MORE HADAMARDABLE GRAPHS

MOHAMMAD ADM[1,2], SHAUN FALLAT[1], KAREN MEAGHER[1], SHAHLA NASSERASR[1,3], MAHSA[1], AND BIDY[1]

Abstract. We aim to give new Hadamard diagonalizable graphs.

1. Introduction

An $n \times n$ matrix is a Hadamard matrix if its entries are all equal to either 1 or -1, and

$$H^t H = nI_n.$$

There are many open questions regarding Hadamard matrices.

$$H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and $H_{2^k} = H_2 \otimes H_{2^{k-1}}$. Thus there is a Hadamard matrix for every power of 2. In general, if H_1 and H_2 are both Hadamard matrices then $H_1 \otimes H_2$ is also a Hadamard matrix.

The famou Hadamard conjecture is that a Hadamard matrix of order $4k$ exists for every positive integer k.

We say that a graph is *Hadamardable* if there exists a Hadamard matrix H such that the Laplacian matrix for the graph is diagonalized by H. This implies that the columns of the Hadamard matrix are eigenvectors for the graph.

2. A generalized construction

There are a few known constructions for Hadamardable matrices.

The complete graph K_n is Hadamardable, provided that a Hadamard matrix of order n exists.

Hadamardable implies the graph is regular and all its Laplacian eigenvalues are even integers.

Theorem 2.1. *If X and Y are both Hadamardable graphs of order n and m respectively then*

(1) X^c is Hadamardable;
(2) $X \cup X$ is Hadamardable (this is the disjoint union);
(3) $X \vee X$ is Hadamardable;
(4) $X \square Y$ is Hadamardable, provided that $n = m$.

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[1] Your University.

[3]Department of Mathematics and Computer Science, Brandon University, Brandon, MB R7A 6A9, Canada.

The *direct product* of graphs X and Y is the graph with vertex set $X \times Y$ and two vertices (x_1, y_1) and (x_2, y_2) are adjacent if x_1 and x_2 are adjacent in X, and y_1 and y_2 are adjacent in Y. This product is denoted by $X \times Y$ and

$$A(X \times Y) = A(X) \otimes A(Y).$$

Lemma 2.2. *If $v_1, v_2, \dots, v_n$ is a set of eigenvectors for A, and $w_1, w_2, \dots, w_m$ is a set of eigenvectors for B then $v_i \otimes w_j$ is a set of eigenvectors for $A \otimes B$.* □

This easily implies the following.

Lemma 2.3. *If X and Y are both Hadamardable graphs then $X \times Y$ is also Hadamardable.*

Proof. Let H_X is the Hadamard matrix that diagonalizes X, and H_Y is the Hadamard matrix that diagonalizes Y. Then $h_X(i) \otimes h_Y(j)$ are eigenvectors for $X \times Y$. The matrix with $h_X(i) \times h_Y(j)$ as its columns is a Hadamard matrix that diagonalizes $X \times Y$. □

For two X and Y on the same vertex set, that have no edges in common, define $X + Y$ to be the graph on the same vertex set as X and Y and with edge set the union of the edges in X and the edges in Y.

Lemma 2.4. *Let X and Y be graphs on the same vertex set that have no edges in common. If X and Y are Hadamardable with the same Hadamard matrix then $X + Y$ is also Hadamardable.*

Proof. Since $A(X+Y) = A(X)+A(Y)$, if H is a Hadamard matrix that diagonalizes both X and Y, then H also diagonalizes $X + Y$. □

The constructions in 2.1 are generalizations of this result.
For example,

$$A(X \cup X) = I_2 \times A(X).$$

I_2 can be considered to be the Laplacian matrix for the graph with two vertices, each with a loop but no other adjacencies. This is Hadamard diagonalizable with the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Let X be a graph on n vertices, then

$$A(X \vee X) = (I_2 \times A(X)) + ((J_2 - I_2) \otimes J_n).$$

The matrix $J_2 - I_2$ is Hadamardable diagonalizable with the same matrix at I_2 (so H_2 diagaonalizes both matrices). Further, assume that H is a Hadamard matrix that diagonalizes X, then H also diagonalizes J_n. So both $I_2 \times A(X)$ and $(J_2 - I_2) \otimes J_n$ can be diagonalized by $H_2 \otimes H$.

If H is a Hadamard matrix of order n, the both J_n and I_n are diagonalizable with H, as is the matrix $J_n - I_n$. If X is a Hadamardable graph with Hadamard matrix H, then the matrix

$$(I_n \otimes X) + ((J_n - I_n) \otimes (J_n - I_n))$$

is also Hadamardable with matrix $H \otimes H$. This is the graph formed by taking the join of X with itself $|X|$ times and between each pair of copies of X a perfect matching is removed.

If X is a Hadamardable graph with Hadamard matrix H, then the matrix

$$(I_2 \otimes X) + ((J_2 - I_2) \otimes (J_n - I_n))$$

is also Hadamardable. This is the adjacencies matrix for the join of X with X and perfect matching is removed between copied of X.

3. Orthogonal Arrays graphs

The *block graph for an orthogonal array* for an orthogonal array denoted by $OA(m, n)$ has a vertex for each column of the array and the vertices are adjacent if the two columns have the same entry in the same position. This graph is denoted by $X_{OA(m,n)}$.

The orthogonal array graph is strongly regular, for an $OA(m, n)$ the parameters of the orthogonal array graph are given in the theorem below (directly copied from my book).

Theorem 3.1. *If $OA(m, n)$ is an orthogonal array where $m < n+1$, then its block graph $X_{OA(m,n)}$ is strongly regular, with parameters*

$$\big(n^2, \quad m(n-1); \quad (m-1)(m-2)+n-2, \quad m(m-1)\big),$$

and modified matrix of eigenvalues

$$\left(\begin{array}{c|cc} 1 & m(n-1) & (n-1)(n+1-m) \\ m(n-1) & n-m & m-n-1 \\ (n-1)(n+1-m) & -m & m-1 \end{array}\right). \qquad \square$$

The Lacplacian eigenvalues are $0, n(m-1), mn$. So we will consider orthogonal arrays with n even.

MacNeish's construction (**?**, Section 6.4.2) can be used to build a $OA(m, n^2)$ from an $OA(m, n)$. If the columns of the $OA(m, n)$ are denoted by c_i, then the columns of the $OA(m, n^2)$ are given by $c_i + kc_j$ for $k \in \{0, \dots, n-1\}$.

For example, the following $OA(3, 2)$

$$OA_1 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

can be used to construct this $OA(3, 4)$

$$OA_2 = \begin{bmatrix} 0011 & 0011 & 2233 & 2233 \\ 0101 & 2323 & 0101 & 2323 \\ 0110 & 2332 & 2332 & 0110 \end{bmatrix}.$$

Consider an $OA(m, n)$, denoted by OA, further, denote the columns of OA by $c_i, c_2, \dots, c_{n^2}$. For each row in the orthogonal array, define an $n^2 \times n^2$ matrix M_i with $i \in \{1, \dots, m\}$. The rows and columns of M_i are indexed by the columns in OA and the (c_i, c_j)-entry of M_i is 1 if c_i and c_j agree in row i, and zero otherwise. Call the matrices M_i with $i \in \{1, \dots, m\}$ the *row matrices* of $OA(m, n)$. Note that M_i and M_j are Schur idempotents, so $M_i \circ M_j = \delta_{i,j} M_i$

In an orthogonal array two distinct columns either have 1 row in which they have the same entry (we say the columns *intersect*) or they do not agree in any rows. Thus the (c_i, c_j)-entry of the matrix

$$\sum_{i=1}^{m} M_i$$

is m if $i = j$, 1 if columns c_i and c_j intersect and 0 otherwise. The adjacency matrix of the orthogonal array graph $X_{OA(m,n)}$ is

$$-mI_{n^2} + \sum_{i=1}^{m} M_i = \sum_{i=1}^{m} (M_i - I).$$

Lemma 3.2. *Let $OA(m, n)$ be an orthogonal array with row matrices M_i. If $OA(m, n^2)$ is the orthogonal array formed from McNeish's construction with $OA(m, n)$, then the row matrices of X are $M_i \otimes M_i$.*

Proof. Let c_i be the columns of the $OA(m, n)$. Then the columns of $OA(m, n^2)$ are $nc_i + c_j$. So columns $nc_i + c_j$ and $nc_k + c_\ell$ intersect in row r if and only if both c_i and c_k, and c_j and c_ℓ intersect in row r. So the r^{th} row matrix for $OA(m, n^2)$ is $M_r \otimes M_r$. □

Corollary 3.3. *Assume that $OA(m, n)$ is an orthogonal array and each row matrix of $OA(m, n)$ is Hadamardable with Hadamard matrix H. If $OA(m, n^2)$ is obtained from $OA(m, n)$ by MacNiesh's construction, then every row matrix for $OA(m, n^2)$ and $X_{OA(m,n^2)}$ are also Hadamardable.*

Proof. Let M_i be the row matrices for $OA(m, n)$. Each of these matrices are Hadamardable with H.

From the previous lemma the row matrices are $M_i \otimes M_i$ and the adjacency matrix of $X_{OA(m,n^2)}$ is

$$\sum_{i=1}^{m} \left((M_i \otimes M_i) - I_{n^4} \right).$$

This is a sum of matrices that are all Hadamardable by H. □

Question 3.1. *Can an orthogonal array graph be Hadamardable while its row matrices are not?*

3.1. **An example.** Define

$$OA_1 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

The orthogonal array graph for OA_1 is the complete graph, so it is Hadamardable. The eigenvectors that diagonalize it are

$$v_0 = (1, 1, 1, 1),\ v_1 = (1, 1, -1, -1),\ v_2 = (1, -1, 1, -1),\ v_3 = (1, -1, -1, 1).$$

Consider the following three matrices:

$$M_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad M_2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad M_3 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Note that B is formed by permutating the second and third rows and columns of matrix A, and C is formed by permuting the second and fourth rows and columns of

A. (These are the same permutations that change vector v_1 in to v_2 and v_3.) Thus these matrices all have every v_i as an eigenvector. So each of M_i is Hadamardable with $H = [v_1, v_2, v_3]$

Recursively define OA_k be the $OA(m, 2^k)$ formed by McNeish's construction on OA_{k-1}. From Corollary 3.3 we have the following result.

Lemma 3.4. *For all k, the graph X_{OA_k} is Hadamardable.*

Conjecture 3.1. *If $X_{OA(m,n)}$ is Hadamardable and $OA(m, n^2)$ is obtained from $OA(m, n)$ by MacNiesh's construction, then $X_{OA(m,n^2)}$ is also Hadamardable.*

3.2. **Latin Square Graph.** An orthogonal array with 3 rows is equivalent to a Latin square. Each column of such an array has three letters, and each column describes an entry in a Latin square; the first two letters give the row and the column and the three letter is the entry is the given row and column.

A *transversal* in an $n \times n$ Latin square is a set of n entries in the squares so that no two of the entries are in the same row, or in the same column or have the same value. A *complete set of transversals* is a partition of the entries of the Latin square into disjoint transversals.

The orthogonal array graph for an orthogonal array with 3 rows is the *Latin square* graph of the corresponding Latin square.

Conjecture 3.2. *If the Latin square graph for a Latin square L is Hadamardable, then L has a complete set of transversals.*

4. Further work

Note that Setion 2 shows the variety of matrices that can be diagonalized by a Hadamard matrix. Each of A, B, C can be any 01 matrix that is diagonalized by a given Hadamard matrix.