Computing $q(K_{m,n})$

Let G be a bipartite graph with parts (X, Y) such that $0 < |X| = m \leq n = |Y|$. Define $\mathcal{B}(G)$ to be the set of all real $m \times n$ matrices B whose rows and columns are indexed by X and Y, respectively and for which $B_{x,y} \neq 0$ if and only if $x \sim y$. We have the following:

Theorem 1. *For any non-empty bipartite graph G, if there is a $B \in \mathcal{B}(G)$ whose set of rows and set of columns are orthonormal, then $q(G) = 2$.*

Proof. Assume that there is such a $B \in \mathcal{B}(G)$. Then since the row and the column spaces of B have the same dimension, we must have $m = n$. Also, the following matrix is in $\mathcal{S}(G)$:

$$A = \begin{pmatrix} 0 & B \\ B^\top & 0 \end{pmatrix}.$$

But we have

$$A^2 = \begin{pmatrix} I_{n,n} & 0 \\ 0 & I_{n,n} \end{pmatrix} = I.$$

This implies that A has exactly 2 distinct eigenvalues. Therefore $q(G) = 2$. □

We also observe the following:

Proposition 2. *For any bipartite graph G with parts (X, Y), if $q(G) = 2$ then $|X| = |Y|$.*

Proof. There is a matrix

$$A = \begin{pmatrix} D_1 & B \\ & \\ B^\top & D_2 \end{pmatrix} \in \mathcal{S}(G),$$

which has two distinct eigenvalues and in which D_1 ad D_2 are diagonal. By shifting and scaling, we can assume that the eigenvalues of A are $-1, 1$. Therefore $A^2 = I$. But then we should have

$$A^2 = \begin{pmatrix} D_1^2 + BB^\top & D_1B + BD_2 \\ & \\ B^\top D_1 + D_2B^\top & B^\top B + D_2^2 \end{pmatrix} = I.$$

This implies that $BB^\top$ and $B^\top B$ are diagonal. Therefore the rows and columns of B must be orthogonal (and so linearly independent). Thus B must be a square matrix and so $|X| = |Y|$. □

Lemma 3. *For any $n \geq 1$, there is a real orthogonal $n \times n$ matrix all of whose entries are non-zero.*

Proof. The lemma is easy for $n = 1, 2$. Assume, therefore, that $n > 2$. Then it is not difficult to see that the matrix $B = I - \frac{2}{n}J$ is a real orthogonal matrix all of whose entries are non-zero. □

Using Theorem 1 and Lemma 3, it is easy to see the following:

Corollary 4. *For any $1 \leq m \leq n$ we have*

$$q(K_{m,n}) = \begin{cases} 2 & : m = n \\ & \\ 3 & : m < n \end{cases}$$

Proof. If $m = n$, it is enough to normalize the real orthogonal matrix in Lemma 3 and use it in Theorem 1. If $m < n$, then according to Proposition 2, we have $q(G) \geq 3$. On the other hand, the adjacency matrix of $K_{m,n}$ has 3 distinct eigenvalues, which completes the proof. □

Corollary 5. *Let the non-empty graph G be any union of complete and complete bipartite graphs as follows:*

$$G = \left(\bigcup_{i=1}^{r} K_{m_i} \right) \bigcup \left(\bigcup_{j=1}^{s} K_{n_j,n_j} \right).$$

Then $q(G) = 2$.

Proof. According to Lemma 3, there are real symmetric orthonormal matrices $B_1, \ldots, B_r$ and $C_1, \ldots, C_s$ where B_i is $m_i \times m_i$ and C_j is $n_j \times n_j$, all of whose entries are non-zero. Then

$$A = \begin{bmatrix} B_1 & & & & & & & \\ & B_2 & & & & & & \\ & & \ddots & & & & & \\ & & & B_r & & & & \\ & & & & \begin{matrix} 0 & C_1 \\ C_1 & 0 \end{matrix} & & & \\ & & & & & \begin{matrix} 0 & C_2 \\ C_2 & 0 \end{matrix} & & \\ & & & & & & \ddots & \\ & & & & & & & \begin{matrix} 0 & C_s \\ C_s & 0 \end{matrix} \end{bmatrix} \in \mathcal{S}(G),$$

and it is easy to see that

$$A^2 = I;$$

therefore $q(G) = 2$. □