## Computing $q(K_{m,n})$

Let G be a bipartite graph with parts (X,Y) such that  $0 < |X| = m \le n = |Y|$ . Define  $\mathcal{B}(G)$  to be the set of all real  $m \times n$  matrices B whose rows and columns are indexed by X and Y, respectively and for which  $B_{x,y} \ne 0$  if and only if  $x \sim y$ . We have the following:

**Theorem 1.** For any non-empty bipartite graph G, if there is a  $B \in \mathcal{B}(G)$  whose set of rows and set of columns are orthonormal, then q(G) = 2.

*Proof.* Assume that there is such a  $B \in \mathcal{B}(G)$ . Then since the row and the column spaces of B have the same dimension, we must have m = n. Also, the following matrix is in  $\mathcal{S}(G)$ :

$$A = \left( \begin{array}{cc} 0 & B \\ B^{\top} & 0 \end{array} \right).$$

But we have

$$A^2 = \left(\begin{array}{cc} I_{n,n} & 0\\ 0 & I_{n,n} \end{array}\right) = I.$$

This implies that A has exactly 2 distinct eigenvalues. Therefore q(G) = 2.

We also observe the following:

**Proposition 2.** For any bipartite graph G with parts (X,Y), if q(G)=2 then |X|=|Y|.

*Proof.* There is a matrix

$$A = \begin{pmatrix} D_1 & B \\ & & \\ B^\top & D_2 \end{pmatrix} \in \mathcal{S}(G),$$

which has two distinct eigenvalues and in which  $D_1$  ad  $D_2$  are diagonal. By shifting and scaling, we can assume that the eigenvalues of A are -1, 1. Therefore  $A^2 = I$ . But then we should have

$$A^{2} = \begin{pmatrix} D_{1}^{2} + BB^{\top} & D_{1}B + BD_{2} \\ B^{\top}D_{1} + D_{2}B^{\top} & B^{\top}B + D_{2}^{2} \end{pmatrix} = I.$$

This implies that  $BB^{\top}$  and  $B^{\top}B$  are diagonal. Therefore the rows and columns of B must be orthogonal (and so linearly independent). Thus B must be a square matrix and so |X| = |Y|.

**Lemma 3.** For any  $n \ge 1$ , there is a real orthogonal  $n \times n$  matrix all of whose entries are non-zero.

*Proof.* The lemma is easy for n=1,2. Assume, therefore, that n>2. Then it is not difficult to see that the matrix  $B=I-\frac{2}{n}J$  is a real orthogonal matrix all of whose entries are non-zero.

Using Theorem 1 and Lemma 3, it is easy to see the following:

Corollary 4. For any  $1 \le m \le n$  we have

$$q(K_{m,n}) = \begin{cases} 2 : m = n \\ 3 : m < n \end{cases}$$

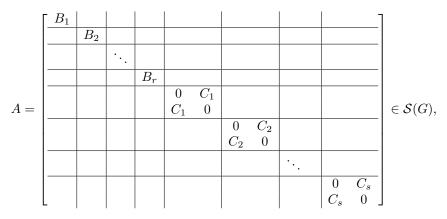
*Proof.* If m = n, it is enough to normalize the real orthogonal matrix in Lemma 3 and use it in Theorem 1. If m < n, then according to Proposition 2, we have  $q(G) \ge 3$ . On the other hand, the adjacency matrix of  $K_{m,n}$  has 3 distinct eigenvalues, which completes the proof.

Corollary 5. Let the non-empty graph G be any union of complete and complete bipartite graphs as follows:

$$G = \left(\bigcup_{i=1}^r K_{m_i}\right) \bigcup \left(\bigcup_{j=1}^s K_{n_j, n_j}\right).$$

Then q(G) = 2.

*Proof.* According to Lemma 3, there are real symmetric orthonormal matrices  $B_1, \ldots, B_r$  and  $C_1, \ldots, C_s$  where  $B_i$  is  $m_i \times m_i$  and  $C_j$  is  $n_j \times n_j$ , all of whose entries are non-zero. Then



and it is easy to see that

$$A^2 = I;$$

therefore q(G) = 2.