# Minimum Number of Distinct Eigenvalues of Cycles

In this document we prove that for any even cycle $C_n$, $q(C_n)$ is equal to the diameter, i.e. $n/2$ and for any odd cycle $C_n$, $q(C_n)$ is equal to the diameter plus one, i.e. $\lfloor n/2 \rfloor + 1$.

Assume, first, that $n \geq 4$ is even and consider the sequence of $n \times n$ matrices $\{I = B^{(0)}, B^{(1)}, B^{(2)}, \ldots\}$, in which, for any $k \geq 1$, we define

$$B^{(k)}_{i,j} = \begin{cases} B^{(k-1)}_{i,j-1} + B^{(k-1)}_{i,j+1} & : \quad 1 < j < n \\ B^{(k-1)}_{i,2} - B^{(k-1)}_{i,n} & : \quad j = 1 \\ B^{(k-1)}_{i,n-1} - B^{(k-1)}_{i,1} & : \quad j = n. \end{cases}$$

It is easy to see that

$$B^{(1)} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ -1 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \in \mathcal{S}(C_n).$$

Let $A = B^{(1)}$. We prove the following

**Lemma 1.** *For any $k \geq 0$, we have $A^k = B^{(k)}$.*

*Proof.* We prove the lemma by induction on $k$. The base case is trivial. Assume the lemma holds for $k$. For any $i, j$, with $1 < j < n$, we have

$$\begin{aligned}(A^{k+1})_{i,j} &= (A^k A)_{i,j} = (B^{(k)} A)_{i,j} \\ &= (B^{(k)})_{i,1} A_{1,j} + \cdots + (B^{(k)})_{i,j-1} A_{j-1,j} + (B^{(k)})_{i,j} A_{j,j} + (B^{(k)})_{i,j+1} A_{j+1,j} + \cdots + (B^{(k)})_{i,n} A_{n,j} \\ &= (B^{(k)})_{i,j-1} + (B^{(k)})_{i,j+1} = (B^{(k+1)})_{i,j}.\end{aligned}$$

Also, if $j = 1$, we have

$$\begin{aligned}(A^{k+1})_{i,1} &= (A^k A)_{i,1} = (B^{(k)} A)_{i,1} \\ &= (B^{(k)})_{i,1} A_{1,1} + (B^{(k)})_{i,2} A_{2,1} + (B^{(k)})_{i,3} A_{3,1} + \cdots + (B^{(k)})_{i,n-1} A_{n-1,1} + (B^{(k)})_{i,n} A_{n,1} \\ &= (B^{(k)})_{i,2} - (B^{(k)})_{i,n} = (B^{(k+1)})_{i,1};\end{aligned}$$

and similarly

$$(A^{k+1})_{i,n} = (B^{(k)})_{i,n-1} - (B^{(k)})_{i,1} = (B^{(k+1)})_{i,n},$$

which completes the proof. □

For any $n \times n$ matrix $X$, we define $\Delta_j(X)$ to be the $j$-th "super-diagonal" of $X$, $j = 1, \ldots, n$. More precisely,

$$\Delta_j(X) = (X_{1,j}, X_{2,j+1}, \ldots, X_{n-j+1,n}).$$

Using Lemma 1 and induction on $k$, it is not hard to show that for any $k$, and $1 \leq j \leq n$,

$$\Delta_j(A^k) = ((A^k)_{1,j}, (A^k)_{1,j}, \ldots, (A^k)_{1,j});$$

i.e. all the elements on the super-diagonals are equal (to the entry on the first row). For any $k$ define $\operatorname{supp}(A^k)$ to be the set of indices $j \in \{1, \ldots, n\}$ for which $(A^k)_{1,j} \neq 0$. Let $n = 2m$.

**Lemma 2.** *Assume $k \leq m-1$.*

*(a) If $k$ is even, then*

$$\mathit{supp}(A^k) = \{1, 3, \dots, k+1\} \cup \{n-1, n-3, \dots, n-k\}.$$

*(b) If $k$ is odd, then*

$$\mathit{supp}(A^k) = \{2, 4, \dots, k+1\} \cup \{n, n-2, \dots, n-k+1\}.$$

*Furthermore, $(A^k)_{1,j} = -(A^k)_{1,n-j+2}$, mod $n$, for any $j \leq k+1$.*

*Proof.* The proof is by induction on $k$. The lemma is clearly true for $k = 0, 1$. Assume the lemma holds for $k$, and let $k+1$ be even. By Lemma 1, the first row of $A^{k+1}$ is as follows:

$$\begin{aligned}
&[(A^{k+1})_{1,1}\,,\ \dots,\ (A^{k+1})_{1,k+1}\,,\ (A^{k+1})_{1,k+2}\,,\ \dots,(A^{k+1})_{1,n-k}\,,\ (A^{k+1})_{1,n-k+1}\,,\ \dots,\ (A^{k+1})_{1,n}]\\
&\quad = [(A^k)_{1,2} - (A^k)_{1,n}\,,\ (A^k)_{1,1} + (A^k)_{1,3}\,,\ \dots,\\
&\qquad (A^k)_{1,k} + (A^k)_{1,k+2}\,,\ (A^k)_{1,k+1} + (A^k)_{1,k+3}\,,\ \dots,\\
&\qquad (A^k)_{1,n-k-1} + (A^k)_{1,n-k+1}\,,\ (A^k)_{1,n-k} + (A^k)_{1,n-k+2}\,,\ \dots,\\
&\qquad (A^k)_{1,n-2} +\ (A^k)_{1,n}\,,\ \dots,\ (A^k)_{1,n-1} - (A^k)_{1,1}\,].
\end{aligned}$$

Therefore, using part (b) of the lemma and the induction hypothesis, it suffices to show that

$$(A^k)_{1,k+1} + (A^k)_{1,k+3} = -(A^k)_{1,n-k-1} - (A^k)_{1,n-k+1}.$$

To see this, note that since $k < m-1$, we have $k+3 < n-k+1$ and so $(A^k)_{1,k+3} = 0$ and we have $n-k-1 > k+1$ and so $(A^k)_{1,n-k-1} = 0$. Now by induction hypothesis $(A^k)_{1,k+1} = -(A^k)_{1,n-k+1}$ which completes the proof of part (a).

Proof of part (b) is similar. □

The following is an easy consequence of Lemma 2.

**Corollary 3.** *(a) If $m$ is even, then*

$$\mathit{supp}(A^m) = \mathit{supp}(A^{m-2}) = \{1, 3, \dots, m-1\} \cup \{n-1, n-3, \dots, m+2\}.$$

*(b) If $m$ is odd, then*

$$\mathit{supp}(A^m) = \mathit{supp}(A^{m-2}) = \{2, 4, \dots, m-1\} \cup \{n, n-2, \dots, m+3\}.$$

*Furthermore, $(A^m)_{1,j} = -(A^m)_{1,n-j+2}$, mod $n$, for any $j \leq m-1$.*

Now we are ready to prove the main result.

**Theorem 4.** *For any even $n \geq 4$, we have*

$$q(C_n) = n/2 = \mathit{diam}(C_n).$$

*Proof.* Assume $m$ is even; the other case is similar. First note that, according to Lemma 3, there is a real constant $a_{m-2}$ such that

$$\mathrm{supp}(A^m + a_{m-2}A^{m-2}) = \{1, 3, \dots, m-3\} \cup \{n-1, n-3, \dots, m\} = \mathrm{supp}(A^{m-4}).$$

Now using Lemma 2, one can find real constants $a_{m-4}, \dots, a_2, a_0$ such that

$$\mathrm{supp}(A^m + a_{m-2}A^{m-2} + a_{m-4}A^{m-4} + \cdots + a_2A^2 + a_0I) = \emptyset.$$

In other words, $A$ satisfies in the polynomial $p(t) = t^m + a_{m-2}t^{m-2} + \cdots + a_2t^2 + a_0$. This means that the minimal polynomial of $A$ is of degree at most $m$. Therefore, $A$ has at most $m$ distinct eigenvalues. We have, thus, proved that $q(C_n) \leq m$.

On the other hand, since the diameter of $C_n$ is $m$, no matrix in $\mathcal{S}(G)$ can satisfy in a polynomial of degree $< m$. Thus $q(C_n) \geq m$, which completes the proof. □

Theorem 4 is not true for odd cycles. According to Shaun's conclusion, we have

$$\operatorname{diam}(C_n) \leq q(C_n) \leq \operatorname{diam}(C_n) + 1. \tag{1}$$

We prove the following

**Theorem 5.** *For any odd $n \geq 3$, we have*

$$q(C_n) = \lceil n/2 \rceil + 1 = diam(C_n) + 1.$$

*Proof.* Assume $m = \lceil n/2 \rceil$. Suppose $A \in \mathcal{S}(C_n)$. It is not hard to see by induction on $k$ that for any $k \leq m$ and for any polynomial $p(t)$ of degree $k$, we have

$$(p(A))_{i,i+k} = A_{i,i+1}\ A_{i+1,i+2}\ \cdots\ A_{k,k+1}, \quad \text{for all } i = 1, \ldots, n-k.$$

In particular, if $p(t)$ is a polynomial of degree $m$, then we have

$$(p(A))_{1,m+1} = A_{1,2}\ A_{2,3}\ \cdots\ A_{m,m+1}. \tag{2}$$

On the other hand, if $q(C_n) = m$, then $p(A) = 0$, for some polynomial $p(t)$ of degree $m$. According to (2), then, we must have

$$A_{1,2}\ A_{2,3}\ \cdots\ A_{m,m+1} = 0,$$

which implies that at least one of the entries $A_{1,2},\ A_{2,3},\ \cdots,\ A_{m,m+1}$ must be zero. But this contradicts the fact that $A \in \mathcal{S}(C_n)$. Thus $q(C_n) > m$. Noting (1) then will complete the proof. □