Applications of Graph Theory to Covering Arrays

Karen Meagher

joint work with Lucia Moura and Brett Stevens

kmeagher@math.uregina.ca

University of Regina

Covering Arrays

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tests	bedroom	hall	bathroom	kitchen
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Any two columns are *qualitatively independent* (every possible pair occurs in some row.)

Covering Arrays on Graphs

A **covering array on a graph** G, denoted $CA(n, G, k)$, is:

- ⋆ an $n \times r$ array where $r = |V(G)|$,
- ⋆ with entries from $\mathbb{Z}_k$ (k is the alphabet),
- ⋆ columns for **adjacent** vertices are qualitatively independent (have all possible pairs).

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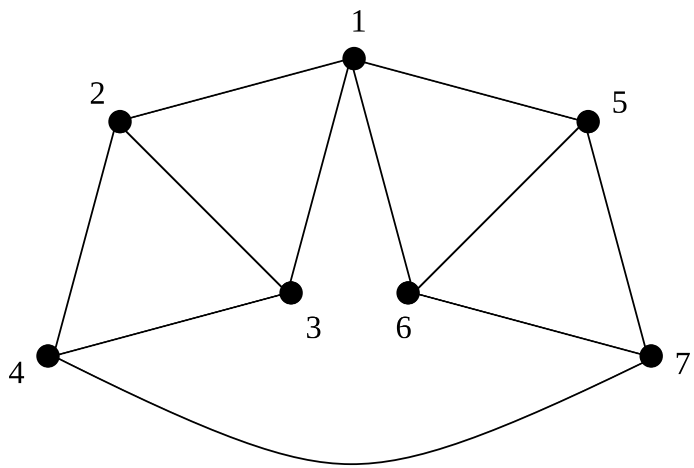

1	2	3	4	5	6	7
0	0	0	0	0	0	0
0	1	1	0	1	1	1
1	0	1	1	0	1	1
1	1	0	1	1	0	1
1	1	1	1	1	1	0

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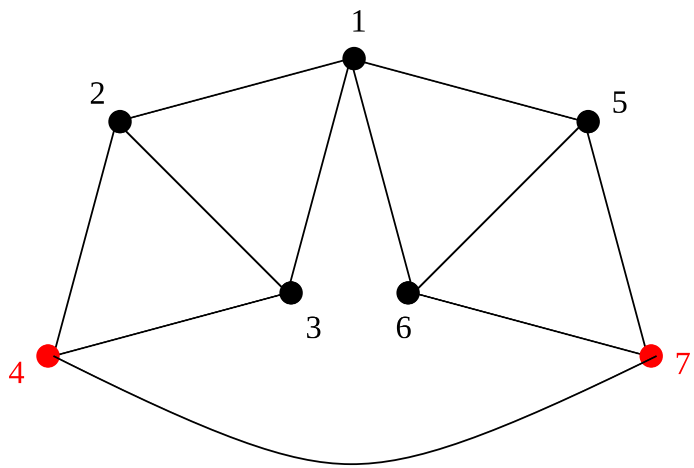

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0	1	1	0	1	1	1
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Graph Homomorphisms

A homomorphism of graphs G and H is a map

$$f : V(G) \to V(H)$$

and if vertices $u, v \in G$ are adjacent then vertices $f(u), f(v) \in H$ are also adjacent

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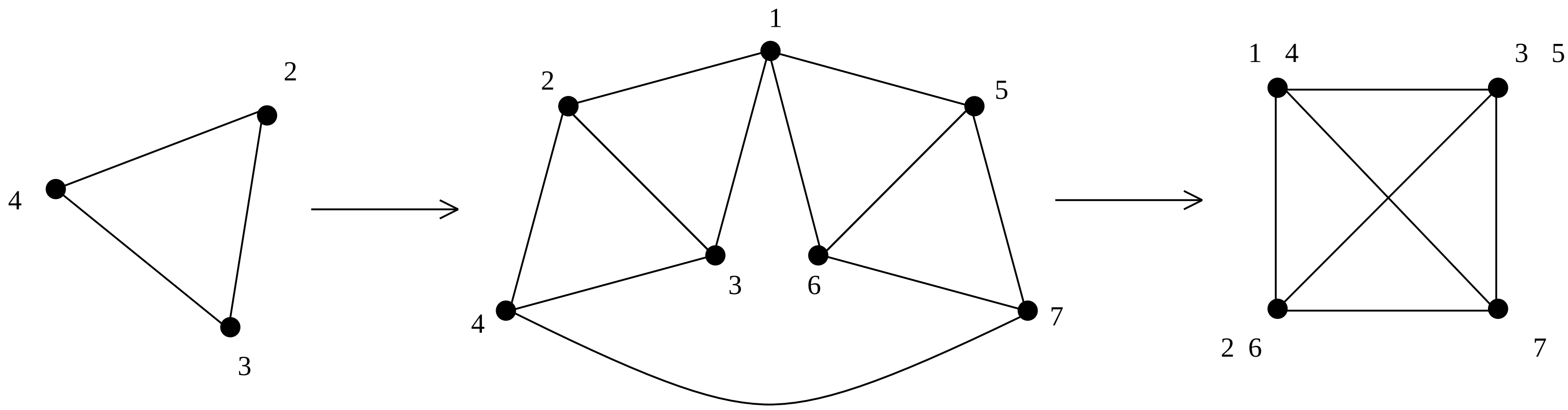
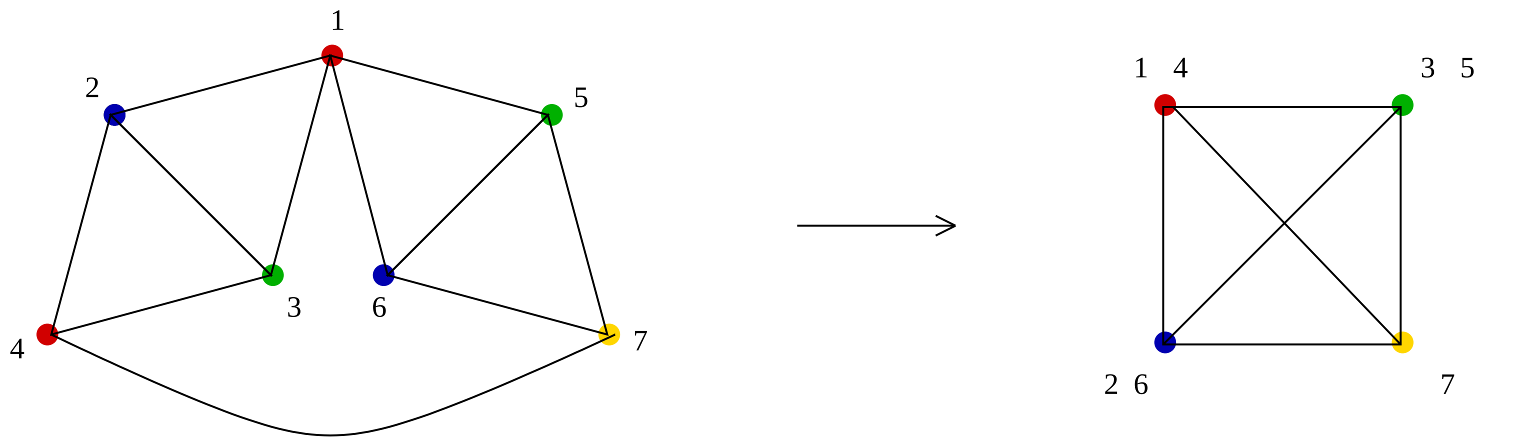


A Construction



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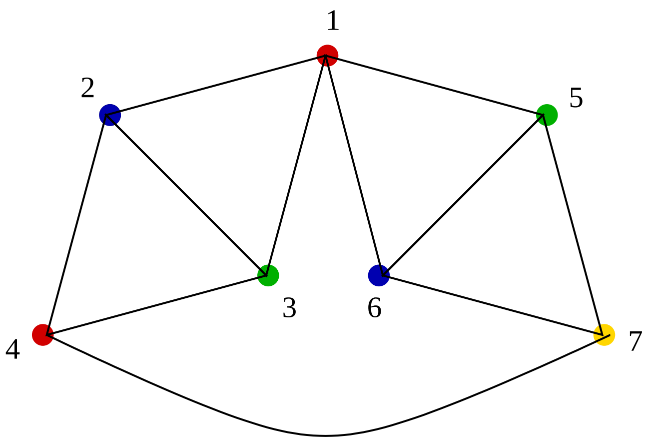

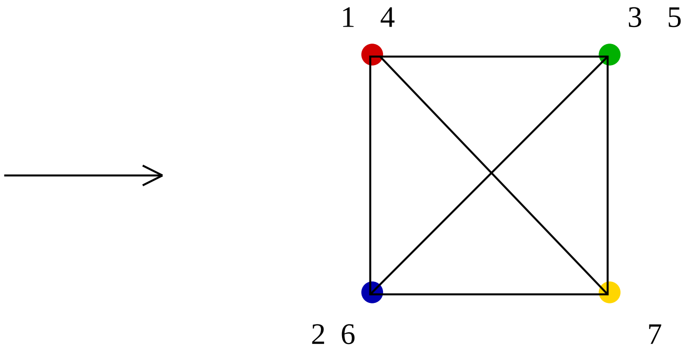

red	blue	green	yellow
0	0	0	0
0	1	1	1
1	0	1	1
1	1	0	1
1	1	1	0

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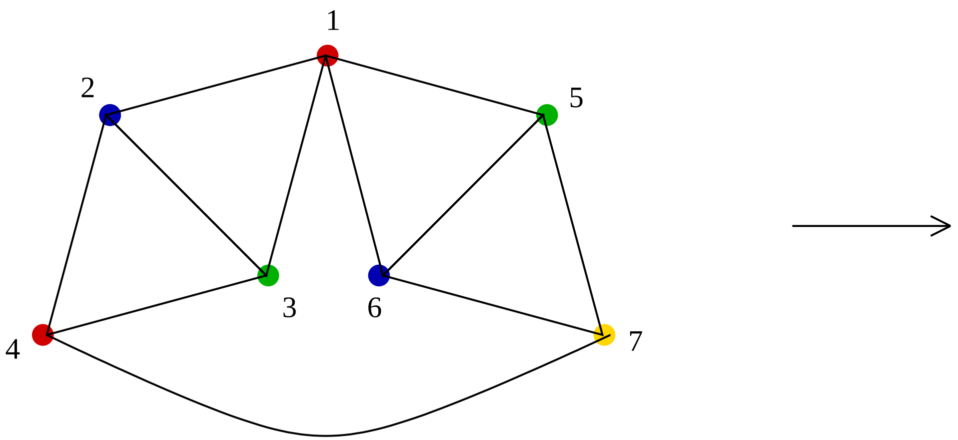

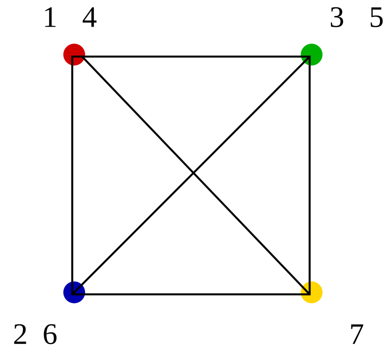

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Bounds from Complete Graphs

For a graph G, the size of a **maximum clique**, denoted $\omega(G)$, is the largest integer such that

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There is a lower bound from maximum clique size

$$CAN(K_{\omega(G)}, k) \leq CAN(G, k) \leq CAN(K_{\chi(G)}, k),$$

and an upper a bound from the chromatic number.

Partitions

	1	2	3	4
1	0	0	0	0
2	0	1	1	1
3	0	2	2	2
4	1	0	1	2
5	1	1	2	0
6	1	2	0	1
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$$1 \longrightarrow \{\{1,2,3\},\{4,5,6\},\{7,8,9\}\}$$

$$2 \longrightarrow \{\{1,4,7\},\{2,5,8\},\{3,6,9\}\}$$

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The columns of a covering array with a k-alphabet and n rows determine k-partitions of an n-set.

Qualitative Independence

Let A, B be k-partitions of an n-set,

$$A = \{A_1, A_2, \ldots, A_k\} \text{ and } B = \{B_1, B_2, \ldots, B_k\}.$$

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Two partitions formed from the columns in a covering array are qualitatively independent.

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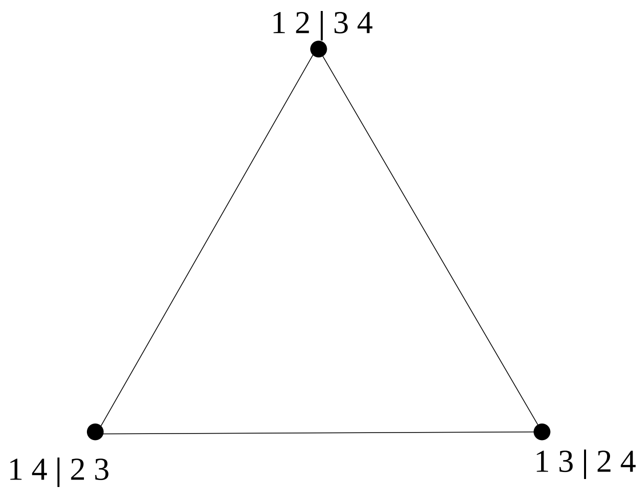
The graph $QI(4,2)$:



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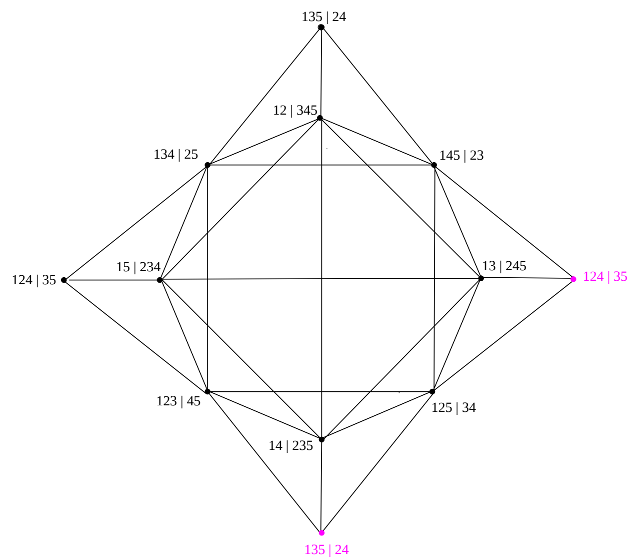
The graph $QI(5, 2)$:



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Theorem (M. and Stevens, 2002) The minimal size of a covering array on a graph G is

$$\min\{n : G \to QI(n,k).\}$$

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A clique in $UQI(k\ell, k)$ is equivalent to a **balanced** covering array
(each letter occurs the same number of times in each column.)

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- This bounds the size of a clique by $k + 1$.
- A covering array with k^2 rows on a k-alphabet can have no more than $k + 1$ columns.

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There does not exist a balanced covering array with 12 rows and 8 columns with an alphabet of size three.

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- ⋆ Find the maximum independent sets in $UQI(k\ell, k)$.
- ⋆ Find all eigenvalues of $UQI(k\ell, k)$.
- ⋆ Generalize eigenvalue bound for $UQI(12, 3)$ to more cases.