intersecting k-set system on an n-set has cardinality

$$\binom{n-t}{k-t}.$$

Erdős-Ko-Rado Theorem

Theorem (Erdős, Ko and Rado - 1961) For n **sufficiently large,** if $\mathcal{F}$ is a t-intersecting k-set system on an n-set then

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- **Wilson 1984 -** For $n \geq (t+1)(k-t+1)$ the largest t-intersecting set system is a trivially t-intersecting set system.
- **Ahlswede and Khachatrian 1997 -** Gave the maximal t-intersecting k-set system for all values of n.

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Theorem (Kleitman and Spencer, Katona - 1973)

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Assume n is even.

- By Sperner's Theorem $|\mathcal{F}| \leq \frac{1}{2}\binom{n}{\frac{n}{2}} = \binom{n-1}{\frac{n}{2}-1}$.
- The set of all $\frac{n}{2}$-sets that contain 1 meets this bound.

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$\mathcal{F}$ is a qualitatively independent set system on an n-set.

Assume n is odd.

- If a set in $\mathcal{F}$ is larger than $\frac{n-1}{2}$, replace it with its complement (this makes n sufficiently large).
- $\mathcal{F}$ is intersecting, by EKR, $|\mathcal{F}| \leq \binom{n-1}{\frac{n-1}{2}}$.
- The set of all $\frac{n-1}{2}$-sets that contain 1 meets this bound.

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- A k-partition of an n-set is a set of k disjoint non-empty subsets (called classes) of the n-set whose union is the n-set.

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- A k-partition of n-set is uniform if each class is size n/k.

Qualitative Independence

Partitions $P = \{P_1, \ldots, P_k\}$ and $Q = \{Q_1, \ldots, Q_k\}$ are qualitatively independent if

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Can we extend Sperner's Theorem and the Erdős-Ko-Rado Theorem to partitions?

Sperner Partition Systems

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- If a partition system is qualitatively independent, then it is also a Sperner partition system.
- A Sperner partition system can be considered as a resolvable Sperner set system.

Sperner's Theorem for Partitions

Theorem (Meagher, Moura and Stevens - 2005) Let $\mathcal{F}$ be a Sperner k-partition system on an n-set.

- If $n = ck$, then
  1. $|\mathcal{F}| \leq \frac{1}{k}\binom{n}{c} = \binom{ck-1}{c-1}$.
  2. Only uniform systems meet this bound.
- If $n = ck + r$ with $0 \leq r < k$, then

$$|\mathcal{F}| \leq \frac{1}{(k-r) + \frac{r(c+1)}{n-c}}\binom{n}{c}.$$

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Theorem (P. L. Erdős and Székely - 2000) For n sufficiently large, the largest intersecting k-partition system on an n-set is a trivially intersecting partition system.

Intersecting Uniform Partitions

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- If $n = ck$, a trivially intersecting uniform k-partition system on an n-set has size

$$\frac{1}{(k-1)!}\binom{ck-c}{c}\binom{ck-2c}{c}\cdots\binom{c}{c}.$$

Erdős-Ko-Rado for Partitions

Theorem (Meagher and Moura - 2004) Let $k, c \geq 1$ and $n = kc$. Let $\mathcal{F}$ be an intersecting uniform k-partition system on an n-set. Then,

1. $|\mathcal{F}| \leq \frac{1}{(k-1)!}\binom{ck-c}{c}\binom{ck-2c}{c}\cdots\binom{c}{c}$.
2. If $\mathcal{F}$ meets this bound, then $\mathcal{F}$ is a trivially intersecting uniform partition system.

Partially Intersecting Partitions

- Partitions $P = \{P_1, \ldots, P_k\}$ and $Q = \{Q_1, \ldots, Q_k\}$ are t-partially intersecting if there exist an i and a j so that

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Conjecture (Czabarka - 2000) For some values of n the largest partially 2-intersecting k-partition system on an n-set is a trivially intersecting partition system.

Uniform Partitions

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If $n = ck$, a trivially partially t-intersecting uniform k-partition system on an n-set has size

$$\binom{ck-t}{c-t}\frac{1}{(k-1)!}\binom{ck-c}{c}\binom{ck-2c}{c}\cdots\binom{c}{c}.$$

EKR for Partitions - version 2

Conjecture (Meagher and Moura - 2005) Let $n = ck$ and $\mathcal{F}$ be a partially t-intersecting uniform partition system. Then,

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2. $\mathcal{F}$ meets this bound only if it is a trivially partially t-intersecting uniform k-partition system.

We have partial results for several values of n, k and t.

Open Problems

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- Prove an Erdős-Ko-Rado type result for partially t-intersecting partitions.
- Develop a theory of extremal partition systems

Adding a Graph Structure

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A covering array on a graph G, denoted $CA(n, G, k)$, is:

- an $r \times n$ array where $r = |V(G)|$,
- with entries from $\mathbb{Z}_k$ (k is the alphabet),
- rows for **adjacent** vertices are qualitatively independent.
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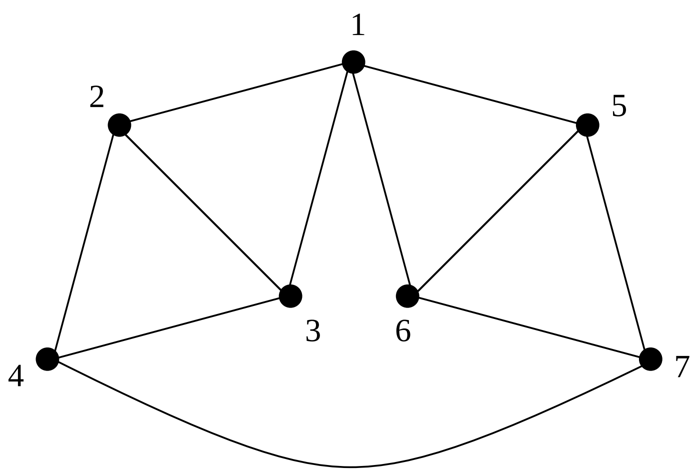

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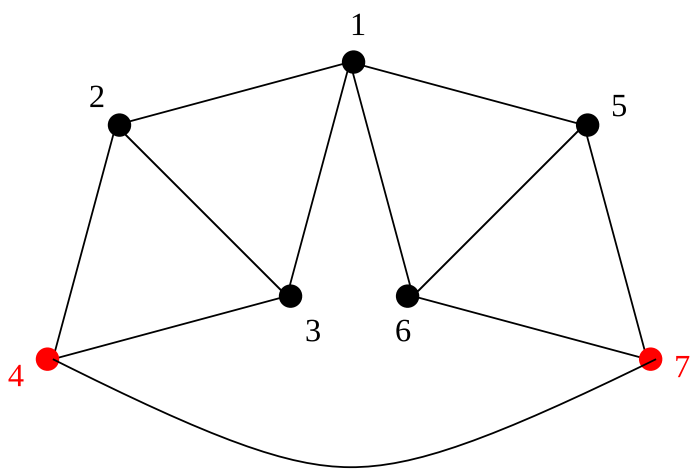

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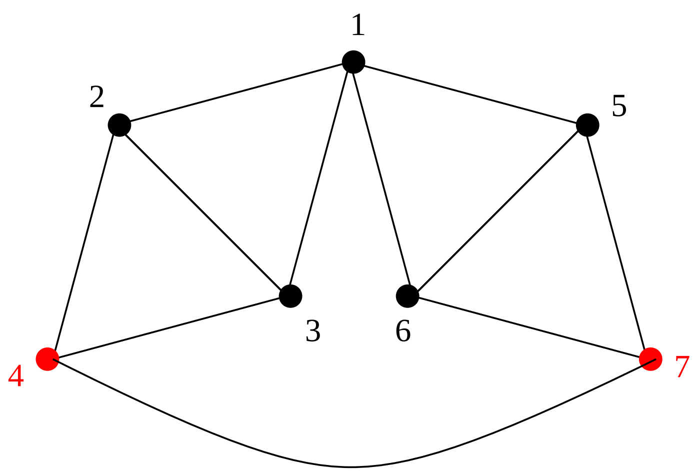

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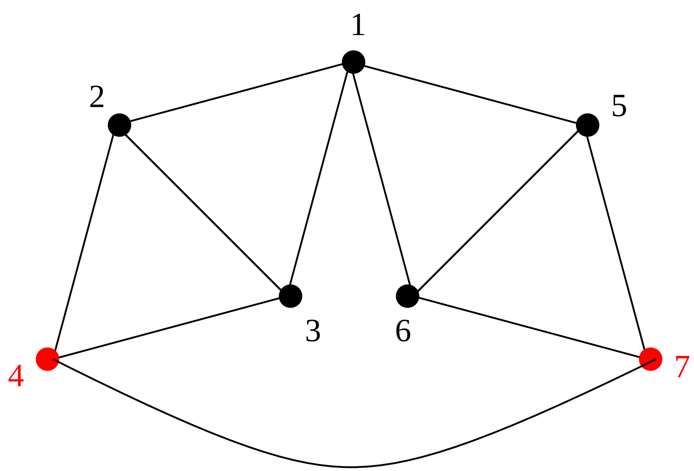

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Covering Arrays on Graphs

A covering array on a graph G, denoted $CA(n, G, k)$, is:

- an $r \times n$ array where $r = |V(G)|$,
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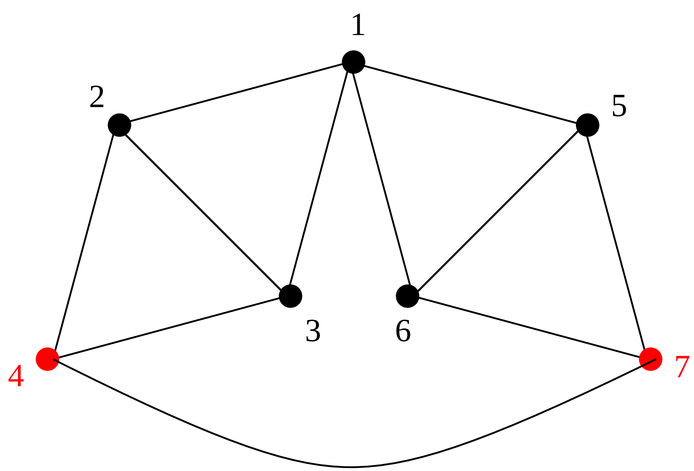

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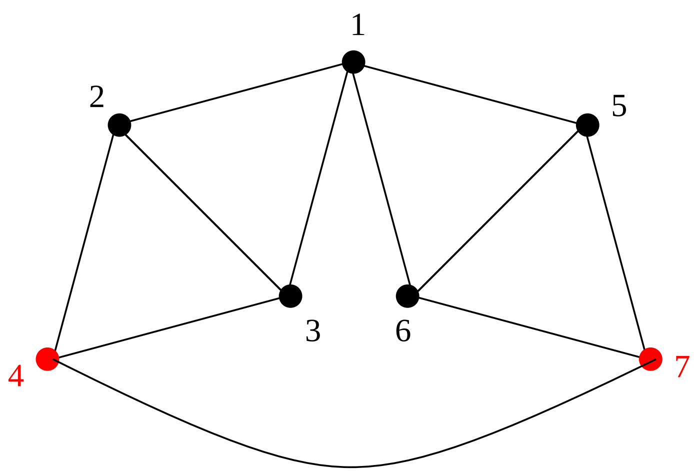

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A complete graph, denoted K_r, is the graph with r vertices and distinct vertices are adjacent.

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- Bshouty and Serroussi (1988) proved that finding $CAN(G, 2)$ for any graph is NP-hard.

Graph Homomorphisms

A homomorphism, $f : G \to H$ is

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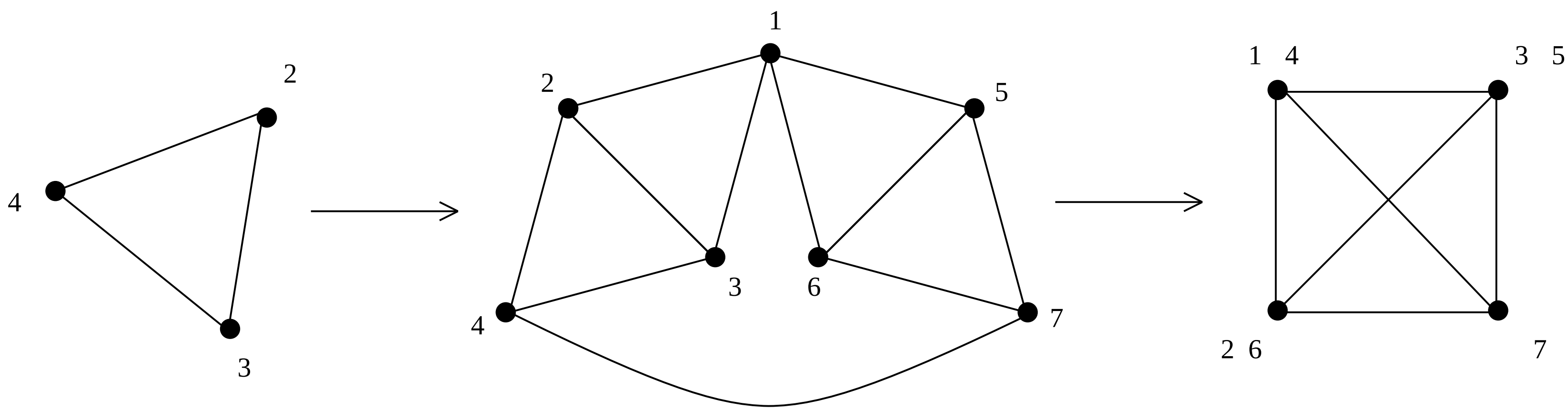

Bounds from Complete Graphs

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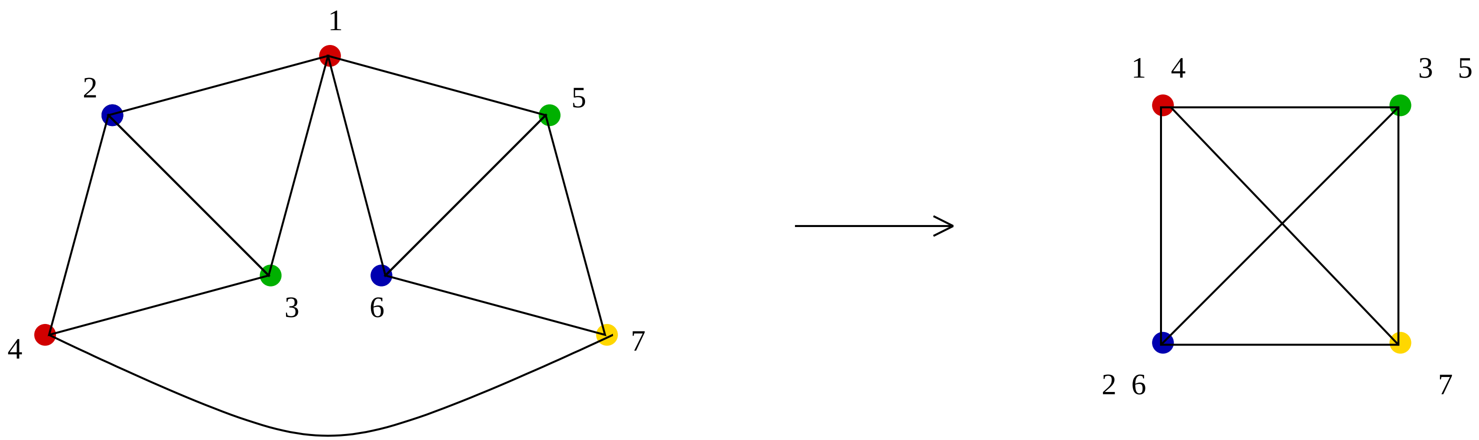

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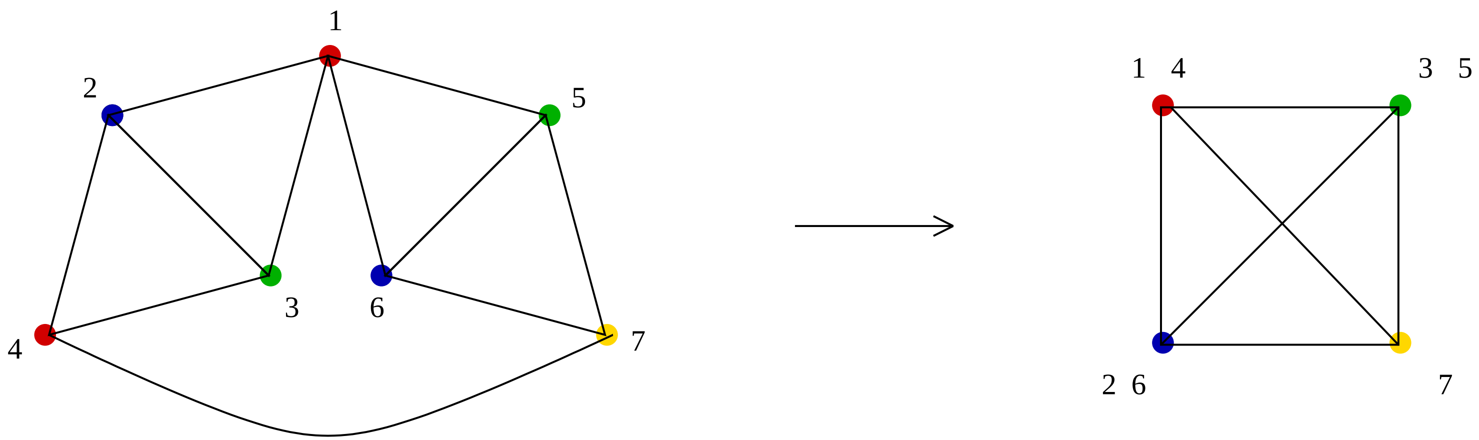

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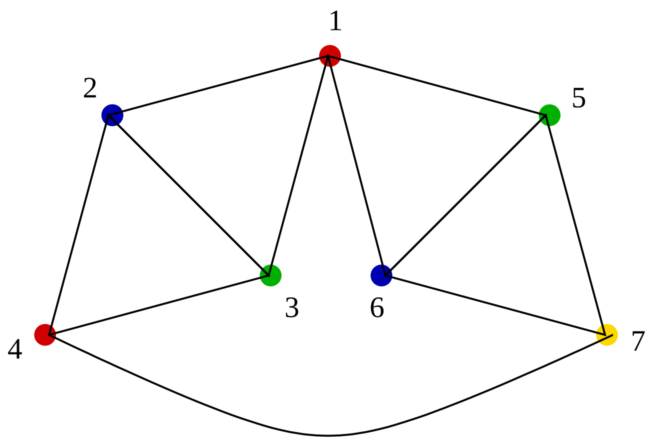

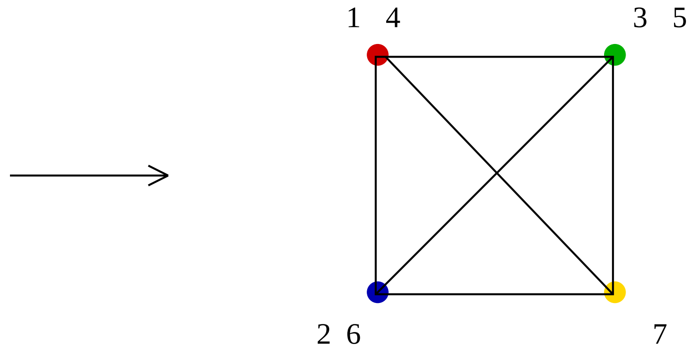

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Define the **qualitative independence graph** $QI(n, k)$ as follows:

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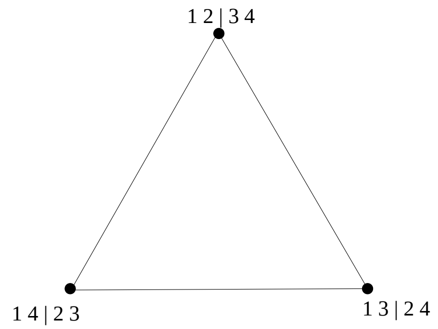

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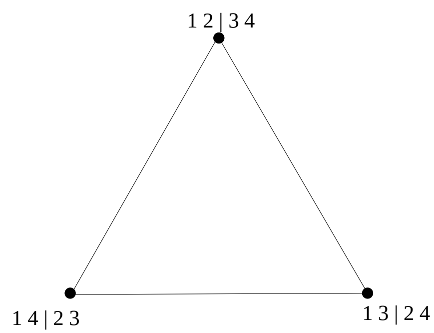
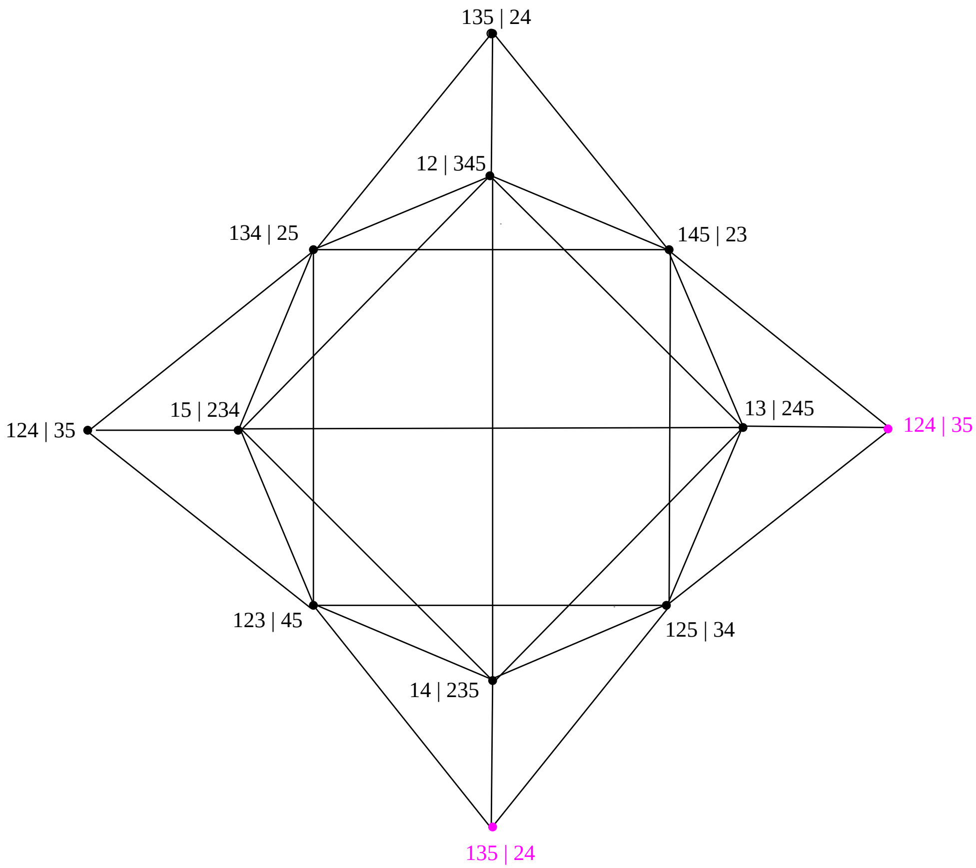


By construction, $CAN(QI(n, k), k) \leq n$.

The Graph $QI(5,2)$



Interesting facts about this graph

- $V(QI(5,2)) = 10$
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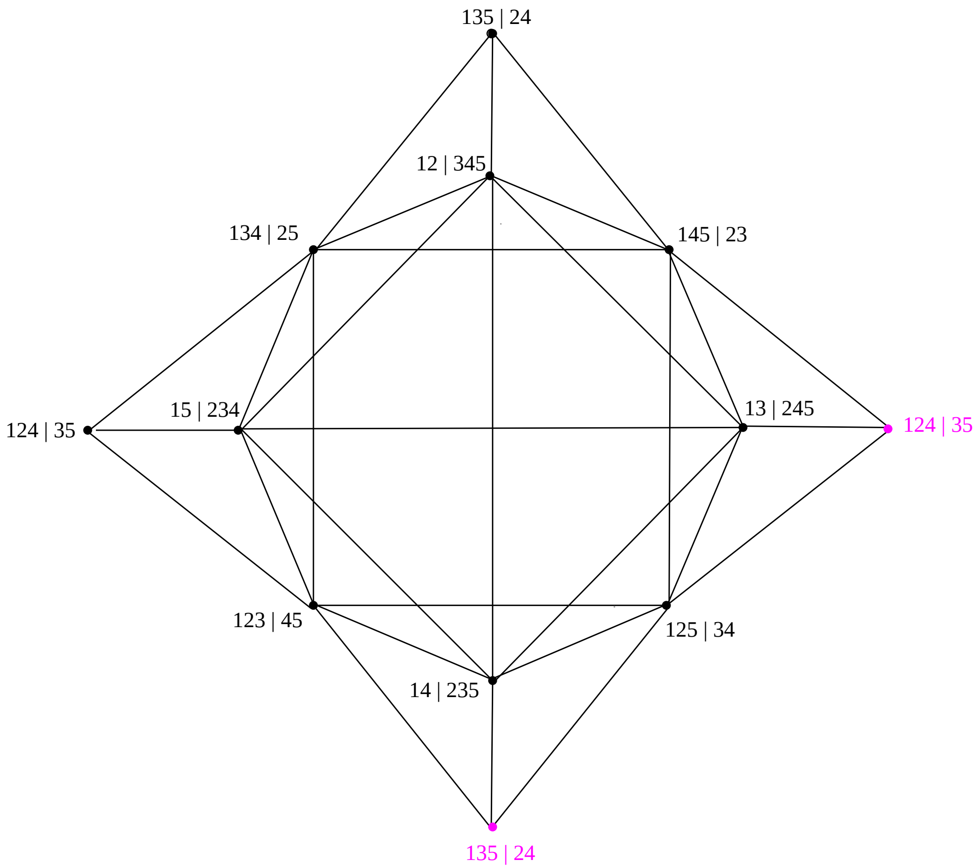

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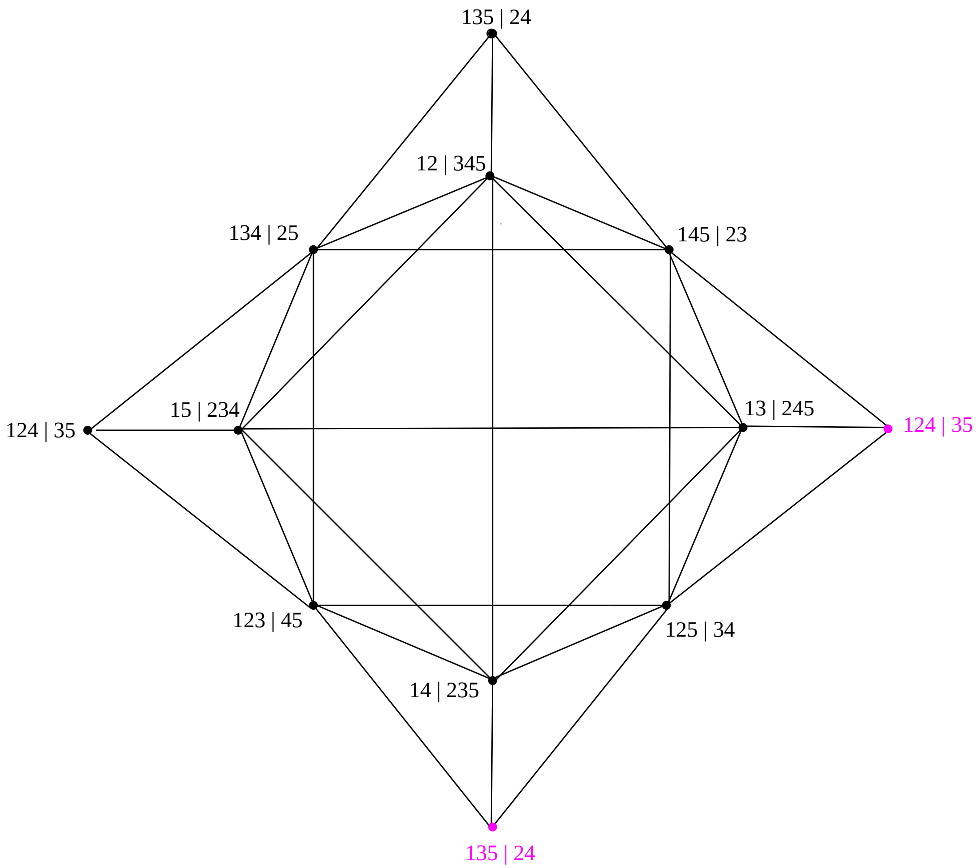

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Theorem (Meagher and Stevens - 2002) A covering array on a graph G with n columns and alphabet k exists if and only if there is a graph homomorphism

$$G \to QI(n,k).$$

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Theorem (Meagher and Stevens - 2002) If $CAN(G,2) \leq n$, then there exists a uniform binary covering array on G with n columns. (Each row has $\lceil \frac{n}{2} \rceil$ 0's and $\lfloor \frac{n}{2} \rfloor$ 1's.)

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  This is a partially 2-intersecting k-partition system on a k^2-set.

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Conjecture (Meagher 2005) $\chi(QI(k^2,k)) = \binom{k+1}{2}$.

Maximum Independent sets

We have two equivalent questions:

- Is this set the largest independent set in $QI(k^2, k)$?
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To solve this, we use algebraic graph theory!

Eigenvalues of Graphs

The adjacency matrix of a graph G on n vertices (labeled $1, 2, \ldots, n$) is an

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Ratio Bound Let G be a vertex transitive graph on n vertices with largest eigenvalue d and least eigenvalue τ. Then

$$\alpha(G) \leq \frac{n}{1 - \frac{d}{\tau}}.$$

Equitable Partitions

Equitable partition for a graph G:

- partition π of $V(G)$ with cells $C_1, C_2, \ldots, C_r$,
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The adjacency matrix of the quotient graph $QI(k^2, k)/\pi$ is

$$\begin{pmatrix} 0 & k!^{k-1} \\ \frac{k!^{k-1}}{k} & k!^{k-1} - \frac{k!^{k-1}}{k} \end{pmatrix}$$

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For $QI(k^2, k)$ and $QI(k^2, k)/\pi$, the largest eigenvalue is $(k!)^{k-1}$ and the smallest eigenvalue is $\frac{-(k!)^{k-1}}{k}$.

Advanced Facts for $QI(k^2, k)$

Using the eigenvalues with the ratio bound,

$$\alpha(QI(k^2,k)) = \frac{1}{(k-1)!}\binom{k^2-2}{k-2}\binom{k^2-k}{k}\cdots\binom{k}{k}.$$

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Theorem (Godsil and Meagher - 2005) Let $n = k^2$ and $\mathcal{F}$ be a partially 2-intersecting uniform k-partition system. Then,

$$|\mathcal{F}| \leq \binom{k^2-2}{k-2}\frac{1}{(k-1)!}\binom{k^2-k}{k}\binom{k^2-2}{k}\cdots\binom{k}{k}$$

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Theorem (Godsil and Newman - 2003) For $k = 3$ all maximum independent sets in $QI(9,3)$ are trivially partially 2-intersecting uniform partition systems.

Their proof used the following fact:

Theorem (Mathon and Rosa - 1985) The graph $QI(9,3)$ is a single graph in an association scheme.

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- In general the extension is a coherent configuration not an association scheme.
- These permutation matrices are representations of the symmetric group acting on cosets for a wreath product
- We do have an association scheme when this representation is multiplicity-free.