Eigenvalues of the Partition Graphs

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1 Bound for Max Clique in the Partition Graph

For positive integers n, k, ℓ with $n = k\ell$, a *uniform k-partition of an n-set* is a partition of an n-set into k classes each of size ℓ. If k does not divide n, it is not possible to have uniform k-partitions of an n-set. In this case, *almost-uniform partitions* are considered. For positive integers n, k, ℓ with $n = k\ell + r$ where $0 \le r < k$, an almost-uniform k-partition of an n-set is a partition of an n-set into k classes, each of size ℓ or $\ell + 1$.

Partitions $P = \{P_1, P_2, ..., P_k\}$ and $Q = \{Q_1, Q_2, ..., Q_k\}$ are called *qualitatively independent* if for all $i, j \in \{1, ..., k\}$

$$P_i \cap Q_j \neq \emptyset.$$

If P and Q are qualitatively independent k-partitions of an n-set then the characteristic vectors of P and Q could be two rows in a covering array with parameters $CA(n, b, k)$.

1.1 Definition (Partition Graph)**.** *Let n, k, ℓ be positive integers such that $n = k\ell + r$ where $0 \le r < k \le \ell$. The* partition graph $P(n, k)$ *is the graph whose vertex set is the set of all almost-uniform k-partitions of an n-set. Vertices are adjacent if and only if the corresponding partitions are qualitatively independent.*

The almost-uniform qualitative independence graphs are vertex transitive. The number of vertices in this graph is

$$|V(AUQI(n,k))| = AU(n,k) = \frac{1}{r!(k-r)!}\binom{n}{\ell}\binom{n-\ell}{\ell}\cdots\binom{n-(k-r-1)\ell}{\ell} \binom{r(\ell+1)}{\ell+1}\binom{(r-1)(\ell+1)}{\ell+1}\cdots\binom{\ell+1}{\ell+1}.$$

A clique of size ω in $P(n,k)$ is a covering array with parameters $CA(n,\omega,k)$.

2 Ratio Bound

2.1 Theorem. *If X is an arc-transitive graph then*

$$\omega(X) \leq 1 - \frac{1}{\tau} \tag{2.1}$$

where τ is the least eigenvalue.

Can use the bound

$$\omega(X) \leq 1 - \frac{1}{\tau'}$$

where τ' is any negative eigenvalue of X.

The graph $P(k^2+i,k)$ with $0 \leq i \leq k$ are arc-transitive.

3 $P(k^2,k)$

Two partitions P and Q in the vertex set of $P(k^2,k)$ are adjacent if and only if every cell of P intersects every cell of Q in exactly one place.

3.1 Lemma. *The graph $P(k^2,k)$ is arc-transitive.*

So the ratio bound for cliques holds. Next we find some eigenvalues of this graph by using an equitable partition.

Let $S_{1,2}$ be the set of all partitions with $1,2$ in the same cell. Then $\{S_{1,2}, V(P(k^2,k))\backslash S_{1,2}\}$ is an equitable partition of the vertices in $P(k^2,k)$. It is equitable because it is the orbit partition of the group $\mathrm{Sym}(2)\times\mathrm{Sym}(k^2-2)$.

The quotient graph is

$$\begin{pmatrix} 0 & d \\ a & d-a \end{pmatrix}$$

where $a = (k!)^{k-1}/k = d/k$ and $d = (k!)^{k-1}$.

The eigenvalues are d and $-d/k$. Putting these into the ratio bound we have that

$$\omega(P(k^2,k)) \leq 1 - \frac{d}{-d/k} = k+1.$$

4 $P(k^2+k,k)$

If two partitions P and Q are adjacent in $P(k^2+k,k)$ then each cell of P has intersection of size 2 with exactly one cell of Q and intersection of size 1 with all other cells of Q.

4.1 Lemma. *The graph $P(k^2+k,k)$ is arc-transitive.*

This means that the ratio bound for cliques hold for $P(k^2+k,k)$,

$$\omega(P(k^2+k,k)) \leq 1 - \frac{d}{\tau}$$

where d is the degree of $P(k^2+k,k)$ and τ is the least eigenvalue of $P(k^2+k,k)$. Further for any eigenvalue λ with $\tau \leq \lambda < 0$ it is true that

$$\omega(P(k^2+k,k)) \leq 1 - \frac{d}{\tau} \leq 1 - \frac{d}{\lambda}.$$

This means that any negative eigenvalue will give a bound on the size of the maximum clique.

Let $S_{1,2}$ be the set of all partitions with $1,2$ in the same cell. Then $\{S_{1,2}, V(P(k^2+k,k))\backslash S_{1,2}\}$ is an equitable partition of the vertices in $P(k^2+k,k)$. It is equitable because it is the orbit partition of the group $\mathrm{Sym}(2)\times\mathrm{Sym}(k^2+k-2)$.

The quotient graph is

$$\begin{pmatrix} a & d-a \\ \frac{d-a}{k+1} & d-\frac{d-a}{k+1} \end{pmatrix}$$

where $a = (\frac{(k+1)!}{2})^{k-1}(k-1)!$ and $d = (\frac{(k+1)!}{2})^k$.

The eigenvalues are d and $\frac{(k+2)a-d}{k+1}$.

4.2 Theorem. *For $k \geq 4$, the maximum clique in $P(k^2+k,k)$ is no bigger than $k+2$.*

Proof. By the ratio bound for cliques we have that

$$\begin{aligned} \omega(P(k^2+k,k)) &\leq 1 - \frac{d}{\frac{(k+2)a-d}{k+1}} \\ &= 1 + \frac{(k+1)(k+1)!}{(k+1)!-2k-4} \end{aligned}$$

For $k \geq 4$

$$\left\lfloor \frac{(k+1)(k+1)!}{(k+1)! - 2k - 4} \right\rfloor = k+1.$$

□

4.3 Corollary. *For* $n \leq k^2 + k$

$$\omega(P(n,k)) \leq k+2$$

5 $P(k^2 + i, k)$ with $0 \leq i \leq k$

This is were some numbers are needed. I would like to know for which i is the bound on the clique size of $P(k^2 + i, k)$ is $k+1$ and for which values of i the bound is $k+2$.

The degree of $P(k^2 + i, k)$ is

$$d = (\frac{k+i}{2})^i \binom{k}{i} i!(k!)^{k-1}$$

(this shoudlbe checked)

The vertices of $P(k^2 + i, k)$ are partitions of $\{1, 2, ..., k^2 + i\}$ with cells of size k or $k+1$. There are 5 orbits from the action of $\mathrm{Sym}(2) \times \mathrm{Sym}(k^2+i-2)$ on the vertices.

a all partitions with 1 and 2 together in a cell of size k.

b all partitions with 1 and 2 together in a cell of size $k+1$.

c all partitions with 1 and 2 in seperate cells and both cells have size k

d all partitions with 1 and 2 in seperate cells and both cells have size $k+1$

e all partitions with 1 and 2 in seperate cells and one cell has size k and the other cell has size $k+1$.

6 TheQuotient Graph

The adjacency matrix for the quotient graph is:

$$\begin{pmatrix} 0 & 0 & \frac{(k-i)(k-i-1)}{k(k-1)} & \frac{i(i-1)}{k(k-1)} & 2\,\frac{i(k-i)}{k(k-1)} \\ 0 & 2\,\frac{1}{k(k+1)} & \frac{(k-i)(k-i-1)}{k(k+1)} & \frac{(i+2)(i-1)}{k(k+1)} & 2\,\frac{(k-i)(i+1)}{k(k+1)} \\ \frac{k-i}{k^2} & \frac{i}{k^2} & \frac{(k-i)(k-i-1)}{k^2} & \frac{i(i-1)}{k^2} & 2\,\frac{i(k-i)}{k^2} \\ \frac{k-i}{(k+1)^2} & \frac{i+2}{(k+1)^2} & \frac{(k-i)(k-i-1)}{(k+1)^2} & \frac{i^2+i-1}{(k+1)^2} & 2\,\frac{(k-i)(i+1)}{(k+1)^2} \\ \frac{k-i}{k(k+1)} & \frac{i+1}{k(k+1)} & \frac{(k-i)(k-i-1)}{k(k+1)} & \frac{i^2-1}{k(k+1)} & \frac{(k-i)(2\,i+1)}{k(k+1)} \end{pmatrix}$$

The eigenvalues of this quotient matrix are 0, 1, $-\frac{-k+i}{(k+1)k}$ and

$$-1/2\left(\frac{k^4-2\,k^2-i^2k-ik+k+i+i^2}{k^2\left(-1+k^2\right)(k+1)}\pm\frac{\sqrt{x}}{k^2\left(-1+k^2\right)(k+1)}\right)$$

with

$$x=9\,k^2+i^2+k^8+4\,k^7-6\,ik-12\,k^4-10\,ik^2+2\,i^3k^2+24\,ik^3-17\,i^2k^2+14\,ik^4+2\,i^3 \\ +4\,i^3k-8\,i^3k^3+14\,i^2k^4+2\,i^2k^5-8\,ik^6-14\,ik^5-6\,k^5+i^4+4\,k^6+i^4k^2-2\,i^4k$$

6.1 Theorem. *The following bounds hold*

$$\begin{aligned} \omega(P(k^2+k/2-1,k)) &\le k+1 \\ \omega(P(k^2+k/2,k)) &\le k+2 \\ \omega(P(k^2+k-2,k)) &\le k+2 \\ \omega(P(k^2+k-3,k)) &\le k+3 \end{aligned}$$

Proof. Let $i=k/2-1$ then the least eigenvalue from the above quotient graph is

$$\tau=-1/8\,\frac{4\,k^3-k^2-5\,k+2+\sqrt{16\,k^6+8\,k^5+89\,k^4+138\,k^3-235\,k^2-148\,k+132}}{\left(k^2-1\right)(k+1)\,k}.$$

Then

$$\tau < -1/8\,\frac{4\,k^3-k^2-5\,k+2+\sqrt{16\,k^6+8\,k^5-23\,k^4-22\,k^3+5\,k^2-12\,k+4}}{(k^2-1)\,(k+1)\,k}$$

$$= -1/8\,\frac{4\,k^3-k^2-5\,k+2+4\,k^3+k^2-3\,k-2)}{(k^2-1)\,(k+1)\,k}$$

$$= -\frac{1}{k+1}$$

if $k > 1$. By Inequality 2.1 we have that

$$\omega(P(k^2+k/2-1,k)) \leq 1-\frac{1}{\tau} < 1-\frac{1}{\tau'} = k+2.$$

for $i = k/2$

$$-1/8\,\frac{4\,k^3-k^2-9\,k+6+\sqrt{16\,k^6+8\,k^5-7\,k^4+18\,k^3-59\,k^2-76\,k+100}}{(k^2-1)\,(k+1)\,k}$$

try something like:

$$\tau < -1/8\,\frac{4\,k^3-k^2-9\,k+6+\sqrt{16\,k^6-56\,k^5-87\,k^4+478\,k^3-131\,k^2-1020\,k+900}}{(k^2-1)\,(k+1)\,k}$$

$$= -1/8\,\frac{4\,k^3-k^2-9\,k+6+4\,k^3-7\,k^2-17\,k+30}{(k^2-1)\,(k+1)\,k}$$

if $i = k-2$ the least evalue is:

$$-1/2\,\frac{k^4-k^3+2\,k^2-4\,k+2+\sqrt{4-64\,k+80\,k^2-12\,k^3-40\,k^4+28\,k^5+5\,k^6-2\,k^7+k^8}}{(k^2-1)\,k^2\,(k+1)}$$

for $i := k-1$;

$$-1/2\,\frac{k^2+\sqrt{k^4+16\,k+16}}{(k+1)^2\,k}$$

which give a bound of $k+3$ in Inequality 2.1. (Use the fact that $k^4+16k+16 < k^4$.)