1. If v is a pendant vertex in a graph G, then $r_v(G) = 0$ or 1. Can we characterize when each case occurs?

2. For which graphs is $r_v(G) = 2$ for all $v \in V(G)$?

3. Can we characterize which vertices in a graph have $r_v(G) = 2$?

   Let G be a graph with a cut-vertex v. Then $r_v(G) = 2$ (or v is *rank strong*) if and only if one of the following (or potentially both) occur:

   (a) $r_v(H) = 2$ for H a subgraph of G with the property that $H - v$ is a component of $G - v$.

   (b) There exist two subgraphs H_1 and H_2 of G with the property that $H_1 - v$ and $H_2 - v$ are distinct components of $G - v$ and $r_v(H_1) = 1$ and $r_v(H_2) = 1$.

   The proof of this is from the equation

   $$mr(G) = \sum mr(H_i - v) + \min\{2, \sum r_v(H_i)\}, \qquad (0.1)$$

   where the sum is taken over all components $H_i - v$ of $G - v$ (so H_i is a subgraph of G that contains v and has the property that $H_i - v$ is a component of $G - v$).

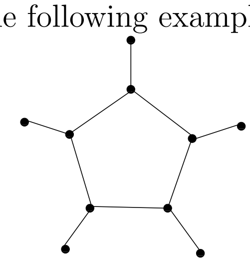
   This is only a partial answer, since we know there are graphs with rank strong vertices that are not cut vertices.

4. (a) For which graphs is $Z(G) = M(G)$?

   (b) For which graphs is $Z^+(G) = M^+(G)$?

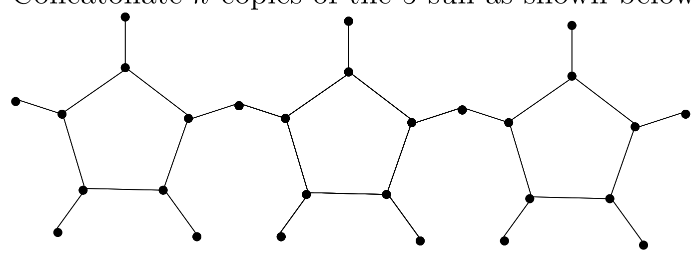
5. (a) How far apart can M and Z be?

   M and Z can be made arbitrarily far apart. Consider the following example:



This is known as the 5-sun, we will denote it by H_5. Then it is not hard to see that $M(H_5) = 2$ and $Z(H_5) = 3$ (for this use two pendant vertices from adjacent vertices and the opposite vertex on the 5-cycle).

Concatonate k copies of the 5-sun as shown below:



This is called a *vertex sum*. The vertex sum of two graphs G and H with $v \in V(G)$ and $v \in V(H)$ (denoted $G \oplus_v H$) is the graph formed by identifying the vertices v in G and H.

The vertex sum of k copies of the 5-sun is called kH_5. Then $M(kH_5) = k + 1$ and $Z(kH_5) = 2k + 1$.

The fact that $M(kH_5) = k + 1$ follows from the three facts below:

i. If v is a pendant vertex then $r_v(G) = 0$ or 1.

ii. If v is a pendant vertex of both G and H then $mr(G \oplus_v H) = mr(G) + mr(H)$. This follows from Equation **??**.

iii. If v is a pendant vertex of both G and H then

$$\begin{aligned} M(G \oplus_v H) &= |V(G \oplus_v H)| - mr(G \oplus_v H) \\ &= (|V(G)| + |V(H)| - 1) - (mr(G) + mr(H)) \\ &= M(G) + M(H) - 1. \end{aligned}$$

Thus $M(kH_5) = 2k - (k - 1) = k + 1$

(b) Can we find a construction that increases Z but not M or where Z grows faster than M?

(c) What about M^+ and Z^+?

6. Can we characterize the vertices in a graph whose deletion increases Z?

7. Is there a nice proof that $Z^+(G) \geq \delta$ where δ is the minimum degree of the vertices in the graph?

8. If H is a subgraph of G then we don't always have that $Z(H) \leq Z(G)$ or $Z^+(H) \leq Z^+(G)$. Can we characterize subgraphs, H, such that $Z(H) \leq Z(G)$ or $Z^+(H) \leq Z^+(G)$? What about $H = K_n$ or $H = C_n$?

9. If G is a strongly regular graph, can we say anything good about $Z(G)$?

10. If G is a circulant/Cayley graph can we say anything good about $Z(G)$?