

Thursday June 30. Karen

U.A.M.

Graph G with adjacency matrix A and define $D = [d_1, \dots, d_n]$

where $d_i = \deg(v_i)$. $U = U(\alpha, \beta, \gamma, \sigma) = \alpha A + \beta I + \gamma J + \sigma D$, $\alpha \neq 0$.

Find $\text{mvr}(G) = \min \{\text{rank}(U) : U \in \text{U.A.M.}\}$

Facts:

$$\textcircled{1} \quad \text{mvr}(G) = \text{mvr}(\bar{G})$$

$$\textcircled{2} \quad \text{mvr}(K_n) = \text{mvr}(nK_1) = 0 \quad |A + I| = |J + \sigma D| \quad (\text{mvr}(G) = 0 \Leftrightarrow G = K_n \text{ or } \bar{G} = K_n)$$

Question: Find a graph G with $\text{mvr}(G) = 1$

Ex.



$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$|A + 0I + 0J + 1D|$$

Generalize to $K_n \cup K_m \leftrightarrow K_{n,1,1,\dots,1}$, $U(1,0,0, \frac{1}{n-1})$ $\begin{array}{c|c} J & 0 \\ \hline 0 & 0 \end{array}$ complete multipartite graph

$$\ast \quad \text{mvr}(K_n \cup K_m) = 1 \text{ if } m+n > 2, \quad U(-2, -2, 1, 0)$$

$$\begin{array}{c|c} -J & J \\ \hline J & -J \end{array}$$

or \bar{G}

Conj: $\text{mvr}(G) = 1 \Leftrightarrow G = K_n \cup K_m \text{ or } K_n \cup mK_1$

Shawn's $\Rightarrow \text{mvr}(G) = 1 \exists \alpha, \beta, \gamma, \sigma \text{ s.t. } \text{rank}(U(\alpha, \beta, \gamma, \sigma)) = 1$

Proof. symmetric \Rightarrow eigenvalues of $U(\alpha, \beta, \gamma, \sigma) \begin{cases} 0 & |G|-1 \text{ (mult)} \\ \lambda \neq 0 & \text{simple} \end{cases}$

$$\begin{array}{c|c} \lambda & 0 \\ \hline 0 & 0 \end{array}$$

H/O $\Rightarrow G \text{ or } \bar{G} \text{ is } K_n \cup K_m \text{ or } K_n \cup mK_1$

G : Connected regular graph

$$A + \beta I + \gamma J$$

If G is regular of degree d , then $\bullet Ae = de$, $e = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$, $\bullet AJ = JA$

Thus, the eigenvalues of $A + \gamma J$ are $n\gamma + \lambda_1, \lambda_2, \dots, \lambda_n$ where $\lambda_1, \dots, \lambda_n$ are unordered eigenvalues of A .

It follows that $\lambda_1 = d$ (common eigenvector is d)

$U = \beta I + (A + \gamma J)$. Suppose λ_2 has the largest multiplicity among the list $\{\lambda_2, \dots, \lambda_n\}$. Set γ s.t. $\gamma n + d = \lambda_2$, $\beta = -\lambda_2$.

Then $\text{rank}(U) = n - \{\text{mult}(\lambda_2) + 1\}$. \square

Suppose

Conclusion: $\forall G$: connected regular with adjacency spectrum $d, \lambda_2, \dots, \lambda_n$.

Let m be the maximum mult. among the eigenvalues of A . Then

$$\text{mult}(G) = n - (m + 1).$$

$$K_n: m = n - 1$$

$$K_{n,n}: \underbrace{n, n, 0, \dots, 0}_{2n-2} \quad 2n - (2n - 2 + 1) = 1$$

Question: What about disconnected regular graphs?

$$G = G_1 \cup G_2$$

$A(G_1)$	
	$A(G_2)$

$$\begin{matrix} n\gamma + d, \lambda_1, \dots, \lambda_{n-1}, \lambda_1^2, \dots, \lambda_{K-1}^2 \\ (A + \gamma J) \end{matrix}$$

$$\begin{matrix} n\gamma + \lambda_1 \\ 0 + \lambda_2 \\ \vdots \\ 0 + \lambda_m \end{matrix}$$

$$A(G_1) \quad d, \lambda_1, \dots, \lambda_{n-1}$$

$$A(G_2) \quad d, \lambda_1^2, \dots, \lambda_{K-1}^2$$

$$0 + \lambda_m$$

Evalues of $A + \beta J$ are $n\beta + d, \lambda_2, \lambda_3, \dots, \lambda_n = d$

Evalues of $\beta I + A + \beta J$ are

$n\beta + d + \beta, \lambda_2 + \beta, \dots, \lambda_{n-1} + \beta, d + \beta$

Example: $\text{mvr}(nk_1) = 0 = n - \max_{\substack{\text{of eval} \\ \text{in } nk_1}} \text{multi} = n - n = 0$

$n\beta + d + \beta$

If max multi is at d , set $\beta = 0 \Rightarrow \text{mvr}(G) \leq n - \# \text{ of components}$
G: regular degree d

Conjecture: $\text{mvr}(G) = n - \left[\max_{\substack{\text{of } A(G)}} \text{mult.} + 1 \right]$ strictly
or $n - [\# \text{ of components}]$ iff d has max. mult.

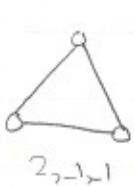
→ always an upper bound

Ex. $\text{mvr}(rk_2) \leq 2r - r = r$, $\text{mvr}(2k_2) = 1$

HW: Conj. $\text{mvr}(rk_2) = r - 1$ $A = I - \frac{1}{r}J$

HW: For which graphs does $\text{mvr}(G) = n - \# \text{ components}$?
regular

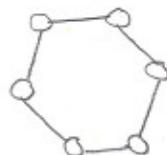
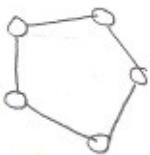
(Which graphs have the degree of the strict max. mult.?)



$2, -1, 1$



$2, -2, 0, 0$



$P + P^\perp: 2 \cos\left(\frac{2k\pi}{n}\right)$

HW: What is $\text{mvr}(G \cup G)$ and $\text{mvr}(G \cup H)$?

Question: Is μ_{ur} monotone on induced subgraphs?
Bachman