Association Schemes and Set-Partition Systems

Karen Meagher

joint work with Chris Godsil

karen.meagher@uregina.ca

University of Regina

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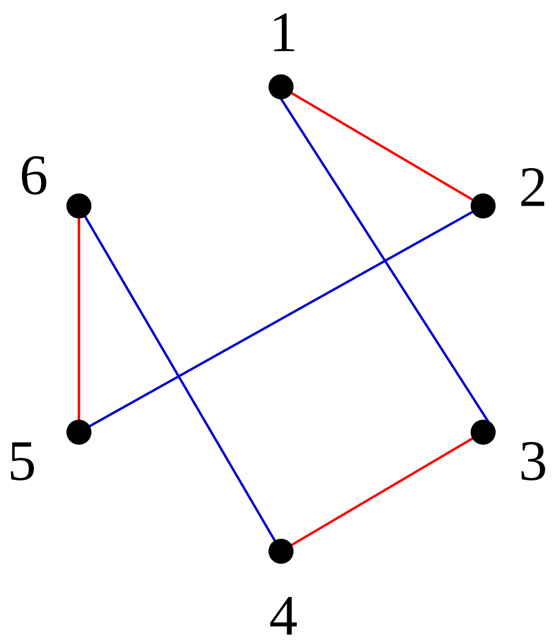

12 | 34 | 56

13 | 25 | 46

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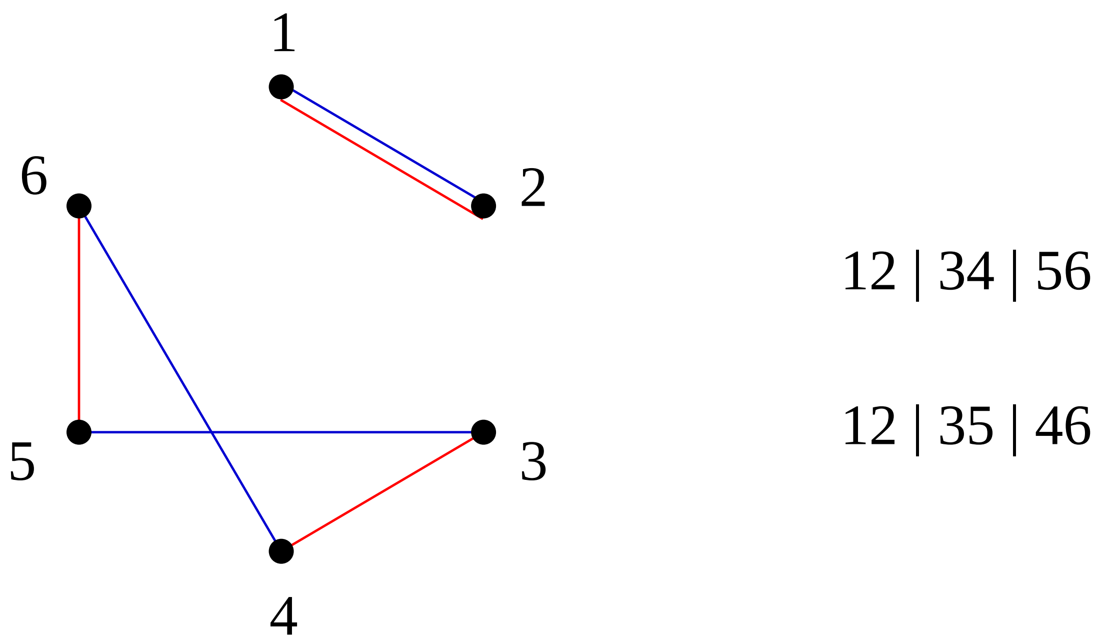

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An example of a uniform 3-partition of a 12-set is

$$P = 1\,2\,3\,4\,|\,5\,6\,7\,8\,|\,9\,10\,11\,12.$$

Meet Tables

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For $P = 123|456|789$ and $Q = 147|256|389$,

	Q_1	Q_2	Q_3
P_1	1	1	1
P_2	1	2	0
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Two tables are **isomorphic** if some permutation of the rows and columns of one table produces the other.

A Scheme?

- $\star$ Let $\{T_1, T_2, \ldots, T_t\}$ be the set of all non-isomorphic meet tables between uniform k-partitions of a $k\ell$-set.

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Is $\mathcal{A}^{k,\ell} = \{A_i : i = 1, \dots, t\}$ an association scheme?

Generalization for Sets

For uniform 2-partitions, $\mathcal{A}^{2,\ell}$:

$$P = 123|456 \text{ and } Q = 124|356,$$

$M_{P,Q} =$

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$\mathcal{A}^{2,\ell}$ is an assoication scheme.

Generalization for Matchings

For perfect matchings, $\mathcal{A}^{k,2}$:

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3×3 **Partitions**

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Table of eigenvalues:

$$\left(\begin{array}{ccccc|c} 1 & 27 & 162 & 54 & 36 & 1 \\ 1 & 11 & -6 & 6 & -12 & 27 \\ 1 & 6 & -6 & -9 & 8 & 48 \\ 1 & -3 & 12 & -6 & -4 & 84 \\ 1 & -3 & -6 & 6 & 2 & 120 \end{array}\right)$$

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A *commutative* homogenous coherent algebra is the Bose-Mesner algebra of an association scheme.

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This is the representation on S_n induced by the trivial representation on G,

$$\mathrm{ind}_{S_n}(1_G).$$

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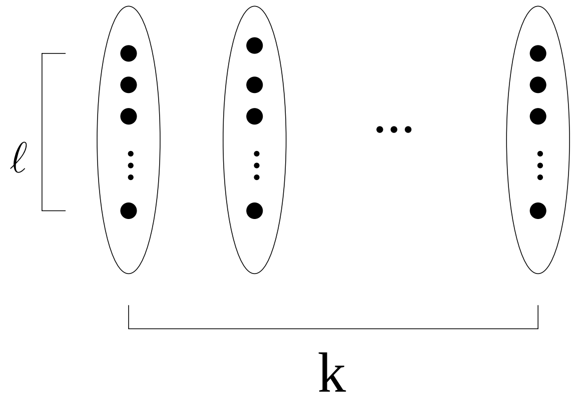

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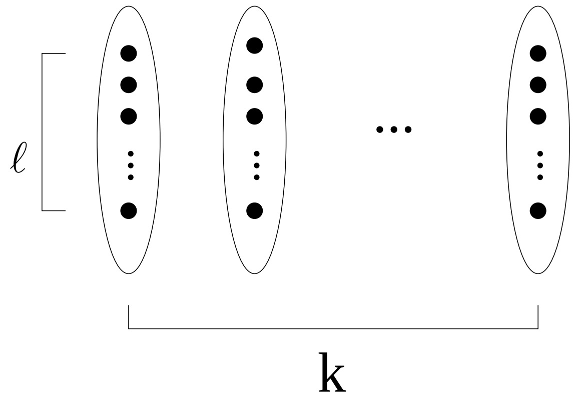

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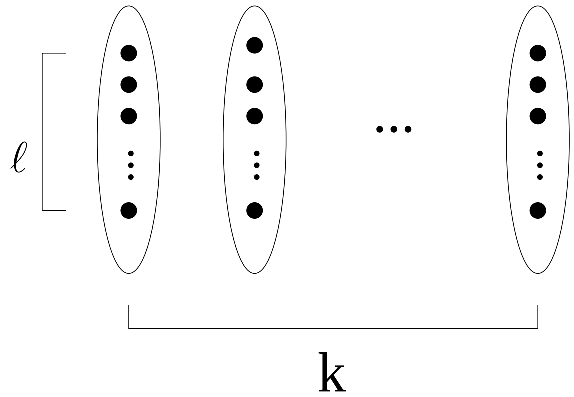

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⋆ Each uniform k-partition of a $k\ell$-set corresponds to a coset in $S_{k\ell}/(S_\ell \wr S_k)$.

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The algebra generated by $\mathcal{A}^{k,\ell}$
is the commutant of the matrices
in the representation
$\mathrm{ind}_{S_n}(1_{S_k \wr S_\ell})$.

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$\mathcal{A}^{k,\ell}$ is an association scheme if and only if $\mathrm{ind}_{S_n}(1_{S_k \wr S_\ell})$ is multiplicity-free.

Two known cases:

Sets

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Perfect matchings

$$\mathrm{ind}_{S_{2k}}(1_{S_2 \wr S_k}) = \sum_{\lambda \vdash k} \chi_{2\lambda}$$

Where $2\lambda = (2\lambda_1, 2\lambda_2, \ldots, 2\lambda_m)$ for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$.

General Partitions

Theorem (Godsil and M. 2006) $\mathrm{ind}_{S_{\ell k}}(1_{S_\ell \wr S_k})$ is multiplicity-free if and only if (ℓ, k) is one of the following pairs:

(a) $(\ell, k) = (2, k)$;

(b) $(\ell, k) = (\ell, 2)$;

(c) (ℓ, k) is one of $(3, 3), (3, 4), (4, 3)$ or $(5, 3)$;

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$\mathcal{A}^{k,\ell}$ is a **homogeneous coherent configuration** for all values of k, ℓ.

The Erdős-Ko-Rado Theorem

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A **trivial intersecting k-set system** is the collection of all k-sets containing a fixed point:

123	124	125	126	134
135	136	145	146	156

A trivial intersecting 3-set system on a 6-set.

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Theorem (Erdős-Ko-Rado 1961)

For $n > 2k$

the maximal intersecting k-set system is a trivial intersecting system.

2-Partially Intersecting

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are **2-partially intersecting** if there exist an i and a j so that

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How big can a
2-partially intersecting uniform k-partition
system be?

A Trivial System

A **trivial 2-partially intersecting uniform k-partition system** is the set of all uniform k-partitions with a cell with a fixed pair.

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⋮		
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Conjecture:

A maximum 2-partially intersecting uniform partition system is a trivial 2-partially intersecting system.

Towards a Proof

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  Proof is a counting argument using "kernel method".
- ⋆ True for 3-partitions of a 9-set
  (Godsil and Newman, 2005)
  Proof uses the structure of the association scheme and the eigenvalues.

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- There is a basis of this irreducible module with trivially intersecting systems.
- The only linear combinations of this basis that produce maximum independent sets give trivially intersecting systems