Koper Summer School - Problem Set Day 2

1. Interlacing and equitable partitions

   (a) Prove that the least eigenvalue of a graph X is greater than or equal to -1 if and only if X is a complete graph, a union of complete graph, or empty.

   (b) Let X be a graph. Show that the orbits of a subgroup of $Aut(X)$ form an equitable partition of X.

   (c) A perfect e-code in a graph X is a set of vertices S such that for each vertex v of X there is a unique vertex in S at distance at most e from v. Prove if X is a regular graph with a perfect 1-code, then -1 is an eigenvalue of $A(X)$.

2. Matrix Theory and Linear Algebra

   (a) (See Theorem 9.4.1 of Godsil and Royle Algebraic Graph Theory) Prove the following statement: Let X be a vertex-transitive graph and π be a partition on $V(X)$ generated by the orbits of some subgroup $Aut(X)$. If π has a class of size 1 (a singleton class), then every eigenvalue of X is an eigenvalue of X/π.

   (b) (See Section 5.3, of Godsil's Algebraic Combinatorics)
   Let $\pi = \{C_1, C_2, \ldots, C_r\}$ be an equitable partition of the vertices of a graph X with the property that C_1 is a singleton class. Then

   $$\frac{p'(X,x)}{p((X/\pi)\backslash C_1, x)} = \frac{|V(X)|\ p(X,x)}{p(X/\pi, x)}$$

   $p(X,x)$ is the characteristic polynomial of x in variable x.
   Prove that
   $$\frac{p'(X,x)}{p(X,x)} = \sum_{\lambda \text{ an eigenvalue}} \frac{m_\lambda}{x-\lambda}.$$

   (c) Use the previous result to find all the eigenvalues and multiplicities for a small Kneser graph (like $K(7,3)$).