Covering Arrays on Graphs

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What is a Covering Array?

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Beer	Olives	Costume	Math Friends	7:30
Beer	Tofu	Formal	Girls Only	10:00
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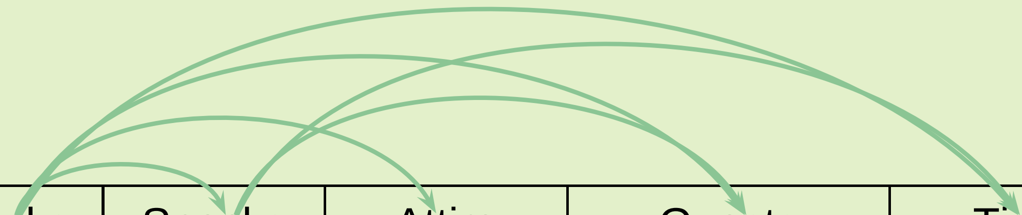

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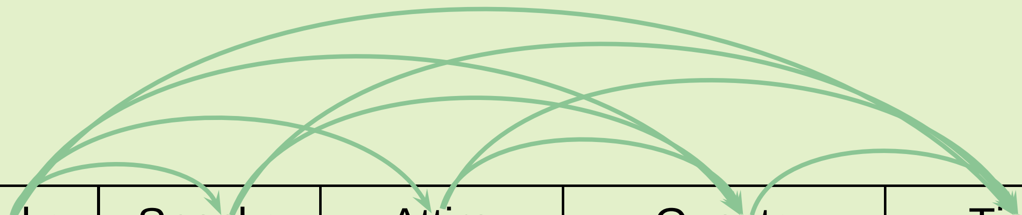

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Covering Arrays on Graphs

A *covering array on a graph* G denoted $CA(n, G, g)$, has:

- $k = |V(G)|$ rows of length n (n is the size)
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- rows for adjacent vertices have all pairs from $\mathbb{Z}_g$

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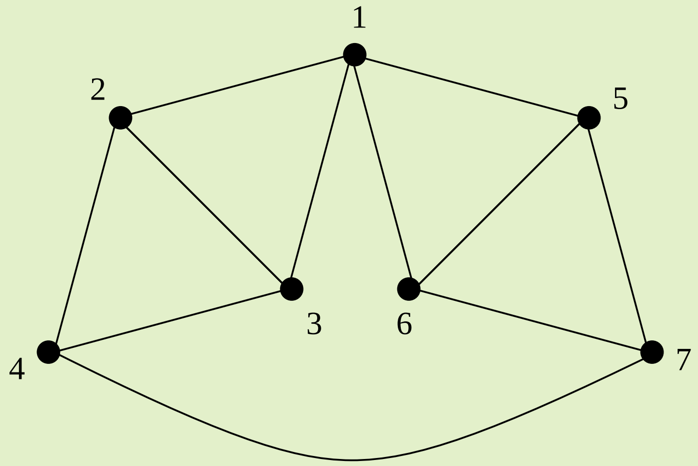

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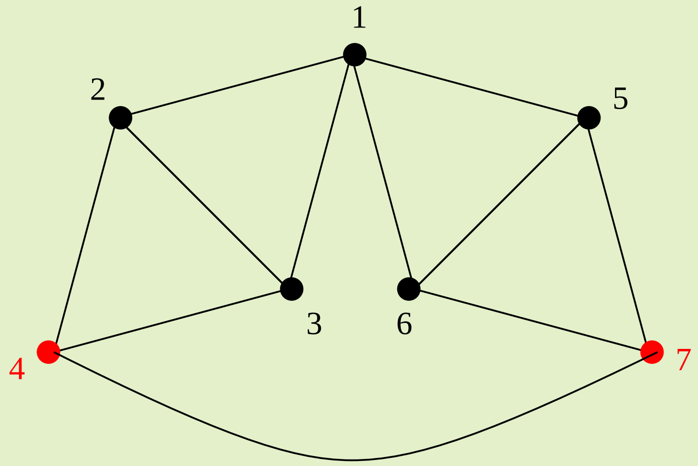

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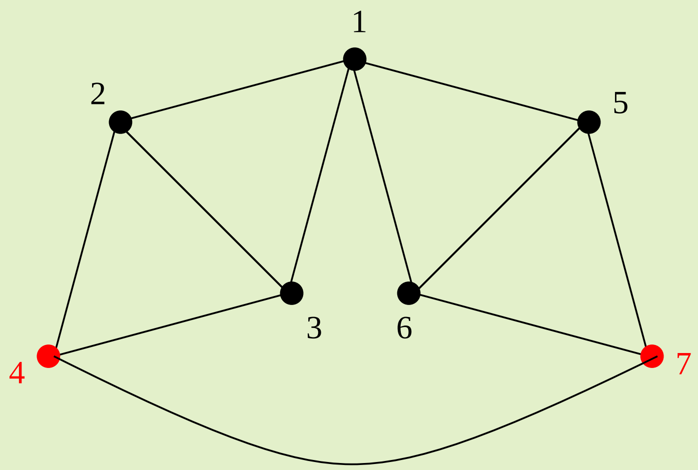

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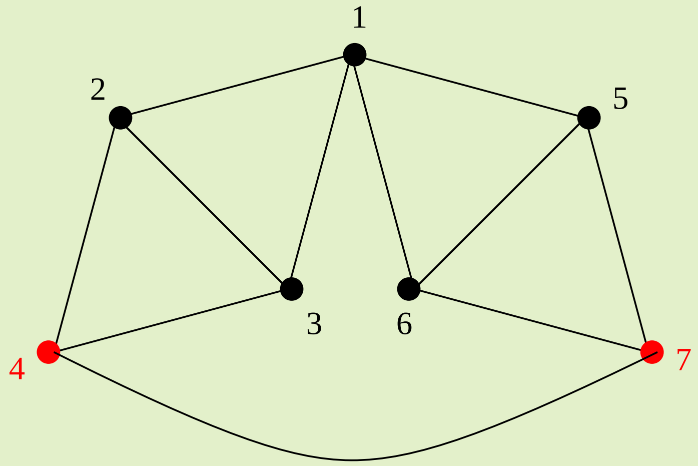

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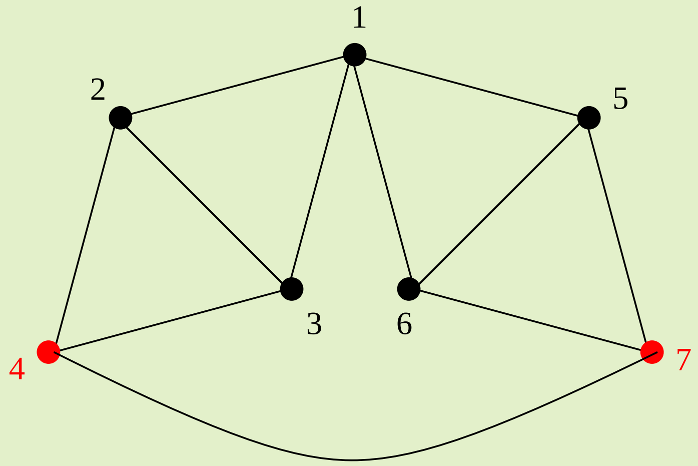

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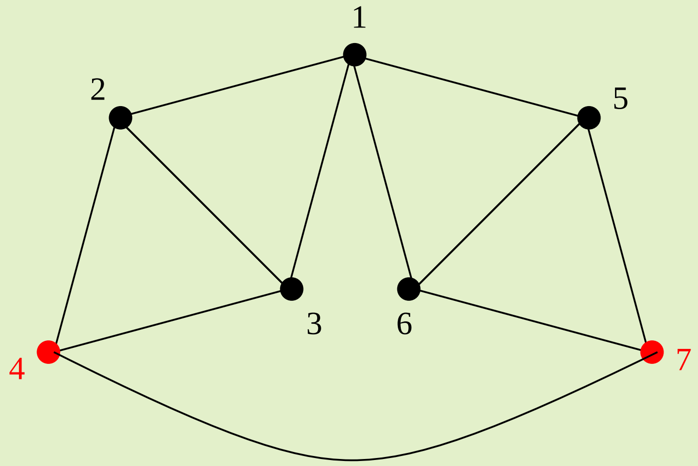

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- Normal covering arrays correspond to covering arrays on complete graphs

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- There is a bound from the chromatic number

$$CAN(G, g) \leq CAN(\chi(G), g).$$

A New Family of Graphs

Define graph $G(n, g)$ by

- Vertices are all sequences which could go into a row of a covering array of size n on $\mathbb{Z}_g$.
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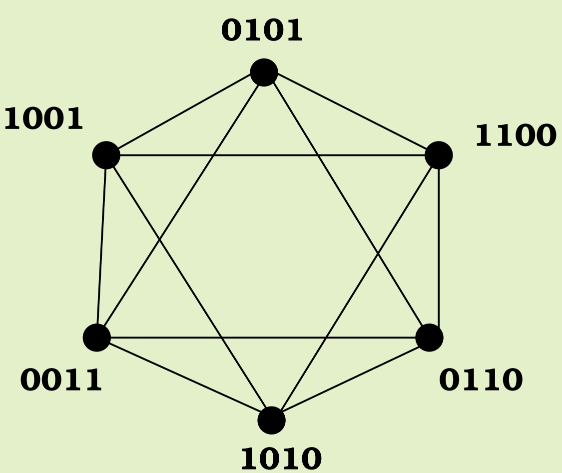

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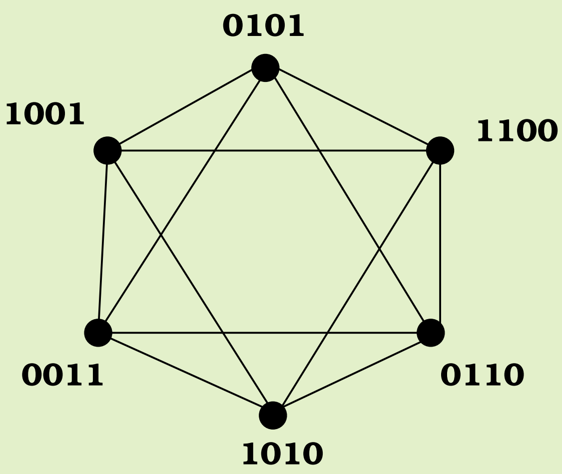

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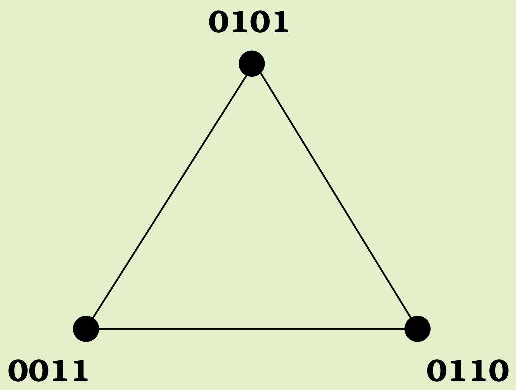
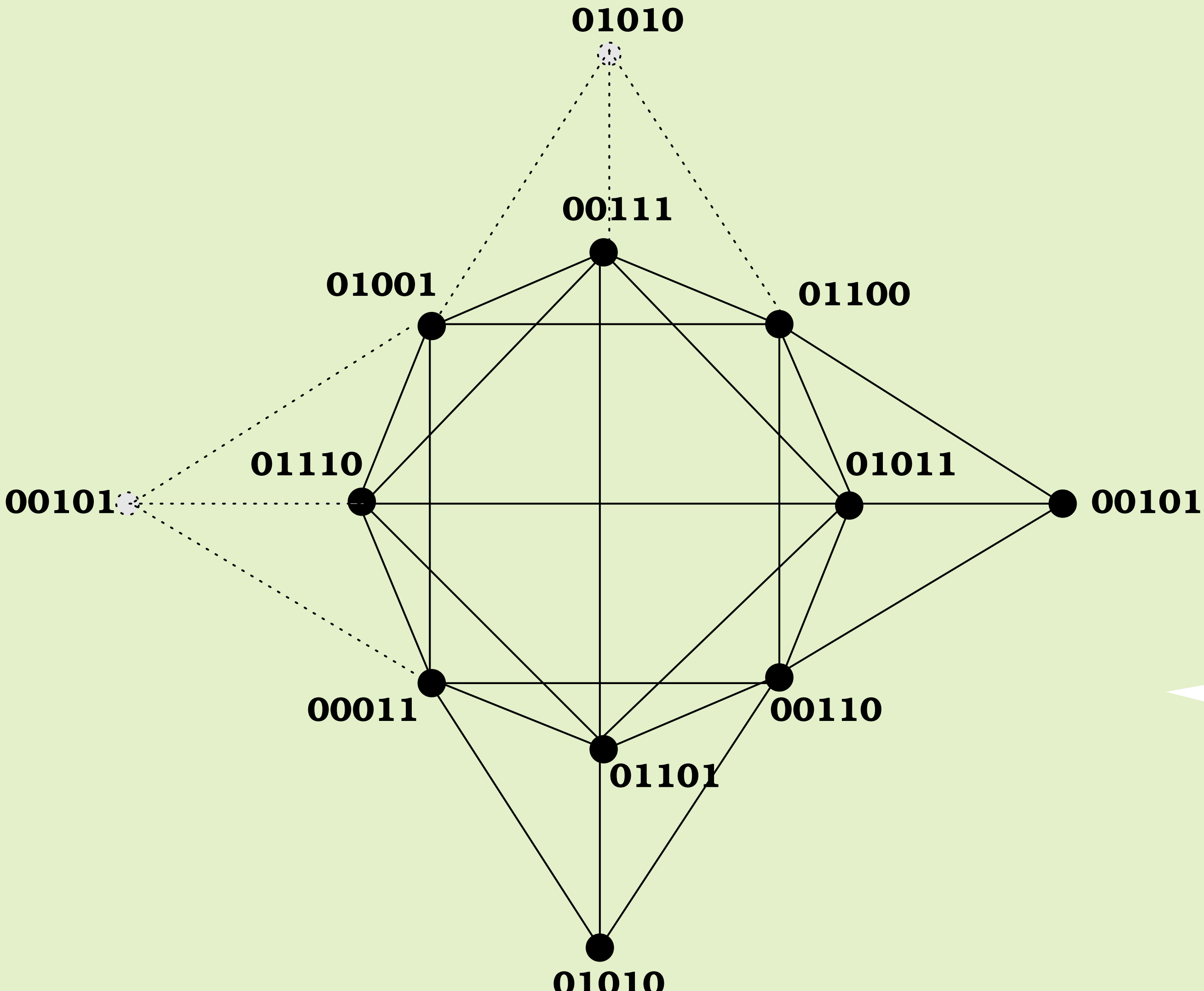


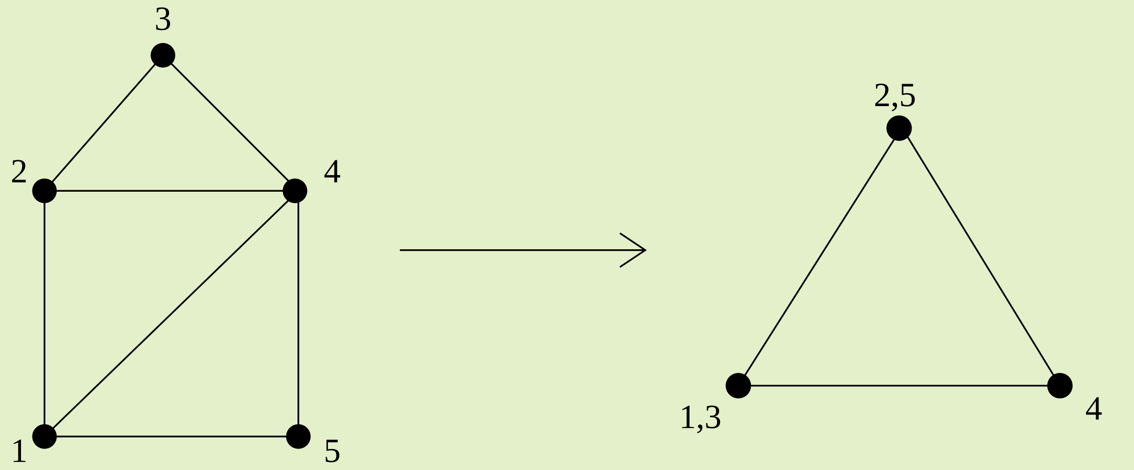
The graph $G(4, 2)$.

The Graph $G(5, 2)$



Graph Homomorphisms

- an edge preserving map between two graphs
- if vertices $u, v \in G$ are connected then vertices $f(u), f(v) \in H$ are also connected



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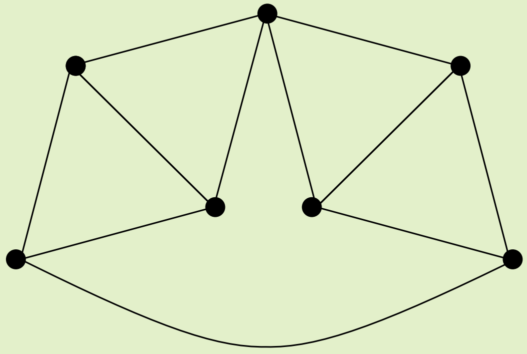

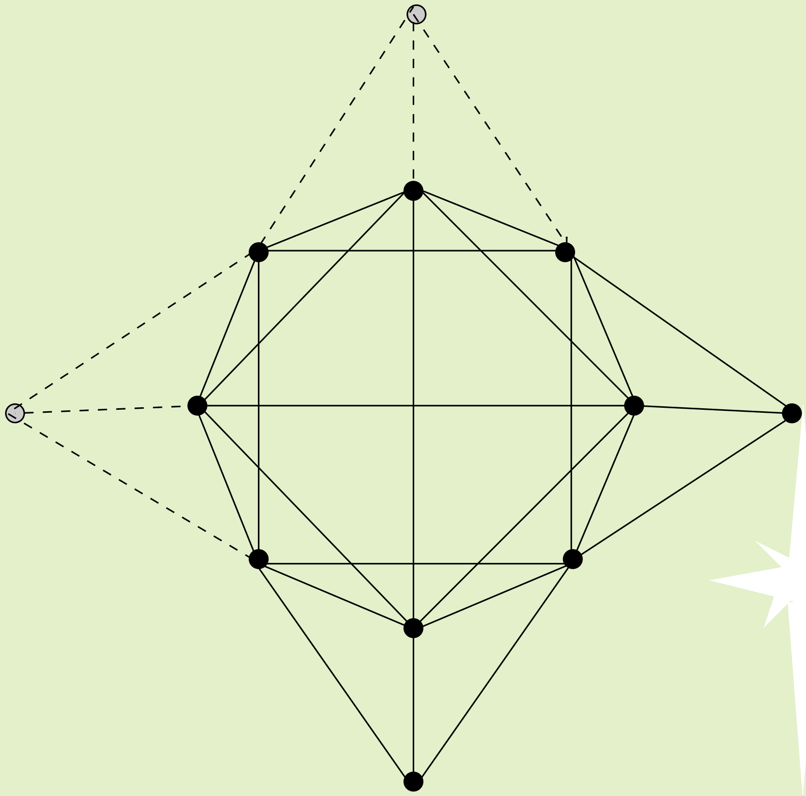

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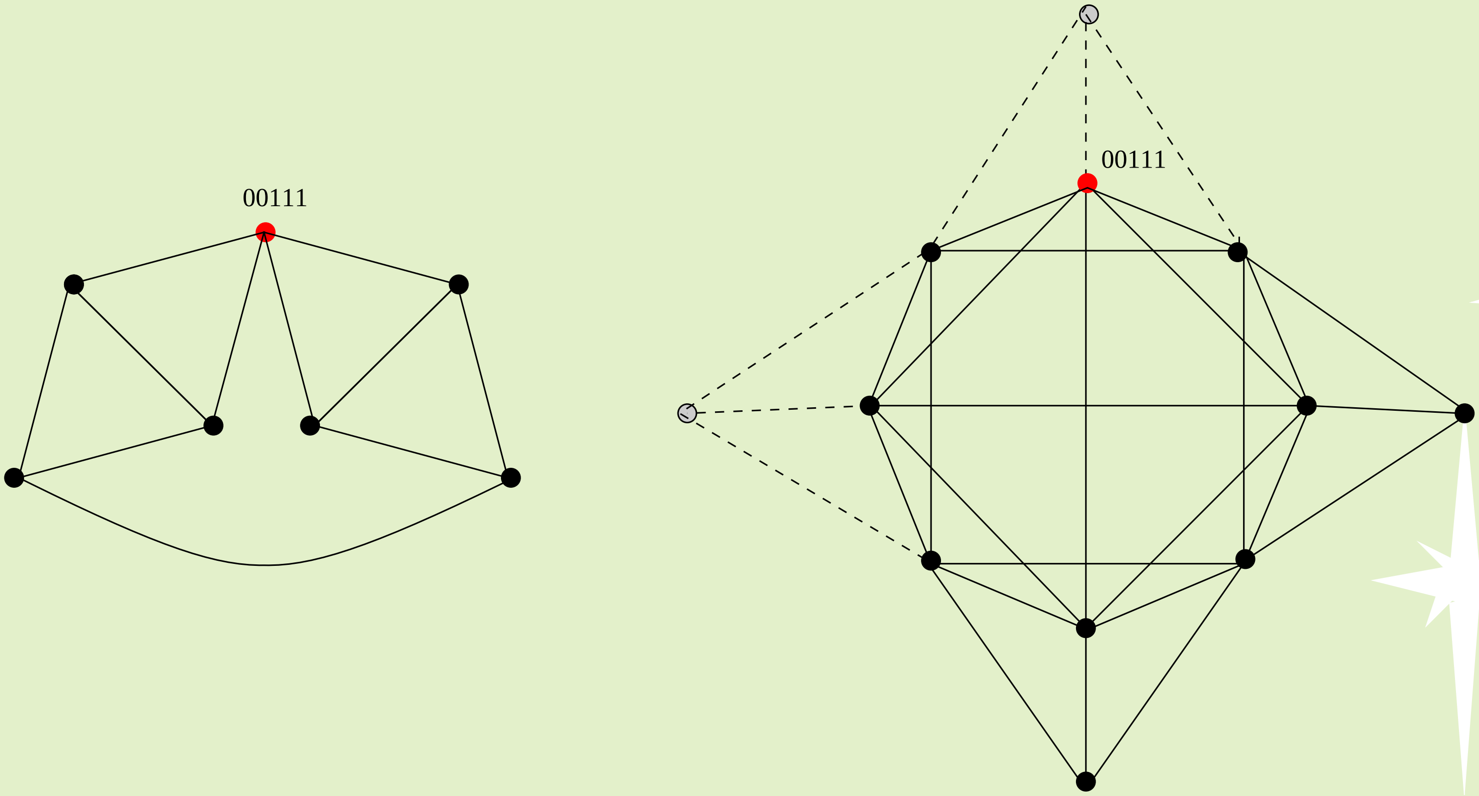

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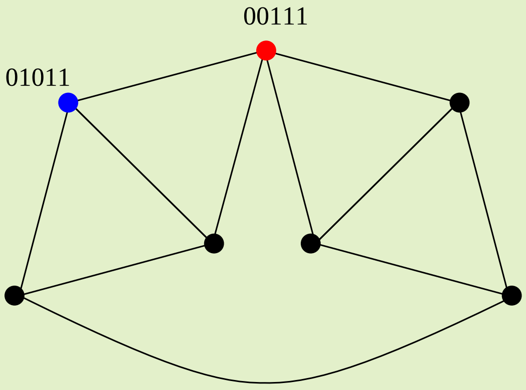

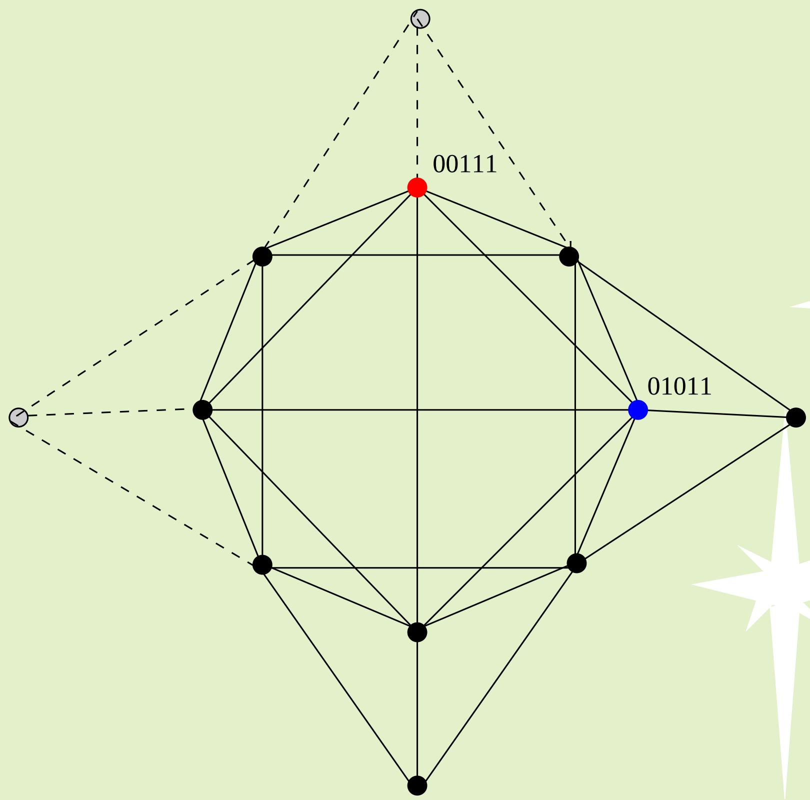

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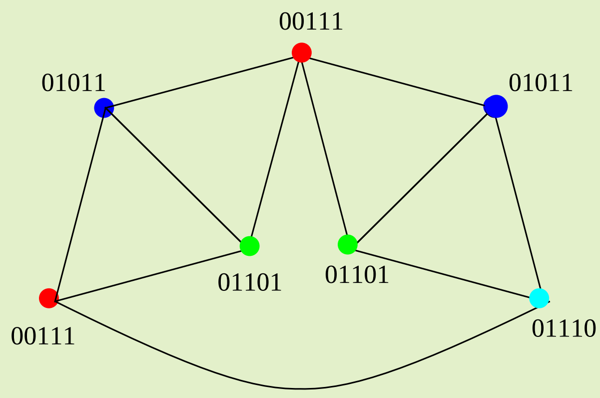

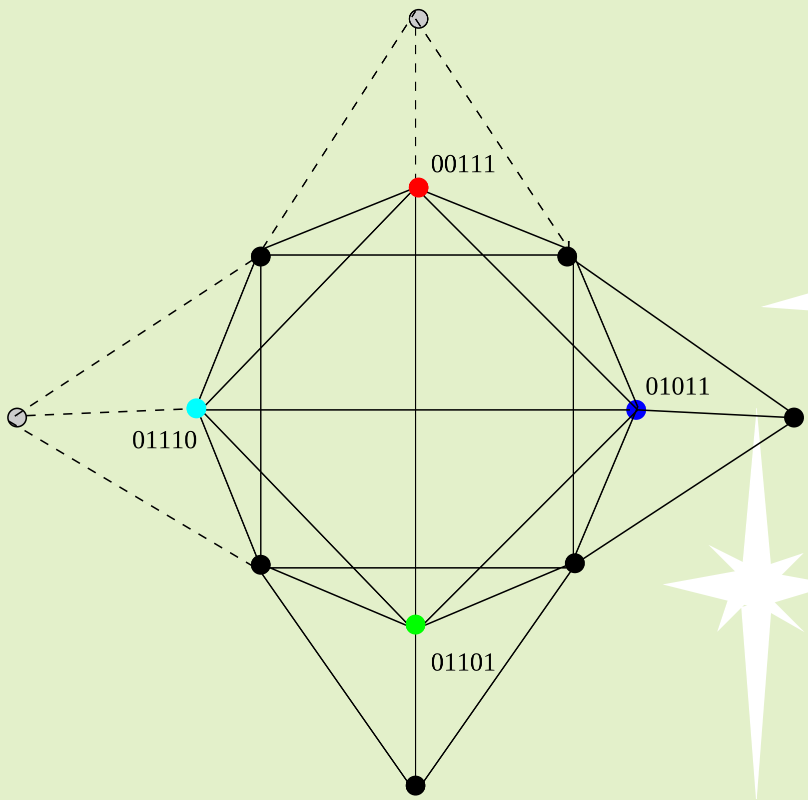

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Facts about $G(n, 2)$

A formula for the chromatic number

$$\chi(G(n,2)) = \left\lceil \frac{1}{2} \binom{n}{\lceil \frac{n}{2} \rceil} \right\rceil.$$

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For n odd the core is graph built only from the vertices with exactly $\frac{n-1}{2}$ 1's.

Facts about $G(g^2, g)$

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There are many open questions.

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- constructions for covering arrays on graph products