Another 3-Part Sperner Theorem

Karen Meagher
Department of Combinatorics and Optimization
University of Waterloo, Waterloo, Ontario, Canada
kmeagher@math.uwaterloo.ca

Submitted: ?? ; Accepted: ?? ;
Mathematics Subject Classifications: 05D05

Abstract

1 Introduction

In this paper, we prove a higher order Sperner theorem.These theorems are stated after some notation and background results are introduced.

For i, j positive integers with $i \leq j$, let $[i, j]$ denote the set $\{i, i+1, \ldots, j\}$. For k, n positive integers, set $\binom{[n]}{k} = \{A \subseteq [1, n] : |A| = k\}$. A system $\mathcal{A}$ of subsets of $[1, n]$ is said to be *k-set system* if $\mathcal{A} \subseteq \binom{[n]}{k}$.

Two subsets A, B are *incomparable* if $A \not\subseteq B$ and $B \not\subseteq A$. A set system on an n-set $\mathcal{A}$ is said to be a *Sperner set system*, if any two distinct sets in $\mathcal{A}$ are incomparable.

Sperner's Theorem is concerned with the maximal cardinality of Sperner set systems as well as with the structure of such maximal systems.

Theorem (Sperner's Theorem [10]). *A* Sperner set system $\mathcal{A}$ *of subsets of* $[1, n]$ *consists of at most* $\binom{n}{\lfloor n/2 \rfloor}$ *sets. Moreover, a Sperner set system meets this bound if and only if* $\mathcal{A} = \binom{[n]}{\lfloor n/2 \rfloor}$ *or* $\mathcal{A} = \binom{[n]}{\lceil n/2 \rceil}$.

One application of Sperner's theorem is to give the exactly size of strength-2 binary covering arrays [**?**, klietman, katona]

A *covering array*, denoted $CA(n, r, k)$, is an $r \times n$ array with entries from $\mathbb{Z}_k$ with the property that for any two rows of the array that each of k^2 pairs from $\mathbb{Z}_k \times \mathbb{Z}_k$ occurs in some column. These are also known as strength-2 covering arrays. If $k = 2$ a $CA(n, r, k)$ is a *binary covering array*. The rows of a $r \times n$ binary covering array are length-n 01-vectors and as such correspond to a subset of an n-set. The cover property for binary covering arrays can be characterized in terms of these sets.

Definition 1 (*Qualitatively Independent Subsets*). Two subsets A and B of an n-set are *qualitatively independent subsets* if

$$A \cap B \neq \emptyset, \quad A \cap \overline{B} \neq \emptyset, \quad \overline{A} \cap B \neq \emptyset, \quad \overline{A} \cap \overline{B} \neq \emptyset.$$

A strength-2 binary covering array corresponds to a set system in which any two sets are qualitatively independent. A set system $\mathcal{A}$ in which any two $A, B \in \mathcal{A}$ have the property that $A \cap B \neq \emptyset$ is called an *intersecting set system*. An intersecting Sperner k-set system with $2k \leq n$ is a qualitatively independent set system.

This leads to the following result, found independently by Katona and by Kleitman and Spencer.

Theorem 2 ([**?**, **?**]). *If $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$ is a qualitatively independent set system of an n-set, then*

$$|\mathcal{A}| \leq \binom{n-1}{\lfloor n/2 \rfloor - 1}.$$

Further, this bound is attained by the system of all $\lfloor n/2 \rfloor$-sets which contain a common element.

This theorem gives the exact size of the optimal binary covering array with r rows can be found for all r.

Theorem 3 ([**?**]). *Let r be a positive integer, then*

$$CAN(r, 2) = \min \left\{ n : \binom{n-1}{\lfloor n/2 \rfloor - 1} \geq r \right\}.$$

Both the proof of Theorem **??** given by Katona and by Kleitman and Spencer used the well-known ErdőS-Ko-Rado Theorem.

Theorem (Erdős-Ko-Rado Theorem [?]). *Let k and n be positive integers with $2k < n$. Then for any intersecting k-set system on an n-set $\mathcal{A}$,*

$$|\mathcal{A}| \leq \binom{n-1}{k-1}.$$

Moreover, equality holds if and only if $\mathcal{A}$ is the collection of all k-sets containing some fixed element.

A strength-t binary covering array, denoted t-$CA(n, r, k)$, is an $r \times n$ array with entries from $\mathbb{Z}_k$ with the property that for any set of t rows in the array, each of k^t t-tuples from $\mathbb{Z}_k \times \mathbb{Z}_k$ occurs in some column. The rows of a binary (that is, $k = 2$) strength-t covering array correspond to a set system. Again, we can characterise the covering property in terms of the set system.

Definition 4 (***t-Qualitatively Independent Set System***)**.** A set system $\mathcal{A}$ is a *t-qualitatively independent set system* if for any collection of t distinct sets $\{A_1, A_2, \ldots, A_t\}$ with $A_i \in \mathcal{A}$ or $\overline{A_i} \in \mathcal{A}$ for $i = 1, \ldots, t$ and $A_i \neq \overline{A_j}$ for $i, j \in [1, \ldots, t]$

$$A_1 \cap A_2 \cap \ldots \cap A_t \neq \emptyset$$

We wish to find a higher strength version of Sperner's theorem with the goal of extending Katona's and Kleitman and Spencer's exact bound on strength-2 binary covering arrays to higher strength binary covering arrays.

It is trivial to give a higher order version the Erdős-Ko-Rado Theorem. We say that a set system $\mathcal{A}$ is *strength-t intersecting* if for any $A_1, A_2, \ldots, A_t \in \mathcal{A}$, $\cap_{i=1}^{t} A_i \neq \emptyset$. If we remove the uniform conditiono n the set systems, these are also known as r-wise t-intersecting systems Clearly, for $t \geq 2$ a strength-t intersecting set system is intersecting and Erdős-Ko-Rado Theorem holds for strength-t intersecting set systems.

Extending Sperner's Theorem is more difficult. We will focus on a strength-3 version of this theorem.

Definition 5 (***Strength-3 Sperner Set System***)**.** A set system $\mathcal{A}$ is a *strength-3 Sperner set system* if for any three distinct sets $A, B, C \in \mathcal{A}$ the following hold:

$$A \nsubseteq B \cup C \quad B \nsubseteq A \cup C \quad C \nsubseteq A \cup B$$

$$B \cap C \nsubseteq A \quad A \cap C \nsubseteq B \quad A \cap B \nsubseteq C$$

Clearly, any strength-3 Sperner set system is also a Sperner set system.

The property $B \cap C \nsubseteq A$ implies that for $B, C \in \mathcal{A}$, $B \cap C \neq \emptyset$. So any strength-3 Sperner set system is also an intersecting set system.

Further, the property $A \nsubseteq B \cup C$ implies $A \cap \overline{B} \cap \overline{C} \neq \emptyset$ and $B \cap C \nsubseteq A$ implies that $\overline{A} \cap B \cap C \neq \emptyset$.

If $\mathcal{A}$ is a strength-3 Sperner set system then $\overline{\mathcal{A}}$ is also a strength-3 set system.

Lemma 6. *If $\mathcal{A}$ is a strength-3 Sperner set system, then the set systems*

$$\mathcal{A}_\cap = \{A \cap B : A, B \in \mathcal{A}, A \neq B\}$$

and

$$\mathcal{A}_\cup = \{A \cup B : A, B \in \mathcal{A}, A \neq B\}$$

are Sperner set systems.

Proof. Let $A, B, C, D \in \mathcal{A}$ and $\{A, B\} \neq \{C, D\}$. We can assume without loss of generality that $A \neq C$ and $A \neq D$. If $A \cup B \subseteq C \cup D$ then $A \subset C \cup D$ and $\mathcal{A}$ is not a strength-3 Sperner set system. Similarly, if $A \cap B \subseteq C \cap D$, then $A \cap B \subseteq C$. □

Lemma 7. *If $\mathcal{A}$ is a 3-qualitatively independent set system, then*

$$\{A, \overline{A} : A \in \mathcal{A}\}$$

is a strength-3 Sperner set system.

Theorem 8. *If $\mathcal{A}$ is a strength-3 Sperner set system on an n-set then*

$$\binom{|\mathcal{A}|}{2} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

Proof. The sets $A \cap B$ are unique. If $A \cap B = C \cap D$ then $A \cap B \subseteq C$. Lemma 6 and Sperner's theorem. □

∗ is this right? ∗

All logarithms are base 2.

Theorem 9. *[?]* $\lim_{k\to\infty} \frac{CAN(3,r,k)}{\log r} = \binom{k}{2}$.

∗ check this!! ∗

From the bound in Theorem 8 $\lim_{k\to\infty} \frac{n}{logr} \leq \frac{n}{(n+1)\log 2 - \frac{1}{2}\log n}$. As n goes to infinity this limit goes to 1, as predicted from the previous theorem.

Conjecture. *The largest strength-3 Sperner set system is an $\frac{n}{2}$-set system.*

Conjecture. *Let $\mathcal{A}$ be a largest strength-3 Sperner set system, then $\mathcal{A}$ has the property that for all distinct $A, B \in \mathcal{A}$, $|A \cap B| = n/4$.*

Theorem 10. *If $\mathcal{A}$ is a strength-3 Sperner set system on an n-set then*

$$\binom{|\mathcal{A}|}{2} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor - 2}$$

Proof. Lemma 6 and Sperner's theorem. Also that $|A \cap B| \leq \lfloor \frac{n}{2} \rfloor - 2$. □

There have been extensions of Sperner's Theorem to systems of families of sets [2] and to systems of subsets of a set X with a 2-partition $X = X_1 \cup X_2$ such that no two subsets A, B in the system satisfy both $A \cap X_i = B \cap X_i$ and $A \cap \overline{X_i} \subseteq B \cap \overline{X_i}$ where $i \in \{1, 2\}$ [3, 4, 5]. Our notion of a Sperner partition system is quite different; our result extends Sperner's Theorem from sets to set-partitions. A related extension of the Erdős-Ko-Rado Theorem to set partitions is found in [8].

Bollobás [1] gives a generalization of the LYM Inequality to two families of sets. For positive integers n, m let $\mathcal{A} = \{A_i, B_i : i = 1, \ldots, m\}$ be a set system of subsets from $[1, n]$ with the property that $A_i \cap B_i \neq \emptyset$ and $A_i \not\subseteq A_j \cup B_j$ for $i \neq j$. Then $\sum_{i=1}^{m} \binom{n-|B_i|}{|A_i|} \leq 1$. This result implies both Sperner's Theorem and the LYM Inequality but does not generalize to three families of sets.

References

[1] B. Bollobás. On generalized graphs. *Acta Math. Acad. Sci. Hungar.*, 16:447–452, 1965.

[2] D. E. Daykin, P. Frankl, C. Greene, and A. J. W. Hilton. A generalization of Sperner's theorem. *J. Austral. Math. Soc. Ser. A*, 31(4):481–485, 1981.

[3] P. L. Erdős and G. O. H. Katona. A 3-part Sperner theorem. *Studia Sci. Math. Hungar.*, 22(1-4):383–393, 1987.

[4] G. Katona. On a conjecture of Erdős and a stronger form of Sperner's theorem. *Studia Sci. Math. Hungar.*, 1:59–63, 1966.

[5] D. J. Kleitman. On a lemma of Littlewood and Offord on the distribution of certain sums. *Math. Z.*, 90:251–259, 1965.

[6] D. J. Kleitman and E. C. Milner. On the average size of the sets in a Sperner family. *Discrete Math.*, 6:141–147, 1973.

[7] D. Lubell. A short proof of Sperner's lemma. *J. Combinatorial Theory*, 1:299, 1966.

[8] K. Meagher and L. Moura. Erdős-Ko-Rado theorems for uniform set-partition systems. *Electron. J. Combin.*, 12(1):Research Paper 40, 12 pp. (electronic), 2005.

[9] L. D. Mešalkin. A generalization of Sperner's theorem on the number of subsets of a finite set. *Teor. Verojatnost. i Primenen*, 8:219–220, 1963.

[10] E. Sperner. Ein Satz über Untermengen einer endlichen Menge. *Math. Z.*, 27:544–548, 1928.