Another 3-Part Sperner Theorem

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Abstract

1 Introduction

In this paper, we prove a higher order Sperner theorem. These theorems are stated after some notation and background results are introduced.

For $i, j$ positive integers with $i \leq j$, let $[i, j]$ denote the set $\{i, i+1, \ldots, j\}$. For $k, n$ positive integers, set $\binom{[n]}{k} = \{A \subseteq [1, n] : |A| = k\}$. A system $\mathcal{A}$ of subsets of $[1, n]$ is said to be a $k$-set system if $\mathcal{A} \subseteq \binom{[n]}{k}$.

Two subsets $A, B$ are incomparable if $A \nsubseteq B$ and $B \nsubseteq A$. A set system on an $n$-set $\mathcal{A}$ is said to be a Sperner set system, if any two distinct sets in $\mathcal{A}$ are incomparable.

Sperner’s Theorem is concerned with the maximal cardinality of Sperner set systems as well as with the structure of such maximal systems.

Theorem (Sperner’s Theorem [10]). A Sperner set system $\mathcal{A}$ of subsets of $[1, n]$ consists of at most $\binom{n}{\lfloor n/2 \rfloor}$ sets. Moreover, a Sperner set system meets this bound if and only if $\mathcal{A} = \binom{[n]}{\lfloor n/2 \rfloor}$ or $\mathcal{A} = \binom{[n]}{\lceil n/2 \rceil}$.

One application of Sperner’s theorem is to give the exactly size of strength-2 binary covering arrays [?, klietman, katona]

A covering array, denoted $CA(n, r, k)$, is an $r \times n$ array with entries from $\mathbb{Z}_k$ with the property that for any two rows of the array that each of $k^2$ pairs from $\mathbb{Z}_k \times \mathbb{Z}_k$ occurs in some column. These are also known as strength-2 covering arrays. If $k = 2$ a $CA(n, r, k)$ is a binary covering array. The rows of a $r \times n$ binary covering array are length-$n$ 01-vectors and as such correspond to a subset of an $n$-set. The cover property for binary covering arrays can be characterized in terms of these sets.
Definition 1 (Qualitatively Independent Subsets). Two subsets $A$ and $B$ of an $n$-set are qualitatively independent subsets if

$$A \cap B \neq \emptyset, \quad A \cap B \neq \emptyset, \quad \overline{A} \cap B \neq \emptyset, \quad \overline{A} \cap B \neq \emptyset.$$ 

A strength-2 binary covering array corresponds to a set system in which any two sets are qualitatively independent. A set system $\mathcal{A}$ in which any two $A, B \in \mathcal{A}$ have the property that $A \cap B \neq \emptyset$ is called an intersecting set system. An intersecting Sperner $k$-set system with $2k \leq n$ is a qualitatively independent set system.

This leads to the following result, found independently by Katona and by Kleitman and Spencer.

Theorem 2 ([?], [?]). If $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$ is a qualitatively independent set system of an $n$-set, then

$$|\mathcal{A}| \leq \binom{n-1}{\left\lfloor n/2 \right\rfloor - 1}.$$ 

Further, this bound is attained by the system of all $\lfloor n/2 \rfloor$-sets which contain a common element.

This theorem gives the exact size of the optimal binary covering array with $r$ rows can be found for all $r$.

Theorem 3 ([?]). Let $r$ be a positive integer, then

$$\text{CAN}(r, 2) = \min \left\{ n : \binom{n-1}{\left\lfloor n/2 \right\rfloor - 1} \geq r \right\}.$$ 

Both the proof of Theorem ?? given by Katona and by Kleitman and Spencer used the well-known Erdős-Ko-Rado Theorem.

Theorem (Erdős-Ko-Rado Theorem [?]). Let $k$ and $n$ be positive integers with $2k < n$. Then for any intersecting $k$-set system on an $n$-set $\mathcal{A}$,

$$|\mathcal{A}| \leq \binom{n-1}{k-1}.$$ 

Moreover, equality holds if and only if $\mathcal{A}$ is the collection of all $k$-sets containing some fixed element.

A strength-$t$ binary covering array, denoted $t$-$CA(n, r, k)$, is an $r \times n$ array with entries from $\mathbb{Z}_k$ with the property that for any set of $t$ rows in the array, each of $k^t$ $t$-tuples from $\mathbb{Z}_k \times \mathbb{Z}_k$ occurs in some column. The rows of a binary (that is, $k = 2$) strength-$t$ covering array correspond to a set system. Again, we can characterise the covering property in terms of the set system.
Definition 4 (*t-Qualitatively Independent Set System*). A set system $\mathcal{A}$ is a *$t$-qualitatively independent set system* if for any collection of $t$ distinct sets $\{A_1, A_2, \ldots, A_t\}$ with $A_i \in \mathcal{A}$ or $\overline{A}_i \in \mathcal{A}$ for $i = 1, \ldots, t$ and $A_i \neq \overline{A}_j$ for $i, j \in [1, \ldots, t]$

$$A_1 \cap A_2 \cap \ldots \cap A_t \neq \emptyset$$

We wish to find a higher strength version of Sperner’s theorem with the goal of extending Katona’s and Kleitman and Spencer’s exact bound on strength-2 binary covering arrays to higher strength binary covering arrays.

It is trivial to give a higher order version the Erdős-Ko-Rado Theorem. We say that a set system $\mathcal{A}$ is *strength-$t$* intersecting if for any $A_1, A_2, \ldots, A_t \in \mathcal{A}$, $A_1 \cap A_2 \cap \ldots \cap A_t \neq \emptyset$. If we remove the uniform condition on the set systems, these are also known as *$r$-wise $t$-intersecting* systems. Clearly, for $t \geq 2$ a strength-$t$ intersecting set system is intersecting and Erdős-Ko-Rado Theorem holds for strength-$t$ intersecting set systems.

Extending Sperner’s Theorem is more difficult. We will focus on a strength-3 version of this theorem.

Definition 5 (*Strength-3 Sperner Set System*). A set system $\mathcal{A}$ is a *strength-3 Sperner set system* if for any three distinct sets $A, B, C \in \mathcal{A}$ the following hold:

$$A \not\subseteq B \cup C \quad B \not\subseteq A \cup C \quad C \not\subseteq A \cup B$$

$$B \cap C \not\subseteq A \quad A \cap C \not\subseteq B \quad A \cap B \not\subseteq C$$

Clearly, any strength-3 Sperner set system is also a Sperner set system.

The property $B \cap C \not\subseteq A$ implies that for $B, C \in \mathcal{A}$, $B \cap C \neq \emptyset$. So any strength-3 Sperner set system is also an intersecting set system.

Further, the property $A \not\subseteq B \cup C$ implies $A \cap \overline{B} \cap \overline{C} \neq \emptyset$ and $B \cap C \not\subseteq A$ implies that $\overline{A} \cap B \cap C \neq \emptyset$.

If $\mathcal{A}$ is a strength-3 Sperner set system then $\overline{\mathcal{A}}$ is also a strength-3 set system.

Lemma 6. If $\mathcal{A}$ is a strength-3 Sperner set system, then the set systems

$$\mathcal{A}_\cap = \{A \cap B : A, B \in \mathcal{A}, A \neq B\}$$

and

$$\mathcal{A}_\cup = \{A \cup B : A, B \in \mathcal{A}, A \neq B\}$$

are Sperner set systems.

Proof. Let $A, B, C, D \in \mathcal{A}$ and $\{A, B\} \neq \{C, D\}$. We can assume without loss of generality that $A \neq C$ and $A \neq D$. If $A \cup B \subseteq C \cup D$ then $A \subseteq C \cup D$ and $\mathcal{A}$ is not a strength-3 Sperner set system. Similarly, if $A \cap B \subseteq C \cap D$, then $A \cap B \subseteq C$. \qed 

Lemma 7. If $\mathcal{A}$ is a 3-qualitatively independent set system, then

$$\{A, \overline{A} : A \in \mathcal{A}\}$$

is a strength-3 Sperner set system.
Theorem 8. If $\mathcal{A}$ is a strength-3 Sperner set system on an $n$-set then

$$\left(\frac{|\mathcal{A}|}{2}\right) \leq \left(\frac{n}{\lfloor \frac{n}{2} \rfloor}\right)$$

Proof. The sets $A \cap B$ are unique. If $A \cap B = C \cap D$ then $A \cap B \subseteq C$. Lemma 6 and Sperner’s theorem.

* is this right? *
All logarithms are base 2.

Theorem 9. \[?\] $\lim_{k \to \infty} \frac{\text{CAN}(3,r,k)}{\log r} = \left(\frac{k}{2}\right)$.

* check this!! *
From the bound in Theorem 8 $\lim_{k \to \infty} \frac{n}{\log r} \leq \frac{n}{(n+1)\log 2 - \frac{1}{2} \log n}$. As $n$ goes to infinity this limit goes to 1, as predicted from the previous theorem.

Conjecture. The largest strength-3 Sperner set system is an $\frac{n}{2}$-set system.

Conjecture. Let $\mathcal{A}$ be a largest strength-3 Sperner set system, then $\mathcal{A}$ has the property that for all distinct $A, B \in \mathcal{A}$, $|A \cap B| = n/4$.

Theorem 10. If $\mathcal{A}$ is a strength-3 Sperner set system on an $n$-set then

$$\left(\frac{|\mathcal{A}|}{2}\right) \leq \left(\frac{n}{\lfloor \frac{n}{2} \rfloor} - 2\right)$$

Proof. Lemma 6 and Sperner’s theorem. Also that $|A \cap B| \leq \lfloor \frac{n}{2} \rfloor - 2$.

There have been extensions of Sperner’s Theorem to systems of families of sets [2] and to systems of subsets of a set $X$ with a 2-partition $X = X_1 \cup X_2$ such that no two subsets $A, B$ in the system satisfy both $A \cap X_i = B \cap X_i$ and $A \cap \overline{X}_i \subseteq B \cap \overline{X}_i$ where $i \in \{1, 2\}$ [3, 4, 5]. Our notion of a Sperner partition system is quite different; our result extends Sperner’s Theorem from sets to set-partitions. A related extension of the Erdős-Ko-Rado Theorem to set partitions is found in [8].

Bollobás [1] gives a generalization of the LYM Inequality to two families of sets. For positive integers $n, m$ let $\mathcal{A} = \{A_i, B_i : i = 1, \ldots, m\}$ be a set system of subsets from $[1, n]$ with the property that $A_i \cap B_i \not= \emptyset$ and $A_i \not\subseteq A_j \cup B_j$ for $i \not= j$. Then $\sum_{i=1}^{m} \left(\frac{n-|B_i|}{|A_i|}\right) \leq 1$. This result implies both Sperner’s Theorem and the LYM Inequality but does not generalize to three families of sets.

References


