Koper Summer School - Problem Set Day 1

1. EKR Theorem

   (a) Find and learn Katona's proof of the Erdős-Ko-Rado Theorem. Can you use this proof to give the characterization of the sets that meet the bound? (Warning this is in Algebraic Graph theory by Godsil and Royle, but there is an error!)

2. Facts about the Kneser graph

   (a) Prove if $n \geq 2k+1$, then $K(n,k)$ is connected. What is the maximum distance between vertices in K(n,k)? (This will depend on the values of n and k, maybe first think about when $n \geq 3k-1$.)

   (b) Show K(n,k) is edge transitive. Is $K(n,k)$ ever distance transitive?

   (c) Prove that $Aut(K(n,k)) = \mathrm{Sym}(n)$. So show there are no other automorphisms of $K(n,k)$ other than permuting the elements in $\{1,\ldots,n\}$.

   (d) A graph X is a *strongly regular graph* if the following hold:

      i. X is k-regular

      ii. if u and v are adjacent the u and v have exactly a common neighbours.

      iii. if u and v are not adjacent the u and v have exactly c common neighbours.

   For which values of k is $K(n,k)$ a strongly regular graph?

   (e) Make a conjecture for the chromatic number of $K(n,k)$. It is easy to find a good colouring of $K(n,k)$, but harder to prove that this colouring is the smallest possible - don't try to prove your colouring is minimal, there is a reference below.

3. Graph Isomorphism

   (a) Show that two graphs X and Y are isomorphic if and only if there is a permutation matrix P such that

   $$P^T\ A(X)\ P = A(Y).$$

   (Two graphs are *isomorphic* if there is a bijection $f : V(X) \to V(Y)$ so that $f(u)$ is a adjacent to $f(v)$ in Y if and only if u is adjacent in v in X.)

(b) Show if γ is an automorphism of a graph X then $A(X)$ and $A(\gamma(X))$ have the same eigenvalues.

(c) Let ϕ be an isomorphism between graphs X and Y. Show that $dist_X(x, y) = dist_Y(x^\phi, y^\phi)$.

4. Properties of Eigenvalues

   (a) Show that the degree of a regular graph is the largest eigenvalue and its multiplicity is the number of components.

   (b) What are the eigenvalues of the complete graph? What are the eigenvalues of tqhe union of ℓ copies of the complete graph?

   (c) Show that a graph has its least eigenvalue greater than or equal to -1 if and only if the graph is either the complete graph or the empty graph.

   (d) What are the eigenvalues of a complete bipartite graph?

   (e) If X is a k-regular graph with eigenvalues $k, \lambda_1, \ldots, \lambda_{n-1}$ what are the eigenvalues of the complements of X, $\overline{X}$?

   (f) How can you tell if a regular graph is the join of graphs from its eigenvalues? (The join of Y and Z, denoted $Y \vee Z$ is the graph formed by taking the disjoint union of Y and Z and adding all the edges between a vertex in Y to a vertex in Z.)