

# Lecture notes

## Zero Forcing number

Nov. , 2010

$Z(G)$  is a combinatorial parameter which gives an upper bound for the maximum nullity of a graph. Roughly speaking a zero forcing set is a bunch of black vertices in a graph which can force the other vertices to the black.

Def<sup>n</sup>

- colour-change rule

If  $G$  is a graph with each vertex either white or black,  $u$  is a black vertex of  $G$  and exactly one neighbour  $v$  of  $u$  is white, then change the colour of  $v$  to black. In this case we say  $u$  forces  $v$ .

$(u \rightarrow v)$

- Given a colouring of  $G$ , "the derived colouring" is the result of applying the colour-change rule

with no more changes (forces) are possible.

- A zero forcing set for a graph  $G$  is a subset of vertices " $Z$ " such that if initially the vertices in  $Z$  are coloured black and the remaining vertices are coloured white, the derived colouring of  $G$  is all black.
- $Z(G)$  is the minimum of  $|Z|$  over all zero forcing sets  $Z \subseteq V(G)$ .

EX:



$\{1, 3\}$  is a zero forcing set for  $P_4$  but

$\{2\}$  is not a zero forcing set for it.

Since we can colour  $P_4$  by any end point as an initial black vertex,  $Z(P_4) = 1$

Now let see how this parameter bounds the max nullity of the graph.

According to what you have seen last week as a theorem;

Thm. IF there is a set of indices  $I$ ,  $|I|=k$ , such that there is no nonzero vector  $x$  in nul space of matrix  $A$  with zeros in position  $I$  then  $\dim(\text{nul}(A)) \leq k$ .

Now Lets look at the following nice proposition:

prop Let  $Z$  be a zero forcing set of  $G=(V, E)$  and  $A \in S(G)$ . There is no vector  $x \neq 0$  in  $\ker(A)$  with coordinates determined by  $Z$ , equal zero.

[once we prove the above, we can conclude that

$m(G) \leq |Z|$  for every  $Z$ . Thus  $m(G) \leq z(G)$ ]



observations:  $z(G) \geq \delta$  ,  $z(G) \geq \omega(G) - 1$

EX :  $z(C_n) = ?$   $z(K_n) = ?$   $z(P_n) = ?$   $z(P) = ?$   
Propositions : ↓  
peterson

Let  $G$  be a strongly regular graph then

$$M(G) \geq \lfloor \frac{|G|}{2} \rfloor$$

$G$  has three eigenvalues, one is  $k$  of multiplicity 1, assume, that  $\lambda$  is its eigenvalue of max multiplicity  $m$ ,  $A_G - \lambda I \in S(G)$  and has

corank  $m$  and  $m \geq \lfloor \frac{|G| - 1}{2} \rfloor = \lfloor \frac{|G|}{2} \rfloor$

Thus

$$M(P) \geq \lfloor \frac{10}{2} \rfloor = 5 \quad \text{and also } z(P) = 5.$$

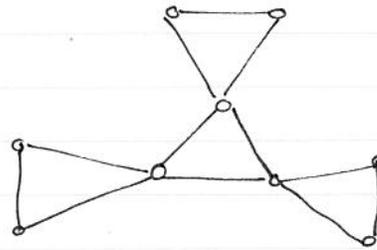
$$\text{Thus } 5 \leq M(P) \leq 5 \Rightarrow M(P) = 5$$

Q1, Can we give a formula or algorithm to find  $z(G)$ , when  $G$  is a strongly regular graph.

The corona of  $G$  with  $H$ ;  $G \circ H$  is obtained by taking one copy of  $G$  and  $|G|$  copies of  $H$  and joining all the vertices in the  $i$ th copy of  $H$  to the  $i$ th vertex of  $G$ .

EX:

$$G = K_3 \circ P_2$$



$$Z(K_3 \circ P_2) = 5,$$

since thus  $M(G) \leq 5$  on the other hand,

$$mr(G) \leq cc(G) = 4.$$

$$\text{Thus } M(G) = |G| - mr(G) \geq |G| - cc(G) = 9 - 4 = 5$$

$$\Rightarrow M(G) = 5$$

It is known that

$$Z(K_t \circ K_s) = M(K_t \circ K_s)$$

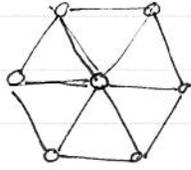
$$Z(P_t \circ K_s) = M(P_t \circ K_s)$$

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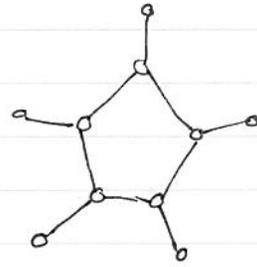
$$Z(K_t \circ P_s) = M(K_t \circ P_s)$$

Of course for too many graphs <sup>the</sup> equality does not hold

EX :



$$M(G) = 2$$
$$Z(G) = 3$$



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Q2: For which families of graphs the following equality is hold,  $Z(G) = M(G)$ .

It has been proved that for  $K_n, P_n, C_n, T$ , and any graph with  $|G| \leq 6$  and also the graphs are listed in the table #1 of the paper which appears as the first link in the DMRC webpage, the equality is hold.

One the most immediate question about any graph parameter is if or not the parameter is minor monotone.

(if yes for a given graph with a minor of known  $Z$ , we can find a bound for the  $Z(G)$ ).

Def<sup>n</sup>

a graph  $H$  is a minor of another graph  $G$  if a graph isomorphic to  $H$  can be obtained from  $G$  by contracting some edges, deleting some edges which is lead to deleting some vertices.

thus any subgraph of a graph is a minor of the original graph.

To this goal let see what happen for  $Z$  if we delete a vertex.

Lemma

If  $v$  is a vertex in graph  $G$  then

$$Z(G-v) - 1 \leq Z(G) \leq Z(G-v) + 1.$$

proof:

Let  $S$  be a min zero forcing set for  $G$  and

$S_v$  be a min zero forcing set for  $G-v$ .

obviously  $S_v \cup \{v\}$  is a zfs for  $G$

$$\text{Thus } Z(G) \leq Z(G-v) + 1.$$

to prove the remaining inequality note that  
 if we apply a zero forcing process on  $G-v$   
 with initial set of black vertices  $S \setminus \{v\}$ , then  
 the derived coloring of this process has ~~the~~ derived  
~~coloring~~ of a set of black vertices  $F$  then  
 $|N_G(v) \setminus F| \leq 1$  since  $v$  at most can  
 force one vertex to black.

Thus  $S \cup (N_G(v) \setminus F)$  is a ZFS for  
 $G-v$ . Thus

$$z(G-v) \leq |S| + |N_G(v) \setminus F| \leq z(G) + 1.$$

This completes the proof of Lemma.