

Question: Is mur monotone on induced subgraphs?
Bachman

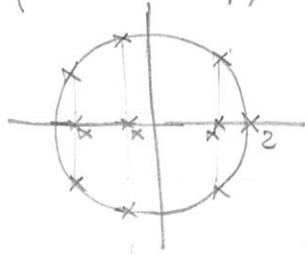
Thursday July 7th Shaun

Thm: G -regular of degree d

If $\sigma(A) = \{d, \lambda_2, \lambda_3, \dots, \lambda_n\} \rightarrow$ multi-set
 \uparrow
adj. matrix of G , then

$$\text{mur}(G) = n - (1 + \max \text{mult in } \{\lambda_2, \dots, \lambda_n\})$$

(Cor1) $\text{mur}(C_n) = n-3$
 $n \geq 3$



(Cor2) $\text{mur}(\overline{C}_n) = n-3$

Paths (P_n) $\text{mur}(P_3) = 1$

Conjecture 1) $\text{mur}(P_n) = n-2$ true for $n \leq 3, 4$

Fact: $\text{mur}(P_n) \geq n-2$

$$U = A + \beta I + \gamma J + \delta D =$$

$$\begin{bmatrix} 8+1 & & & \\ 8 & 8+1 & & \\ & 8 & 8+1 & \\ & & \ddots & \\ & & & 8+1 & 8+1 \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 8+1 & & & & \\ -1 & * & & & \\ 0 & 1 & * & & \\ 0 & 0 & -1 & * & \\ 0 & & & 0 & -1 \\ 0 & & & & 0 & -1 & * \end{bmatrix}$$

$n-2$

Conjecture 2) $\text{mur}(G) \leq |G| - 2$ (Fact: $\text{mur}(G) \leq |G|-1$)

$$U = A + \beta I + \gamma J + \delta D$$

$$\begin{bmatrix} \gamma + \beta & \gamma + 1 & \gamma & \gamma \\ \gamma + \beta + 2\gamma & \gamma + 1 & \gamma & \gamma \\ \beta + \gamma + 2\gamma & \gamma + 1 & \gamma & \gamma \\ \beta + \gamma + \delta & \gamma + 1 & \gamma & \gamma \end{bmatrix}$$

Laplacian

$$P_4: \beta = 0, \gamma = -\frac{1}{2}, \delta = 1$$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Fan

$$(D - A + \beta J)$$

$$e = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = 0, \lambda_2, \dots, \lambda_n$$

sym. \Rightarrow eigenvectors are orthogonal

$$(D - A + \beta J)e = (\beta n)e$$

$$(D - A + \beta J)x_i = \lambda_i x_i$$

$$\Rightarrow \sigma(D - A + \beta J) = \{\beta n, \lambda_2, \dots, \lambda_n\}$$

choose β s.t. $\beta n = \lambda_2$

Then $D - A + \beta J$ will have a mult. eigenvalue, using I (SI)

this mult. eigen. can be zero.

Generalize: $\text{mur}(G) \leq n - \frac{\text{max mult. of } +1}{D-A}$

Thm: $G: \sigma(L(G)) = \{0, \lambda_2, \dots, \lambda_n\}$, then

$\text{mur}(G) \leq n - (1 + \text{max mult}\{\lambda_2, \dots, \lambda_n\})$. Further, equality holds if G is regular

$$\sigma(L(G)) = d - \sigma(A(G))$$

Q. $\text{murm}(G_1 \cup G_2)$ in terms of $\text{murm}(G_1)$ & $\text{murm}(G_2)$?

Hw: 1) $\text{murm}(\text{trees})$

2) $\text{murm}(G \cup v)$ in terms of $\text{murm}(G)$

3) $\text{murm}(G \vee v)$ (join)

4) $\text{murm}(G + 1) \rightarrow$ an edge with a new vertex

$$U' = \left[\begin{array}{c|cc} U(G) & 8 \\ \hline 8 & 8 & \beta+8 \end{array} \right] = \left[\begin{array}{c|c} U & 8_1 \\ 8_1 & \beta+8 \end{array} \right]$$


Is 8_1 in the col U ?

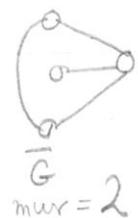
If $\beta+8 \neq 0$, then

$$\text{rank}(U') = 1 + \text{rank}(U(G)) - \frac{8^2}{\beta+8} \geq 1 + \text{murm}(G).$$

\downarrow
Schur Comp.

$$\text{murm}(G') \leq 2 + \text{murm}(G)$$

$$\text{murm}(P_3 + v) = 2$$



Q. When does adding an isolated vertex not change the mur?

Ex. $\text{mur}(K_n \cup K_1) = \text{mur}(K_n \cup K_1 \cup K_1)$

$$\text{rank} \begin{bmatrix} U & \left| \begin{matrix} Ux \\ x^T U \end{matrix} \right. \\ \hline x^T U & x^T Ux \end{bmatrix} = \text{rank}(U) \quad (\text{the only way})$$

$Ux \in$

$$\text{rank} \begin{bmatrix} & \left| \begin{matrix} & \\ & x^T Ux \\ \hline & x^T Ux + \epsilon \end{matrix} \right. \end{bmatrix} = \text{rank}(U) + 1$$

* If G is regular, by adding an isolated vertex, rank can only go up by 0 or 1, (and not 2).
it can't go down because of not having D.