DIFFERENTIAL GEOMETRY OF CURVES AND SURFACES

6. Gauss' Theorema Egregium

Question. *How can we decide if two given surfaces can be obtained from each other by "bending without stretching"?*

The simplest example is a flat strip, say of paper, which can be rolled (without stretching!) to give a cylinder; a bit more precisely, a piece of the plane can be deformed to a piece of a cylinder. On the other hand, it is intuitively clear that we cannot deform without stretching a piece of a plane to a piece of a sphere (or of an ellipse, of a hyperboloid, of a torus etc. — just try to wrap such an object!) In general, the question is hard. Let's try to translate the question into a more formal language:

- bending = *map between surfaces*
- without stretching = *preserving the distances between two points*, which is the same as *local isometry* (see the last section in chapter 4 of the notes)

Gauss came up with the idea that an answer to our question can be given by measuring curvature of curves on a surface, more precisely, the principal curvatures, but in a very ingenious way. Because if we compare the plane and the cylinder we see that their principal curvatures are $k_1^{\text{plane}} = k_2^{\text{plane}} = 0$ whereas $k_1^{\text{cylinder}} = 1$, $k_2^{\text{cylinder}} = 0$. Even though they are locally isometric, the plane and the cylinder have different principal curvatures. Nevertheless, their product is the same! Recall that the latter product is what we called the Gauss curvature. In fact, Gauss' observation was that what we noticed in the case of the deformation plane→cylinder is a very general phenomenon: if there is a deformation without stretching of the surface S_1 to the surface S_2, then the product of the principal curvatures at any point on S_1 is preserved by the deformation. Another instructive (and easy to visualize, thanks to the computer generated animation available online) example is the continuous deformation of the helicoid in the catenoid. If you follow a point on the helicoid during this process, the Gauss curvature doesn't change, even though the surface changes its shape radically during the process. Gauss' theorem can be stated as follows:

Theorema Egregium.[1] *If $f : S_1 \to S_2$ is a local isometry, then the Gauss curvature of S_1 at P equals the Gauss curvature of S_2 at $f(P)$.*

Remark. 1. The theorem can only be used to rule out (local) isometries between surfaces. By this we mean that the converse of the theorem is not true: one can find a diffeomorphism (that is, a differentiable bijective map) between surfaces which preserves the Gauss curvature at any point, but is not an isometry. You will see such an example in Homework no. 5. In the same spirit: we cannot deduce that the plane and the cylinder (or the helicoid and the catenoid) are locally isometric just because they have the same Gauss curvature!

2. In fact the importance of the theorem goes beyond the application mentioned above (that is, rule out locally isometric surfaces). Many authors consider it "the most important single theorem in differential geometry". But it's not easy to explain why, so we refer the reader to Spivak's monograph [Sp], the end of chapter 3, part B (page 143 - 144, "What does Theorema Egregium really mean?").

The rest of the chapter is devoted to the proof of this theorem. The key point is to consider a local parametrization (U, φ) and produce a formula for $K(\varphi(Q))$ (where Q is in U) which depends only on the functions E, F, and G (the coefficients of the first fundamental form).

[1] Gauss' original paper where this theorem was stated and proven was written in Latin. He says that this is a *remarkable theorem*, which in Latin means *theorema egregium* (egregium=remarkable).

First we note that for any Q in U, the vectors φ'_u, φ'_v, and N furnish a basis of $\mathbb{R}^3$. Thus we can consider the expansions

$$
\begin{aligned}
\varphi''_{uu} &= \Gamma^1_{11}\varphi'_u + \Gamma^2_{11}\varphi'_v + L_1 N\\
\varphi''_{uv} &= \Gamma^1_{12}\varphi'_u + \Gamma^2_{12}\varphi'_v + L_2 N\\
\varphi''_{vu} &= \Gamma^1_{21}\varphi'_u + \Gamma^2_{21}\varphi'_v + L_2 N\\
\varphi''_{vv} &= \Gamma^1_{22}\varphi'_u + \Gamma^2_{22}\varphi'_v + L_3 N\\
N'_u &= a_{11}\varphi'_u + a_{21}\varphi'_v\\
N'_v &= a_{12}\varphi'_u + a_{22}\varphi'_v.
\end{aligned} \tag{1}
$$

The numbers Γ^k_{ij} are called the *Christoffel symbols* of the parametrization (U, φ). Altogether, they are in number of $2 \times 2 \times 2 = 8$, but in fact, since $\varphi'_{uv} = \varphi'_{vu}$, we have

$$\Gamma^1_{12} = \Gamma^1_{21} \text{ and } \Gamma^2_{12} = \Gamma^2_{21}.$$

We take dot product of the first four equations with N and use equations (3), page 12, chapter 5 of the notes. We obtain

$$L_1 = e,\ L_2 = f,\ L_3 = g,$$

where e, f, g are the coefficients of the second fundamental form. The Christoffel symbols can be determined by taking again the first four equations and making dot products with φ'_u and φ'_v. We obtain:

$$
\begin{aligned}
\Gamma^1_{11}E + \Gamma^2_{11}F &= \varphi'_{uu}\cdot\varphi'_u = \frac{1}{2}E_u\\
\Gamma^1_{11}F + \Gamma^2_{11}G &= \varphi'_{uu}\cdot\varphi'_v = F_u - \frac{1}{2}E_u
\end{aligned} \tag{2}
$$

$$
\begin{aligned}
\Gamma^1_{12}E + \Gamma^2_{12}F &= \varphi'_{uv}\cdot\varphi'_u = \frac{1}{2}E_v\\
\Gamma^1_{12}F + \Gamma^2_{12}G &= \varphi'_{uv}\cdot\varphi'_v = \frac{1}{2}G_u
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
\Gamma^1_{22}E + \Gamma^2_{22}F &= \varphi'_{vv}\cdot\varphi'_u = F_v - \frac{1}{2}G_u\\
\Gamma^1_{22}F + \Gamma^2_{22}G &= \varphi'_{vv}\cdot\varphi'_v = \frac{1}{2}G_v
\end{aligned} \tag{4}
$$

Each of the three groups of equations can be considered as a linear system of two equations with two unknowns. The determinant of each system is $EG - F^2$ which is strictly positive (why?). We can solve each system and obtain concrete expressions for Γ^k_{ij} in terms of E, F, G and their derivatives.

Now we consider the obvious equation

$$(\varphi''_{uu})'_v - (\varphi''_{uv})'_u = 0.$$

If we use (1), we deduce

$$
\begin{aligned}
&\Gamma^1_{11}\varphi''_{uv} + \Gamma^2_{11}\varphi''_{vv} + eN'_v + (\Gamma^1_{11})'_v\varphi'_u + (\Gamma^2_{11})'_v\varphi'_v + e'_vN\\
&= \Gamma^1_{12}\varphi''_{uu} + \Gamma^2_{12}\varphi''_{vu} + fN'_u + (\Gamma^1_{12})'_u\varphi'_u + (\Gamma^2_{12})'_u\varphi'_v + f'_uN
\end{aligned}
$$

We use again (1) to substitute all second order derivatives. Then we equate the coefficient of φ'_v. We obtain:

$$\begin{aligned}&\Gamma^1_{11}\Gamma^2_{12}+\Gamma^2_{11}\Gamma^2_{22}+ea_{22}+(\Gamma^2_{11})'_v\\ &=\Gamma^1_{12}\Gamma^2_{11}+\Gamma^2_{12}\Gamma^2_{12}+fa_{21}+(\Gamma^2_{12})'_u.\end{aligned}$$

But we actually know a_{21} and a_{22}, by the Weingarten equations (see chapter 5, section 4 of the notes). By making the replacements we obtain:

$$\begin{aligned}(\Gamma^2_{12})'_u-(\Gamma^2_{11})'_v+\Gamma^1_{12}\Gamma^2_{11}+\Gamma^2_{12}\Gamma^2_{12}-\Gamma^2_{11}\Gamma^2_{22}-\Gamma^1_{11}\Gamma^2_{12}\\ =-E\frac{eg-f^2}{EG-F^2}=-EK\end{aligned}$$

and this gives the desired

formula for K in terms of E, F, G and their partial derivatives of first and second order with respect to u and v.

To finish the proof, we consider the local isometry $f : S_1 \to S_2$. Take P a point of S_1. Because $d(f)_P : T_PS_1 \to T_{f(P)}S_2$ is a linear isomorphism, f is a local diffeomorphism at P, which implies that there exists a local parametrization (U,φ) of S_1 around P such that $(U, f\circ\varphi)$ is a local parametrization of S_2 around $f(P)$. For any Q in U, the canonical basis of $T_{f(\varphi(Q))}S_2$ consists of

$$(f\circ\varphi)'_u(Q)=d(f)_{\varphi(Q)}(\varphi'_u(Q)),\ (f\circ\varphi)'_v(Q)=d(f)_{\varphi(Q)}(\varphi'_v(Q)).$$

Moreover, if we denote $P=\varphi(Q)$, we have

$$\mathrm{I}_{f(P)}(d(f)_P(w))=\mathrm{I}_P(w),$$

for all $w=w_1\varphi'_u(Q)+w_2\varphi'_v(Q)$ in T_PS_1. We also have

$$d(f)_P(w)=w_1(f\circ\varphi)'_u(Q)+w_2(f\circ\varphi)'_v(Q),$$

and

$$\mathrm{I}_P(w)=E(Q)w_1^2+2F(Q)w_1w_2+G(Q)w_2^2$$

as well as

$$\mathrm{I}_{f(P)}(d(f)_P(w))=\tilde{E}(Q)w_1^2+\tilde{F}(Q)w_1w_2+\tilde{G}(Q)w_2^2.$$

We deduce $E(Q)=\tilde{E}(Q)$, $F(Q)=\tilde{F}(Q)$, and $G(Q)=\tilde{G}(Q)$ for any $Q=(u,v)$ in U. Combined with the formula for K from above, these imply that the curvature of S_1 at $\varphi(Q)$ equals the curvature of S_2 at $f(\varphi(Q))$, for any Q in U. □

References

[dC] M. P. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice Hall 1976

[Gr-Abb-Sa] A. Gray, E. Abbena, and S. Salamon, *Modern Differential Geometry with MATHEMATICA*, 3rd edition

[Sp] M. Spivak, *A Comprehensive Introduction to Differential Geometry*, vol. II