DIFFERENTIAL GEOMETRY OF CURVES AND SURFACES

7. Geodesics and the Theorem of Gauss-Bonnet

7.1. Geodesics on a Surface. The goal of this section is to give an answer to the following question.

Question. *What kind of curves on a given surface should be the analogues of straight lines in the plane?*

Let's understand first what means "straight" line in the plane. If you want to go from a point in a plane "straight" to another one, your trajectory will be such a line. In other words, a straight line $\mathcal{L}$ has the property that if we fix two points P and Q on it, then the piece of $\mathcal{L}$ between P and Q is the shortest curve in the plane which joins the two points. Now, if instead of a plane ("flat" surface) you need to go from P to Q in a land with hills and valleys (arbitrary surface), which path will you take? This is how the notion of *geodesic* line arises:

"Definition" 7.1.1. Let S be a surface. A curve $\alpha : I \to S$ parametrized by arc length is called a *geodesic* if for any two points $P = \alpha(s_1), Q = \alpha(s_2)$ on the curve which are sufficiently close to each other, the piece of the trace of α between P and Q is the shortest of all curves in S which join P and Q.

There are at least two inconvenients concerning this definition: first, why did we say that P and Q are "sufficiently close to each other"?; second, this definition will certainly not allow us to do any explicit examples (for instance, find the geodesics on a hyperbolic paraboloid).

Let's see an example, which will at least help us understand the definition. Which are the geodesics on the unit sphere S^2? They are the *great circles*, that is, circles with centre at O (rely on your intuition or wait a bit until you get to the rigorous proof below). Let us

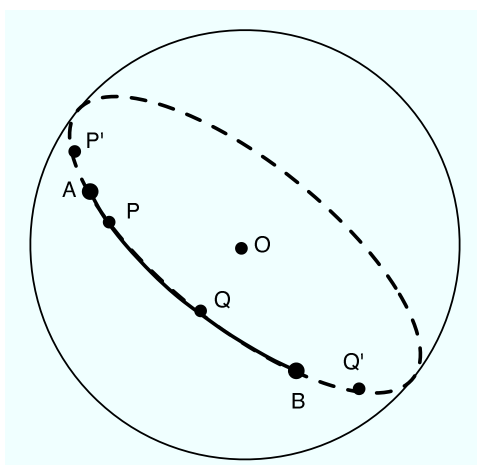

Figure 1. The shortest path between P and Q on the sphere is the (small) piece of the great circle between P and Q. Because this is true for any two points P and Q on the thickened curve, the latter curve is a geodesic.

look at Figure 1. Definition 7.1.1. gives two geodesic segments on S^2 which join A and B: the thickened and the dotted piece of the great circle. The dotted one is the strangest one, because we can find two points P' and Q' on it such that the piece of its trace between P' and Q' is *not* the shortest curve on S^2 between P' and Q'; that's only true if P' and Q' are sufficiently close to each other. Of course, we could go back to Definition 7.1.1 and change it, that is, simply remove the words "which are sufficiently close to each other". Without getting into details, we just mention that we wouldn't like the resulting definition, because it would be too restrictive.

Our next goal is to obtain descriptions of geodesics which can be used in concrete examples. To this end, we will try to understand how changes the length of a curve if we vary the curve without changing its endpoints. It is important to note that we will deal with differentiable curves α defined on *closed* intervals $[a, b]$ (until now, such intervals were always open). By definition, we say that $\alpha : [a, b] \to \mathbb{R}^3$ is a differentiable curve if there exists two numbers c and d such that $c < a < b < d$, and a differentiable curve on (c, d) whose restriction to $[a, b]$ is α.

Theorem 7.1.2. *(i) Let $\alpha : [a, b] \to \mathbb{R}^3$ be a differentiable curve parametrized by arc length and let $\alpha_\lambda : [a, b] \to \mathbb{R}^3$ be a family of curves*[1] *with $-\epsilon < \lambda < \epsilon$ which depends differentiably on λ, such that $\alpha_0 = \alpha$ (see Figure 2). Then we have*

$$\frac{d}{d\lambda}|_0 \ell(\alpha_\lambda) = -\int_a^b \frac{d}{d\lambda}|_0 \alpha_\lambda(s) \cdot \alpha''(s) ds$$

(ii) Assume that all the above curves have the trace contained in the surface S. Then we have

$$\frac{d}{d\lambda}|_0 \ell(\alpha_\lambda) = -\int_a^b \frac{d}{d\lambda}|_0 \alpha_\lambda(s) \cdot \alpha''(s)^T ds,$$

where $\alpha''(s)^T$ denotes the orthogonal projection of $\alpha''(s)$ on the plane $T_{\alpha(s)}S$.

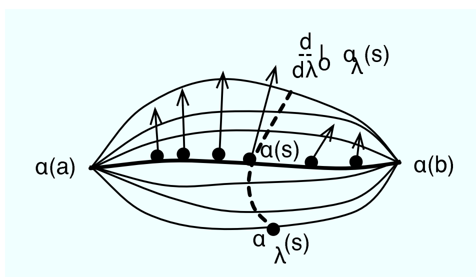

FIGURE 2. The family of curves α_λ, with $-\epsilon < \lambda < \epsilon$ is a variation of α. The dotted curve is $\lambda \mapsto \alpha_\lambda(s)$ and the vector with tail at $\alpha(s)$ is the tangent vector to the latter curve at $\alpha_0(s)$.

Proof. We have

$$\begin{aligned}
\frac{d}{d\lambda}|_0 \ell(\alpha_\lambda) &= \frac{d}{d\lambda}|_0 \int_a^b \|\alpha_\lambda'(s)\| ds \\
&= \int_a^b \frac{d}{d\lambda}|_0 \sqrt{\alpha_\lambda'(s) \cdot \alpha_\lambda'(s)} ds \\
&= \int_a^b \frac{1}{2\sqrt{\alpha_0'(s) \cdot \alpha_0'(s)}} \cdot 2(\frac{d}{d\lambda}|_0 \alpha_\lambda'(s)) \cdot \alpha_0'(s)) ds \\
&= \int_a^b (\frac{d}{d\lambda}|_0 \alpha_\lambda'(s)) \cdot \alpha'(s) ds
\end{aligned}$$

[1]Except α_0, which is the same as α, the curves α_λ are not necessarily parametrized by arc length.

We also have that
$$\frac{d}{d\lambda}|_0 \alpha'_\lambda(s) = (\frac{d}{d\lambda}|_0 \alpha_\lambda(s))'$$
because differentiation w.r.t. s and λ can be interchanged with each other. We use the integration by parts formula and write further
$$\frac{d}{d\lambda}|_0 \alpha_\lambda(s) \cdot \alpha'(s)|_{s=a}^{s=b} - \int_a^b \frac{d}{d\lambda}|_0 \alpha_\lambda(s) \cdot \alpha''(s) ds$$
which gives the desired result, as both $\alpha_\lambda(a)$ and $\alpha_\lambda(b)$ are constant. This is how we prove (i). To prove (ii), we decompose
$$\alpha''(s) = \alpha''(s)^T + \alpha''(s)^N$$
where $\alpha''(s)^N$ is perpendicular to $T_{\alpha(s)}S$; we also note that $\frac{d}{d\lambda}|_0 \alpha_\lambda(s)$ is in $T_{\alpha(s)}S$, so
$$\alpha''(s) \cdot (\frac{d}{d\lambda}|_0 \alpha_\lambda(s)) = \alpha''(s)^T \cdot (\frac{d}{d\lambda}|_0 \alpha_\lambda(s)).$$
□

We deduce the following corollary.

Corollary 7.1.3. *A curve $\alpha : I \to S$ parametrized by arc length is a geodesic if and only if*
$$\alpha''(s)^T = 0,$$
for all s in I (as before, $\alpha''(s)^T$ denotes the orthogonal projection of $\alpha''(s)$ to $T_{\alpha(s)}S$). Equivalently, for any s in I, the vector $\alpha''(s)$ is perpendicular to the tangent plane at $\alpha(s)$ to S.

Note. The corollary is for us the main characterization of a geodesic, which will be used throughout the course. Most textbooks use this as a definition. Our Definition 7.1.1 is certainly more intuitive, but less useful and less clear (what exactly means "P and Q *sufficiently* close to each other"?)

The idea of the proof of Corollary 7.1.3. If α is a geodesic, then we pick two points P and Q on the curve, sufficiently close to each other, and vary the piece of α between P and Q; the length of the latter curve is minimal, thus the derivative with respect to the variation parameter λ is equal to 0. From Theorem 7.1.2 (ii), we can deduce that $\alpha''(s)^T = 0$, for all s. The converse (if $\alpha''(s)^T = 0$ for all s, then α is a geodesic) is harder to prove (see for instance [dC], Section 4-6.)

Examples. 1. Great circles on a sphere are geodesics. Because if $\alpha : \mathbb{R} \to S^2$ is a parametrization by arc length of such a circle, then for any s in $\mathbb{R}$, the vector $\alpha''(s)$ is parallel to the radius of $\alpha(s)$ (being perpendicular to $\alpha'(s)$), thus it is perpendicular to the tangent plane at $\alpha(s)$ to S^2.

2. One can also determine the geodesics on a cylinder C (see Chapter 4, Figure 4). To this end we use the local isometry f from the plane to the cylinder described in HW no. 4, Question no. 4. The point is that in general, a local isometry between two surfaces maps geodesics to geodesics (see Definition 7.1.1 and remember that a local isometry preserves lengths of curves). Thus geodesics on the cylinder are images of straight lines under f (the "rolling" map); it's easy to see that they are just helices[2] on the cylinder.

In general, finding the geodesics on a given surface is not easy. In the following we will show how to determine geodesics contained in the image of a local parametrization, by solving a system of differential equations.

[2]See Chapter 2.

Proposition 7.1.4. *Let (U, φ) be a local parametrization of the surface S and let $\alpha : I \to S$ be a curve parametrized by arc length, whose trace is contained in $\varphi(U)$. Write*

$$\alpha(s) = \varphi(u(s), v(s)),$$

where $s \mapsto u(s)$ and $s \mapsto v(s)$ are real functions of s. Then α is a geodesic if and only if[3]

$$(1)\quad u''(s) + \Gamma^1_{11}(u(s), v(s))(u'(s))^2 + 2\Gamma^1_{12}(u(s), v(s))u(s)'v'(s) + \Gamma^1_{22}(u(s), v(s))(v'(s))^2 = 0$$
$$v''(s) + \Gamma^2_{11}(u(s), v(s))(u'(s))^2 + 2\Gamma^2_{12}(u(s), v(s))u'(s)v'(s) + \Gamma^2_{22}(u(s), v(s))(v'(s))^2 = 0.$$

Here Γ^k_{ij} are the Christoffel symbols associated to (U, φ) (see Chapter 6 of the notes).

Proof (sketch). In each space $T_{\varphi(Q)}S$ we have the basis $\varphi'_u(Q), \varphi'_v(Q)$. By Lemma 3.3.3, we have

$$\alpha'(s) = u'(s)\varphi'_u(u(s), v(s)) + v'(s)\varphi'_v(u(s), v(s)).$$

Consequently, if we omit writing $(u(s), v(s))$, we have

$$(2)\quad \alpha''(s) = u''(s)\varphi'_u + u'(s)(\varphi''_{uu}u'(s) + \varphi''_{uv}v'(s)) + v''(s)\varphi'_v + v'(s)(\varphi''_{vu}u'(s) + \varphi''_{vv}v'(s)).$$

Now $\varphi''_{uu}, \varphi''_{uv}, \varphi''_{vu}$ and φ''_{vv} can be expressed by Equations (1), Chapter 6. We plug them into (2) and take into account that the equation $\alpha''(s)^T = 0$ means that the coefficients of φ'_u and φ'_v are both zero. These give the two equations (1). $\square$

From the general theory of differential equations we can deduce the following geometric information about geodesics: if P is a point on S and w a vector in T_PS, with $\|w\| = 1$, then there exists a unique geodesic $\alpha : I \to S$ with

$$\alpha(0) = P \text{ and } \alpha'(0) = w.$$

Here I is a "small" interval around 0. An interesting question is how "big" can be made the interval I? And an interesting answer is: if S is compact (with respect to the subspace topology induced from $\mathbb{R}^3$), then we can always take $I = \mathbb{R}$; that is, we can extend the domain of any geodesic to the whole $\mathbb{R}$. We will not address this point here (this is a consequence of the theorem of Hopf-Rinow, see for instance [dC, Section 5-3]).

7.2. The Theorem of Gauss-Bonnet. The sum of the angles of a triangle is equal to π. Equivalently, in the triangle represented in Figure 3, we have

$$\theta_1 + \theta_2 + \theta_3 = 2\pi.$$

Proof: if you go along the triangle, when you come back to where you started the trip, you will have rotated yourself with 360^0.

Now let's take a triangle on the sphere S^2, whose sides are geodesics (that is, pieces of great circles). Let us try to determine again $\theta_1 + \theta_2 + \theta_3$, this time in Figure 4. We can see there that the sphere is partitioned in the following five pieces: the thickened triangle, its antipodal image, and the slices of angle θ_1, θ_2, and θ_3. The slice of angle θ_3 has area $2\theta_3$ (because the slice of size 2π is the whole sphere, and has area 4π), and similarly for θ_1 and θ_2. If A denotes the area of the thickened triangle, then we have

$$2A + 2\theta_1 + 2\theta_2 + 2\theta_3 = 4\pi \Rightarrow \theta_1 + \theta_2 + \theta_3 = 2\pi - A.$$

So the sum of the (inside) angles of the triangle is this time equal to $\pi + A$.

In this section we will give a formula for the sum of the angles of a geodesic polygon on an *arbitrary* surface. There is a preparatory result we need to prove first. We will consider a special kind of a local parametrization, namely *orthogonal parametrization.* By definition, this is (U, φ) with the property that for any Q in U, the vectors $\varphi'_u(Q)$ and $\varphi'_v(Q)$ are

[3]It is important to note that (1) is a nonlinear system of two differential equations of order two with unknowns $u(s)$ and $v(s)$.

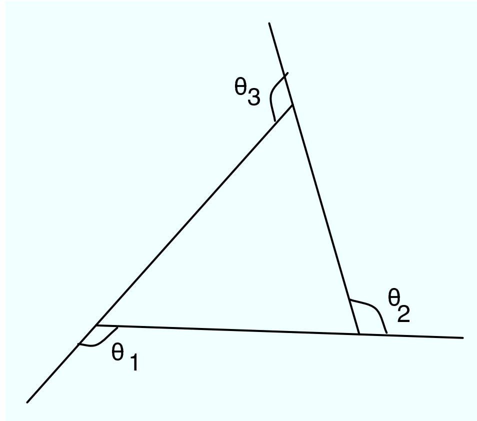

FIGURE 3. $\theta_1 + \theta_2 + \theta_3 = 2\pi$

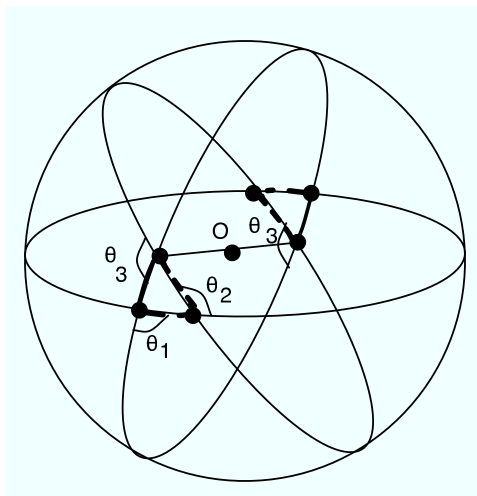

FIGURE 4. $\theta_1 + \theta_2 + \theta_3 < 2\pi$

perpendicular (orthogonal). Equivalently we have $F = 0$ everywhere on U. Consequently, the vectors

$$e_1 = \frac{1}{\sqrt{E}}\varphi'_u,\ e_2 = \frac{1}{\sqrt{G}}\varphi'_v \tag{3}$$

are an orthonormal basis in every tangent space.

Lemma 7.2.1. *Let (U, φ) be an orthogonal parametrization of a surface and $\alpha : (a, b) \to \varphi(U)$ a geodesic parametrized by arc length, of the form*

$$\alpha(s) = \varphi(u(s), v(s)),$$

for all s in (a, b). If $\phi(s)$ denotes the angle from[4] *$\varphi'_u(\alpha(s))$ to $\alpha'(s)$, then we have*

$$\frac{d\phi}{ds} = \frac{1}{2\sqrt{EG}}\left(E'_v\frac{du}{ds} - G'_u\frac{dv}{ds}\right).$$

Proof. First we prove the following claim.

[4]The angle $\phi = \phi(s)$ satisfies $0 \le \phi < 2\pi$, and is uniquely determined by $\alpha' = (\cos\phi)e_1 + (\sin\phi)e_2$

Claim. We have

$$\left(\frac{d}{ds}e_1(\alpha(s))\right)^T = -\frac{d\phi}{ds}e_2(\alpha(s)),$$

where, as usually, the superscript T indicates the orthogonal projection to the tangent space.

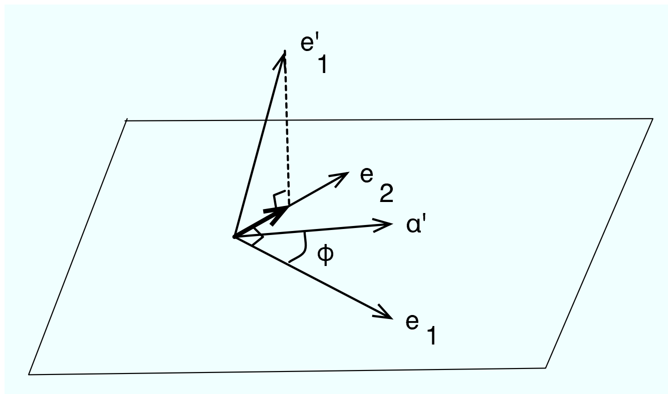

FIGURE 5. Proof of Lemma 7.2.1 (the plane is $T_{\alpha(s)}S$ and the thickened arrow is $(e_1')^T$).

To prove the claim, we differentiate the relation

$$\alpha' \cdot e_1 = \cos\phi$$

and obtain

$$\alpha'' \cdot e_1 + \alpha' \cdot e_1' = -(\sin\phi)\phi' \Rightarrow \alpha' \cdot e_1' = -(\sin\phi)\phi', \tag{4}$$

where we have used that α is a geodesic. On the other hand, because $\|e_1\| = 1$, the vector e_1' is perpendicular to e_1, so the vector $(e_1')^T$ is parallel to e_2, that is, $(e_1')^T = \lambda e_2$, for some number λ (see also Figure 5). Because

$$\alpha' \cdot e_1' = \alpha' \cdot (e_1')^T = \lambda\alpha' \cdot e_2 = \lambda\cos(\pi/2 - \phi) = \lambda\sin\phi,$$

equation (4) implies

$$\lambda = -\phi',$$

and the claim is proved.

From the claim we deduce

$$\frac{d\phi}{ds} = -\frac{d}{ds}(e_1(\alpha(s))) \cdot e_2(\alpha(s)).$$

We replace

$$\frac{d}{ds}(e_1(\alpha(s)) = \frac{d}{ds}(e_1(u(s), v(s)) = (e_1')_u\frac{du}{ds} + (e_1')_v\frac{dv}{ds},$$

then we take into account of (3) and deduce the desired formula (we skip the computations). □

We will also need the notion of surface integral of a function (you may have seen this in the Vector Calculus course). Namely, of (U, φ) is a local parametrization of a surface S, $f : S \to \mathbb{R}$ a function, and R a compact subspace of $\varphi(U)$, then, by definition

$$\int\int_R f d\sigma := \int\int_{\varphi^{-1}(R)} f(\varphi(u, v))\|\varphi_u' \times \varphi_v'\|dA = \int\int_{\varphi^{-1}(R)} f \circ \varphi\sqrt{EG - F^2}dA \tag{5}$$

where the last two integrals are double integrals over the the region $\varphi^{-1}(R)$ in $\mathbb{R}^2$. The last equality comes from the fact that

$$\|\varphi_u' \times \varphi_v'\| = \sqrt{EG - F^2}.$$

We are now ready to state and prove the following version of the theorem of Gauss-Bonnet.

Theorem 7.2.2. (Gauss-Bonnet local). *Let (U, φ) be a local orthogonal parametrization of a surface and let R contained in $\varphi(U)$ be a region bounded by the geodesics $\alpha_k : I_k \to S$, $k = 1, 2, 3$, with*

$$\alpha_1(s_{11}) = \alpha_2(s_{21}),\ \ \alpha_2(s_{22}) = \alpha_3(s_{32}),\ \text{ and } \alpha_3(s_{33}) = \alpha_1(s_{13}),$$

for the numbers

- $s_{13} < s_{11}$ *in* I_1
- $s_{21} < s_{22}$ *in* I_2
- $s_{32} < s_{33}$ *in* I_3

Denote by θ_1, θ_2, θ_3 the external angles of the triangle R (for instance, θ_1 is the angle from[5] *$\alpha_1'(s_{13})$ to $\alpha_3'(s_{33})$ etc., see Figure 6). Then we have*

$$\theta_1 + \theta_2 + \theta_3 = 2\pi - \int\!\!\int_R K d\sigma,$$

where K is the Gauss curvature.

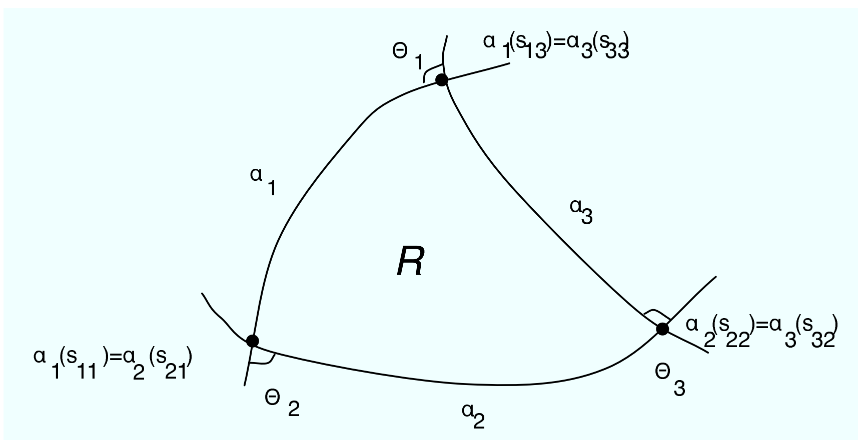

FIGURE 6. The region R is bounded by (pieces of traces of) the geodesics α_1, α_2, and α_3. By θ_1 we mean the angle between the tangent vectors $\alpha_3'(s_{33})$ and $\alpha_1'(s_{13})$; similarly for θ_2 and θ_3.

Proof (sketch). We use Lemma 7.2.1 for the geodesics α_i, where $i = 1, 2, 3$. Denote by ϕ_i the angle between φ_u' and α_i' along α_i. We have

$$\frac{d\phi_i}{ds} = \frac{1}{2\sqrt{EG}} \left(E_v' \frac{du}{ds} - G_u' \frac{dv}{ds} \right).$$

[5] It is important, although not enough emphasized in these notes, that the surface is *oriented*: this means that in any tangent spaces we have a distinguished sense of rotation, and this allows us to measure the *signed* angle between two non-opposite vectors in a tangent space to the surface, which **is in the interval** $(-\pi, \pi)$.

along φ_i. If we integrate we obtain

$$
\begin{aligned}
\phi_1(s_{11}) - \phi_1(s_{13}) &= \int_{s_{11}}^{s_{13}} \frac{1}{2\sqrt{EG}} \left(E_v' \frac{du}{ds} - G_u' \frac{dv}{ds} \right) \\
\phi_2(s_{22}) - \phi_2(s_{21}) &= \int_{s_{21}}^{s_{22}} \frac{1}{2\sqrt{EG}} \left(E_v' \frac{du}{ds} - G_u' \frac{dv}{ds} \right) \\
\phi_3(s_{33}) - \phi_3(s_{32}) &= \int_{s_{32}}^{s_{33}} \frac{1}{2\sqrt{EG}} \left(E_v' \frac{du}{ds} - G_u' \frac{dv}{ds} \right)
\end{aligned} \tag{6}
$$

If we add the left hand sides we obtain

$$(\phi_3(s_{33}) - \phi_1(s_{13})) + (\phi_2(s_{22}) - \phi_3(s_{32})) + (\phi_1(s_{11}) - \phi_2(s_{21})).$$

If we look at Figures 6 and 7, we can easily see that the latter expression is equal to $-\theta_1 - \theta_2 - \theta_3 + 2k\pi$, where k is an integer. If Figure 6 was in a plane (not on an arbitrary surface) and φ_u' was the same, the sum would actually be always $-\theta_1 - \theta_2 - \theta_3 + 2\pi$. One can show (and it's not trivial) that even if the triangle is on a surface, the sum equals $-\theta_1 - \theta_2 - \theta_3 + 2\pi$.

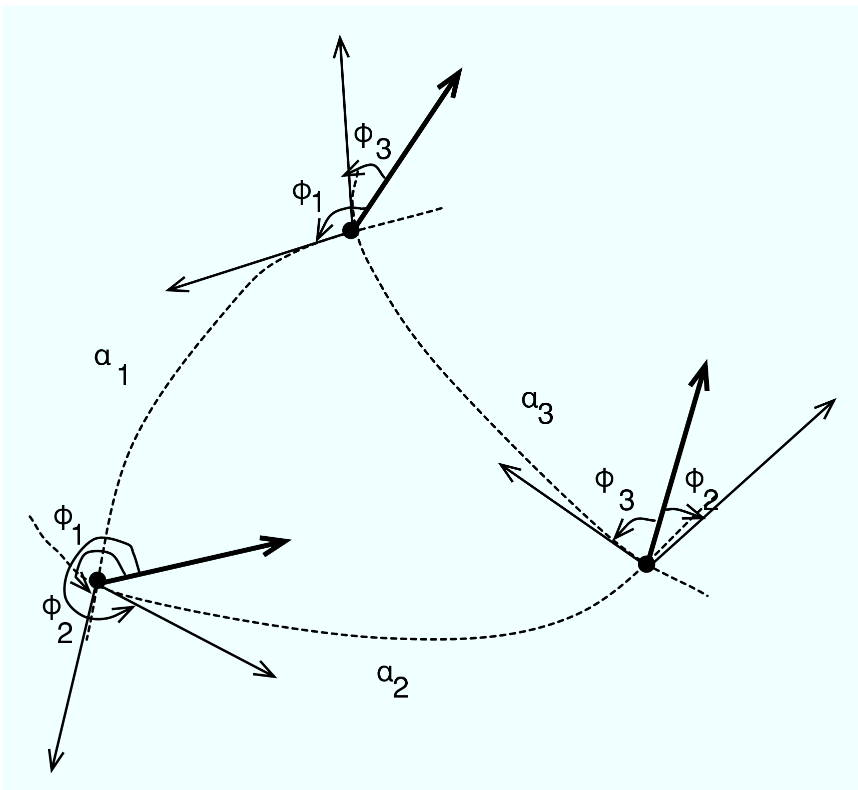

FIGURE 7. The thickened arrows are φ_u' at each of the three points.

Now let's analyze the sum of the right hand sides of Equations (6). This represents the line integral

$$\oint_C \frac{E_v'}{2\sqrt{EG}} du - \frac{G_u'}{2\sqrt{EG}} dv,$$

where C is the (closed) path in $\mathbb{R}^2$ (with coordinates u, v) obtained by joining the paths

$$s \mapsto \varphi^{-1}(\alpha_1(s)),\ s \mapsto \varphi^{-1}(\alpha_2(s)),\ \text{and } s \mapsto \varphi^{-1}(\alpha_3(s)).$$

Green's formula says that

$$\oint_C f du + g dv = \int\int_D \left(\frac{\partial g}{\partial u} - \frac{\partial f}{\partial v}\right) dA.$$

In our case, we obtain

$$\oint_C \frac{E'_v}{2\sqrt{EG}} du - \frac{G'_u}{2\sqrt{EG}} dv = -\int\int_{\varphi^{-1}(R)} \left(\left(\frac{E'_v}{2\sqrt{EG}}\right)'_v + \left(\frac{G'_u}{2\sqrt{EG}}\right)'_u\right) dA.$$

In Chapter 6 we have obtained a formula for the curvature K in terms of the coefficients E, F, and G of the first fundamental form. It turns out that this formula, used in the case $F = 0$, becomes very simple, namely

$$K = -\frac{1}{2\sqrt{EG}} \left(\left(\frac{E'_v}{\sqrt{EG}}\right)'_v + \left(\frac{G'_u}{\sqrt{EG}}\right)'_u\right).$$

This implies

$$\oint_C \frac{E'_v}{2\sqrt{EG}} du - \frac{G'_u}{2\sqrt{EG}} dv = \int\int_{\varphi^{-1}(R)} K\sqrt{EG} dA = \int\int_R K d\sigma,$$

where we have used Equation (5). The theorem is now proved. □

To understand the global version of the theorem of Gauss-Bonnet, we need a few more notions, as follows:

- a <u>*region*</u> R on a surface S is a compact subset of S bounded by finitely many curves
- a <u>*triangle*</u> is a region bounded by three curves
- a <u>*triangulation*</u> of a region R is a finite collection of triangles $T_1, \dots, T_n$ such that
  1. $\bigcup_{i=1}^n T_i = R$
  2. If $T_i \cap T_j \neq \phi$, then either $i = j$, or $T_i \cap T_j$ is a common edge of T_i and T_j, or $T_i \cap T_j$ is a common vertex of T_i and T_j.

Figure 8 should help in understanding these notions. A <u>*geodesic region*</u> is a region bounded by geodesics. A <u>*geodesic triangle*</u> is a triangle whose edges are geodesics. A <u>*geodesic triangulation*</u> is a triangulation by geodesic triangles. Compact surfaces (like the sphere, the ellipsoid, the torus etc. — you are free to think of those as surfaces that have an "inside") are geodesic regions (not bounded by any geodesic). If $T_1, \dots, T_n$ is a triangulation of the region R, then we denote as follows:

- n_2 the number of triangles (actually $n_2 = n$)
- n_1 the number of edges
- n_0 the number of vertices
- $\chi = n_2 - n_1 + n_0$

We will rely (without proof) on the following two propositions.

Proposition 7.2.3. *(a) Any region on a surface (thus any compact surface) admits a triangulation. Any geodesic region has a geodesic triangulation. Moreover, in the latter case we can refine the triangulation in such a way that any triangle is contained in the image of an orthogonal parametrization.*

(b) If R is a region, then the number $\chi = n_2 - n_1 + n_0$ is the same for any triangulation of R. We call it the <u>Euler-Poincaré characteristic</u> *of R, denoted $\chi(R)$.*

Proposition 7.2.4. *Let R be a region in the surface S and $T_1, \dots, T_n$ a triangulation with the property that each T_i is contained in the image of a local parametrization. Let also f be*

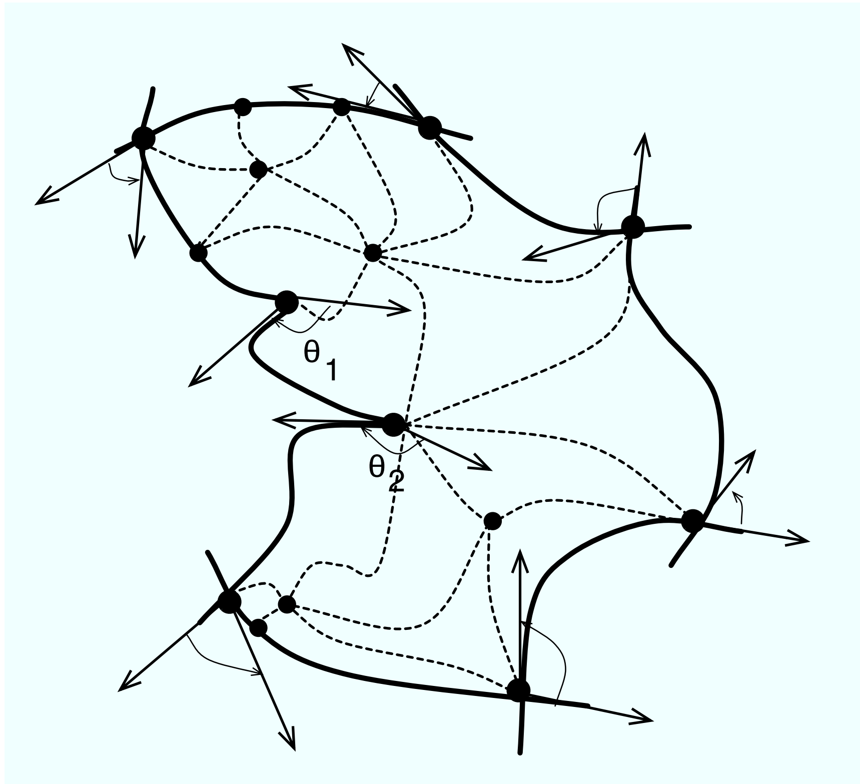

FIGURE 8. A region on a surface and a triangulation of it (with $n_0 = 16$, $n_1 = 35$, and $n_2 = 18$). We take this opportunity to mention that, as explained in the footnote on page 7, the angles θ_1 and θ_2 are negative (the other external angles are all positive).

a differentiable function on S. The number

$$\sum_{i=1}^{n} \int_{T_i} f d\sigma =: \int_R f d\sigma$$

does not depend on the choice of the triangulation (note that each term of the sum above is well defined in view of Equation (5)). We call this the integral of f on R.

We are now ready to state a more general version[6] of the theorem of Gauss-Bonnet.

Theorem 7.2.5. (Gauss-Bonnet global) *Let S be an orientable surface and R a geodesic region in S. Choose the parametrizations of the curves which bound R in such a way that we can go along the contour determined by them in the sense given by the orientation with increasing parameters; let $\theta_1, \theta_2, \ldots, \theta_n$ be the external angles (like in Theorem 7.2.2, see also Figure 8). Then we have*

$$\theta_1 + \theta_2 + \ldots + \theta_n = 2\pi\chi(R) - \int\int_R K d\sigma.$$

[6]This is not the most general version of the theorem, though — cf. [dC], page 274 or [Gr-Abb-Sa], page 918.

Corollary 7.2.6. *If S is a compact orientable surface, then*

$$\int\int_S K d\sigma = 2\pi\chi(S).$$

Rather than the theorem, we will actually prove the corollary, independently of the theorem. Of course, the corollary can be deduced from the theorem by taking $R = S$, which implies that the boundary of R is empty and there are no angles θ_i to be taken into account: but again, this is not what we're going to do.

Proof of Corollary 7.2.6. Let us consider a geodesic triangulation $T_1, \dots, T_n$ of S with the property that each triangle is contained in the image of an orthogonal parametrization whose orientation is compatible with the global orientation of the surface (see Proposition 7.2.3 (a)). As usually, n_0, n_1, n_2 are the number of vertices, edges, respectively triangles of the triangulation. For each $i = 1, 2, \dots, n$ we consider the external angles $\theta_i^1, \theta_i^2, \theta_i^3$ of the triangle T_i (recall that they are in $(-\pi, \pi)$ and they depend on the orientation) and the *internal angles*

$$\alpha_i^1 := \pi - \theta_i^1, \ \alpha_i^2 := \pi - \theta_i^2, \ \alpha_i^3 := \pi - \theta_i^3.$$

Theorem 7.2.2 for the triangle T_i says that

$$\int\int_{T_i} K d\sigma = \alpha_i^1 + \alpha_i^2 + \alpha_i^3 - \pi.$$

We add up all these equalities and obtain

$$\int\int_S K d\sigma = \sum_{i=1}^n (\alpha_i^1 + \alpha_i^2 + \alpha_i^3) - n_2\pi. \tag{7}$$

The sum of angles at every vertex of the triangulation is 2π, thus

$$\sum_{i=1}^n (\alpha_i^1 + \alpha_i^2 + \alpha_i^3) = 2\pi n_0.$$

Every edge of the triangulation belongs to two triangles, thus

$$3n_2 = 2n_1.$$

Now (7) implies

$$\int\int_S K d\sigma = (2n_0 - n_2)\pi = (2n_0 - 2n_1 + 2n_2)\pi = \chi(S)2\pi.$$

The corollary is proved. □

Examples of compact surfaces are: the sphere, the torus with one or several holes (see Figure 9), and any smooth deformations of those. For instance, we can deform the sphere and obtain ellipsoids, but not only — there are many shapes you can get by deforming the sphere (use your imagination, you can get a potato, a cucumber, a tomato etc.)

The equation stated in the corollary is surprising because if we have a compact surface S and we deform it, then

(a) *in the right hand side, $\chi(S)$ does not change under (differentiable) deformations*: for instance, take the triangulated sphere in Figure 10 and make strange shapes out of it (for instance, a cucumber), without making sharp edges or corners and without making self-intersections; the triangulation is preserved during the deformation, as well as the number of triangles, edges, and vertices.

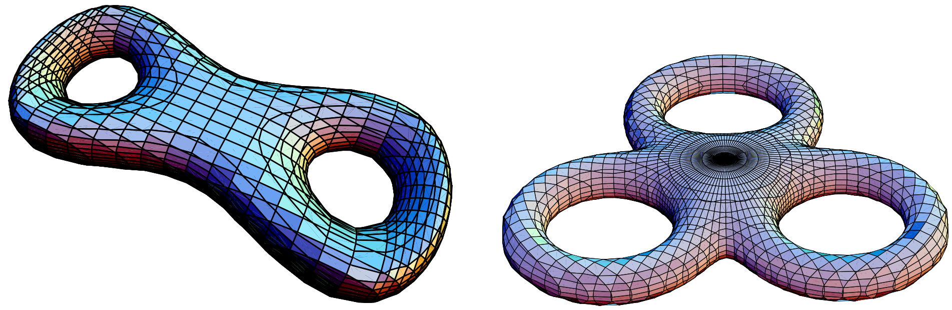

Figure 9. Tori with two, respectively three holes.

(b) *in the left hand side, K changes during the deformation*: again, for the sphere, you can deform it to a thin and long cucumber in such a way that the top is very curved, so K at that point will become much larger than 1.

Nevertheless, in spite of what we said at (b), the "average" of K on S does *not* change under the deformation: and this is unexpected!

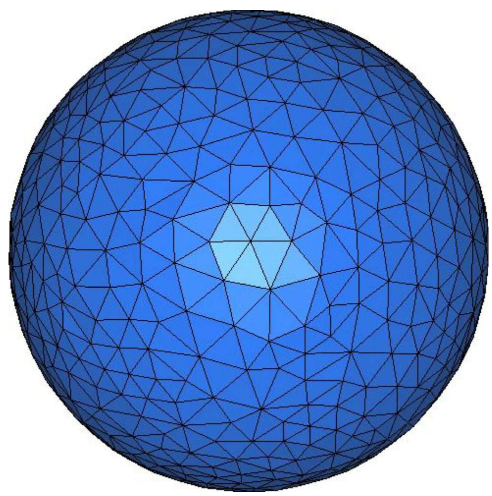

Figure 10. A triangulation of the sphere.

Let us compute the Euler-Poincaré characteristic of the sphere. One way of doing this is by inscribing a tetrahedron in the sphere and joining the four vertices by curves on the sphere. We obtain a triangulation with $n_2 = 4$, $n_1 = 6$, $n_0 = 4$ (see Figure 11), so

$$\chi(S^2) = 4 - 6 + 4 = 2.$$

To calculate the Euler-Poincaré characteristic of the torus, we use the triangulation indicated in Figure 12. We have $n_2 = 24$, $n_1 = 36$, $n_0 = 12$. This implies

$$\chi(T) = 24 - 36 + 12 = 0.$$

One can show that a torus with k holes has the Euler-Poincaré characteristic equal to $2 - 2k$ (so it can be negative).

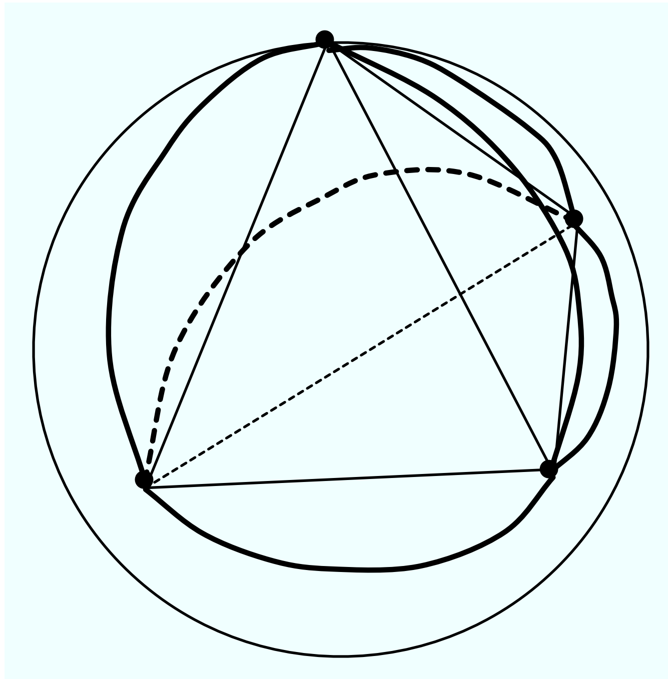

FIGURE 11. A more lucrative triangulation of the sphere: the thickened points and lines are all on the sphere.

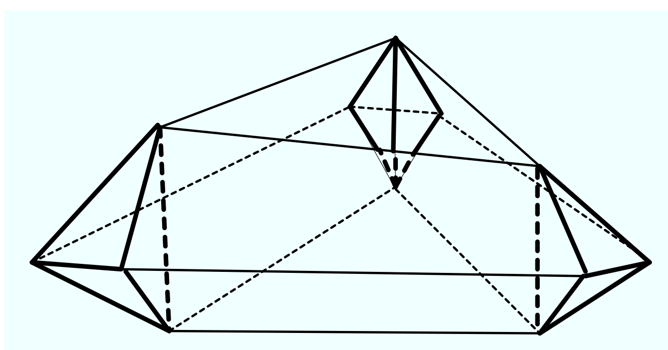

FIGURE 12. A triangulation of the torus: the figure is incomplete, as each quadrilateral has to be divided in two triangles. Warning: not all triangles you see here are elements of the triangulation (only 24 of them).

REFERENCES

[dC] M. P. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice Hall 1976

[Gr-Abb-Sa] A. Gray, E. Abbena, and S. Salamon, *Modern Differential Geometry with MATHEMATICA*, 3rd edition