The minimum number of distinct eigenvalues of a graph

Notation:

- For a graph G on n vertices, $S(G)$ denotes the set of real symmetric matrices $A = [a_{ij}] \in M_n$ such that $a_{ij} \neq 0$, $(i \neq j)$ if and only if vertices i and j are connected by an edge.
- $d(G)$ denotes the diameter of the graph G.
- $q(G)$: The minimum number of distinct eigenvalues of G when minimum is taken over all matrices in $S(G)$.

Some of the known results:

- For any tree T, $q(T) \geq d(T) + 1$; see [5]
- There exist trees with $q(T) > d(T) + 1$; see [1]
- For any positive integer d, there exists a constant $f(d)$ such that for any tree T with diameter d, there is a matrix $A \in S(T)$ with at most $f(d)$ distinct eigenvalues (claimed by B. Shader who also says: "our $f(d)$ is super super exponential").
- $f(d) \geq (9/8)d$ for d large; see [3]
- More work on general graphs; see [2] for example.

Questions:

(1) What about graphs with diameter d? Is there any constant similar to $f(d)$?
(2) Is $f(d)$ linear in d or exponential?
(3) What is $q(T)$ if T is a complete binary tree of height h?

References

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