DIFFERENTIAL GEOMETRY OF CURVES AND SURFACES

5. The Second Fundamental Form of a Surface

The main idea of this chapter is to try to measure to which extent a surface S is different from a plane, in other words, how "curved" is a surface. The idea of doing this is by assigning to each point P on S a unit normal vector $N(P)$ (that is, a vector perpendicular to the tangent plane at P). We are measuring to which extent is the map from S to $\mathbb{R}^3$ given by $P \mapsto N(P)$ (called the Gauss map) different from the constant map, so we are interested in its derivative (or rather, differential). This will lead us to the concept of second fundamental form, which is a quadratic form associated to S at the point P.

5.1. Orientability and the Gauss map. Let S be a regular surface in $\mathbb{R}^3$.

Definition 5.1.1. (a) A *normal vector* to S at P is a vector perpendicular to the plane T_PS. If it has length 1, we call it a *normal unit vector*.

(b) A *(differentiable) normal unit vector field* on S is a way of assigning to each P in S a unit normal vector $N(P)$, such that the resulting map $N : S \to \mathbb{R}^3$ is differentiable[1].

At any point P in S there exist two normal unit vectors, which differ from each other by a minus sign. So a normal unit vector field is just a choice of one of the two vectors at any point P. It is a useful exercise to try to use ones imagination to visualize normal unit vector fields on surfaces like the sphere, the cylinder, the torus, the graph of a function of two variables etc. Surprinsingly enough, such vector fields do not exist on any surface. An interesting example in this respect is the Möbius strip, see Figure 1: assuming that there exists a normal unit vector field on this surface and trying to represent it, we can easily see that it has a jump at some point.

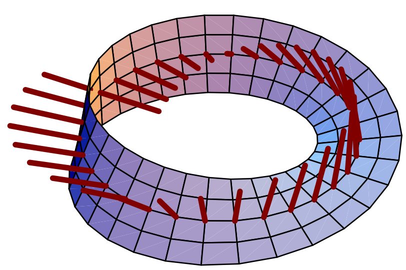

Figure 1. The Möbius strip and a tentative to construct a normal unit vector field on it.

Definition 5.1.2. A surface S is said to be *orientable* if there is a differentiable unit normal vector field N on it. If so, then (the endpoint of) N is on the unit sphere S^2 and the map

$$N : S \to S^2$$

is called the *Gauss map* of S (see also Figure 2).

[1] We haven't defined differentiability of maps from S to $\mathbb{R}^3$. Like in Section 3.2, $N : S \to \mathbb{R}^3$ is differentiable means that for any local parametrization (U, φ) of S, the map $N \circ \varphi : U \to \mathbb{R}^3$ is differentiable. Equivalently, any of the three components of N is a differentiable function from S to $\mathbb{R}$.

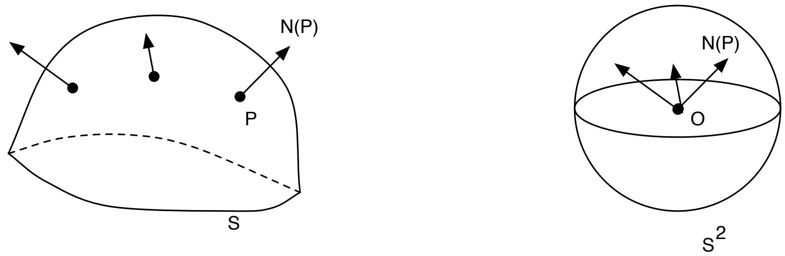

FIGURE 2. To each P in S the Gauss map assigns a point $N(P)$ on the unit sphere S^2.

Remarks. 1. For the original definition of orientability, one can see [dC], Section 2-6. or [Gr-Abb-Sa], Section 11.1 (our Definition 5.1.2 above is equivalent to the original one, but more handy than that one). Intuitively, a surface is orientable if it is possible to define counterclockwise rotations on small pieces of S in a continuous manner.

2. Besides the Möbius strip, there are other examples of non-orientable surfaces, more notably the Klein bottle and the real projective space. These are nicely described in [Gr-Abb-Sa], Chapter 11 (but they are not regular, as they have self-intersections). One can see that none of these examples is a compact regular surface (by the way: give examples of compact surfaces; is the Möbius strip compact? why?). Indeed a fairly recently proved result says that any compact regular surface is orientable (Samelson 1969).

Example. If (U, φ) is a local parametrization of an *arbitrary* surface, then $\varphi(U)$ is orientable. Indeed, we can define

$$N(\varphi(Q)) := \frac{1}{\|\varphi'_u(Q) \times \varphi'_v(Q)\|}\varphi'_u(Q) \times \varphi'_v(Q),$$

for Q in U. In other words, any surface is locally orientable. We also deduce that the graph of a function of two variables is an orientable surface (see also Corollary 3.1.3).

From now on we will make the following assumption.

Assumption. The surface S is orientable and $N : S \to S^2$ is a Gauss map [2].

We are interested in the differential of the map $N : S \to S^2$ at a point P in S. By Definition 3.3.4, this map goes from T_PS to $T_{N(P)}S^2$. Since both tangent planes are perpendicular to $N(P)$, they must be equal. So the differential of N at P is

$$d(N)_P : T_PS \to T_PS,$$

that is, a linear endomorphism of T_PS. It is defined by

$$d(N)_P(w) = \frac{d}{dt}|_0 N(\alpha(t)),$$

for any w in T_PS of the form $w = \alpha'(0)$, where α is a curve $\alpha : (-\epsilon, \epsilon)$ with $\alpha(t)$ in S, $\alpha(0) = P$.

Examples. 1. *The plane* Π. We choose a unit vector N perpendicular to Π. The Gauss map is $N(P) = N$ (constant), for any P in Π. Its differential is

$$d(N)_P(w) = 0,$$

for all w in T_PS.

2. *The sphere* S^2. A unit normal vector to S^2 at $P = (x, y, z)$ is (x, y, z) (since the tangent plane to S^2 at P is perpendicular to OP, see also Figure 3).

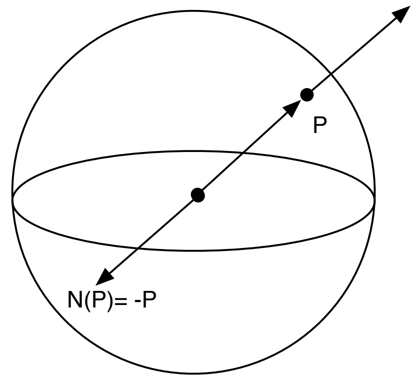

FIGURE 3. Two copies of a unit normal vector at P to the sphere S^2; the third vector is $N(P)$.

Rather than assigning to P the point P itself, we prefer to assign its negative, that is

$$N(x, y, z) = (-x, -y, -z),$$

for all $P = (x, y, z)$ in S^2. Take $w = \alpha'(0)$ in $T_P S$, where $\alpha(t) = (x(t), y(t), z(t))$. Then

$$d(N)_P(w) = \frac{d}{dt}|_0(-x(t), -y(t), -z(t)) = -\alpha'(0),$$

so $d(N)_P(w) = -w$, for all w in $T_P S$.

3. *The cylinder C.* Consider again the example on page 2, Chapter 4 of the notes. See also Figure 4.

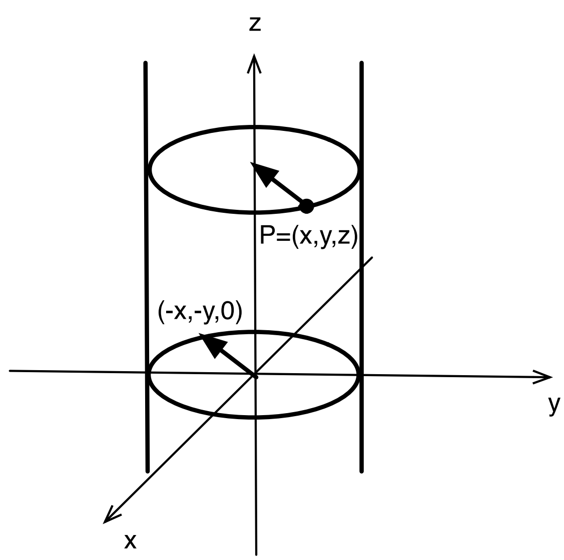

FIGURE 4. The Gauss map of the cylinder C.

We choose

$$N(x, y, z) = (-x, -y, 0),$$

for all (x, y, z) on C. We deduce easily that if P is in C and

$$w = \alpha'(0) = (x'(0), y'(0), z'(0))$$

is in T_PS, then

$$d(N)_P(x'(0), y'(0), z'(0)) = (-x'(0), -y'(0), 0).$$

In other words, if v is in T_PS parallel to the xy plane (that is $v = (\lambda, \mu, 0)$ for some numbers λ and μ), then

$$d(N)_P(v) = -v.$$

If w in T_PS is parallel to the z axis, then

$$d(N)_P(w) = 0.$$

So the eigenvalues of $d(N)_P$ are -1 and 0.

We will see that for any surface S and any point P, the eigenvalues of the map $d(N)_P$ are real numbers. The reason is that $d(N)_P$ is a selfadjoint operator. The goal of the next section is to give the required background concerning the latter notion. This can be considered as a continuation of Section 4.1 of the notes.

5.2. Quadratic forms (part II). This time we consider a two dimensional vector subspace V of $\mathbb{R}^3$.

Lemma 5.2.1. *Let Q be an arbitrary quadratic form on V. Then there exists an orthonormal basis*[3] e_1, e_2 *of V such that for any $v = xe_1 + ye_2$ in V we have*

$$Q(v) = \lambda_1 x^2 + \lambda_2 y^2.$$

Here the numbers λ_1 and λ_2 have the following descriptions:

$$\lambda_1 = \max\{Q(v) \mid v \text{ in } V, \|v\| = 1\}$$
$$\lambda_2 = \min\{Q(v) \mid v \text{ in } V, \|v\| = 1\}.$$

Proof. The subspace $\{v \text{ in } V \mid \|v\| = 1\}$ is compact. The restriction of Q to it is continuos, hence it must have a maximum point, call it e_1. We also denote

$$\lambda_1 = Q(e_1).$$

We take e_2 in V, $\|e_2\| = 1$, e_2 perpendicular to e_1 and we denote

$$\lambda_2 = Q(e_2).$$

We show that $B(e_1, e_2) = 0$. Indeed, for any t in $\mathbb{R}$, the vector $(\cos t)e_1 + (\sin t)e_2$ has length 1, hence the function

$$f(t) = Q((\cos t)e_1 + (\sin t)e_2) = \cos^2 tB(e_1, e_1) + \sin 2tB(e_1, e_2) + \sin^2 tB(e_2, e_2)$$

has a maximum at $t = 0$. This implies $f'(0) = 0$, so

$$B(e_1, e_2) = 0.$$

The only thing which still needs to be shown is that e_2 is a minimum point of Q on $\{v \text{ in } V \mid \|v\| = 1\}$. Indeed, for any $v = xe_1 + ye_2$ in V with $x^2 + y^2 = 1$ we have

$$Q(v) = Q(e_1)x^2 + Q(e_2)y^2 \geq Q(e_2)(x^2 + y^2) = Q(e_2),$$

where we have used that $Q(e_1) \geq Q(e_2)$. The lemma is proved. $\square$

Definition 5.2.2. A linear endomorphism A of V is called *self-adjoint* if

$$A(v) \cdot w = v \cdot A(w),$$

[3] By this we mean that $e_1 \cdot e_2 = 0$, $\|e_1\| = \|e_2\| = 1$.

for all v, w in V. Equivalently, the function $B : V \times V \to \mathbb{R}$ given by

$$B(v, w) = A(v) \cdot w,$$

for all v, w in V, is a symmetric bilinear form on V.

Let $Q : V \to \mathbb{R}$, $Q(w) = A(w) \cdot w$, w in W, be the associated quadratic form.

Theorem 5.2.3. *Let A be a self-adjoint linear endomorphism of V. Then there exists an orthonormal basis e_1, e_2 of V and two numbers λ_1, λ_2 with*

$$A(e_1) = \lambda_1 e_1, \;\; A(e_2) = \lambda_2 e_2.$$

The numbers λ_1, λ_2 are the maximum, respectively minimum of $Q(v) = A(v) \cdot v$, for v in V, $\|v\| = 1$.

Proof. We consider the quadratic form $Q(v) = A(v) \cdot v$, v in V and we use the previous lemma. There exists an orthonormal basis e_1, e_2 with

$$Q(xe_1 + ye_2) = \lambda_1 x^2 + \lambda_2 y^2.$$

Moreover, we have shown that $B(e_1, e_2) = 0$, which means

$$A(e_1) \cdot e_2 = 0,$$

so we must have $A(e_1) = \alpha e_1$, for some number α. But then we have

$$\alpha = A(e_1) \cdot e_1 = Q(e_1) = \lambda_1.$$

We have shown that

$$A(e_1) = \lambda_1 e_1.$$

Similarly we show that

$$A(e_2) = \lambda_2 e_2.$$

The theorem is proved. $\square$

Remark. The numbers λ_1, λ_2 are the eigenvalues of A. They are uniquely determined by A. By contrary, the vectors e_1, e_2 are not unique. For example, if $A(v) = v$, for all v in V, then $\lambda_1 = \lambda_2 = 1$, but e_1, e_2 can be any pair of orthonormal vectors.

We are now ready to get back to the Gauss map of a surface and its differential.

5.3. The second fundamental form. Again S is a regular orientable surface and $N : S \to S^2$ a Gauss map.

5.3.1. Theorem. *For any point P in S, the linear endomorphism $d(N)_P$ of T_PS is self-adjoint.*

Proof. We consider a local parametrization (U, φ) of S with $\varphi(Q) = P$ for some Q in U. Set

$$Q = (u_0, v_0).$$

The vectors $\varphi'_u(Q), \varphi'_v(Q)$ are a basis of T_PS. It is sufficient[4] to show that

$$d(N)_P(\varphi'_u(Q)) \cdot \varphi'_v(Q) = \varphi'_u(Q) \cdot d(N)_P(\varphi'_v(Q)). \tag{1}$$

By the definition of the differential map, we have

$$d(N)_P(\varphi'_u(Q)) = \frac{d}{du}|_{u_0} N(\varphi(u, v_0)) = (N \circ \varphi)'_u(Q),$$

and similarly

$$d(N)_P(\varphi'_v(Q)) = (N \circ \varphi)'_v(Q).$$

[4] Why?

We note that

$$N \circ \varphi \cdot \varphi'_v = 0 \text{ everywhere on } U$$

which implies (by taking partial derivative with respect to u)

$$(N \circ \varphi)'_u \cdot \varphi'_v + N \circ \varphi \cdot \varphi'_{vu} = 0$$

so

$$(N \circ \varphi)'_u(Q) \cdot \varphi'_v(Q) = -N(\varphi(Q)) \cdot \varphi'_{vu}(Q).$$

Similarly,

$$(N \circ \varphi)'_v(Q) \cdot \varphi'_u(Q) = -N(\varphi(Q)) \cdot \varphi'_{uv}(Q).$$

Equation (1) follows from the fact that

$$\varphi'_{uv}(Q) = \varphi'_{vu}(Q).$$

□

The quadratic form corresponding to[5] $-d(N)_P$ is the second fundamental form.

5.3.2. Definition. The <u>*second fundamental form*</u> of S at P is the quadratic form $\mathrm{II}_P : T_PS \to \mathbb{R}$ given by

$$\mathrm{II}_P(v) = -d(N)_P(v) \cdot v,$$

v in T_PS.

Our next goal is to give geometric interpretations to the number $\mathrm{II}_P(v)$.

5.3.3. Definition. Let $\alpha : (-\epsilon, \epsilon) \to \mathbb{R}^3$ be a curve whose trace is contained in S such that $\alpha(0) = P$, parametrized by arc length. The number

$$\kappa_n = \alpha''(0) \cdot N(P)$$

is called the <u>*normal curvature*</u> of α at P with respect to S. Alternatively, we have

$$\kappa_n = \kappa \cos \phi$$

where κ is the curvature of α at the point P and ϕ is the angle between the vectors $N(P)$ and $\alpha''(0)$.

We can also see that κ_n is the length of the orthogonal projection of $\alpha''(0)$ to the straight line determined by $N(P)$, possibly with a negative sign, if the angle ϕ is obtuse (see also Figure 5).

The following result gives a geometric interpretation of the second fundamental form.

5.3.4. Theorem. *Let α be a curve like in Definition 5.3.3. Then we have*

$$\kappa_n = \mathrm{II}_P(\alpha'(0)).$$

Notation. If P is in S and v in T_PS with $\|v\| = 1$, the number

$$\kappa_n(v) := \mathrm{II}_P(v)$$

is called the <u>*normal curvature along*</u> v. This is the normal curvature of *any* curve α like in Definition 5.3.3. whose tangent vector at P is $\alpha'(0) = v$.

Proof of Theorem 5.3.4. Denote $N(s) := N(\alpha(s))$, for s in $(-\epsilon, \epsilon)$. We have

$$N(s) \cdot \alpha'(s) = 0$$

which implies by taking derivatives

$$N'(s) \cdot \alpha'(s) + N(s) \cdot \alpha''(s) = 0.$$

[5]It will become immediately clear why did we want to take the negative sign. By the way, the endomorphism $-d(N)_P$ of T_PS is called the *shape operator*, or the *Weingarten map* of S at P.

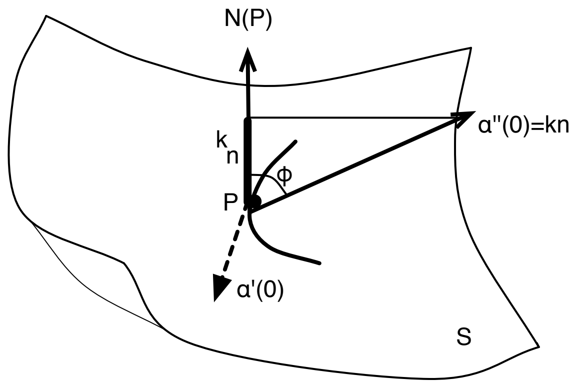

FIGURE 5. The normal curvature interpreted as a projection. Technical detail: the vector $\alpha'(0)$ is “behind” the surface, that’s why we can’t “see” it.

For $s = 0$ we take into account that

$$N'(0) = \frac{d}{ds}|_0 N(s) = d(N)_P(\alpha'(0))$$

and obtain[6]

$$-\mathrm{II}_P(\alpha'(0)) + \kappa_n = 0$$

which implies the desired equality. □

Take v in T_PS with $\|v\| = 1$. Among all curves α like in Definition 5.3.3 with $\alpha'(0) = v$, a “natural” one is the intersection of the surface S with the plane through P containing $N(P)$ and v. Denote the latter by Π, and note it is an *affine* plane. Thus the trace of the curve mentioned above is the intersection $S \cap \Pi$; it is called the *normal section* along v (see also Figure 6).

Let us parametrize the normal section by its arc length (starting at P) and denote by α the resulting parametrized curve. The vector $\alpha'(0)$ is parallel to (i.e. collinear with) v: indeed, it is parallel to both T_PS and Π, and the intersection of these two planes is the line through P of direction v. Since $\|\alpha'(0)\| = \|v\| = 1$, we have $\alpha'(0) = \pm v$. If necessary, one reverses the parametrization of α in order to get the + sign, that is

$$\alpha'(0) = v.$$

Denote by n the normal vector[7] to α and κ the curvature of α at $\alpha(0)$. The vector $\alpha''(0) = \kappa n$ is parallel to the plane Π, so n is parallel to Π and perpendicular to $v = \alpha'(0)$, which implies $n = \pm N(P)$. We deduce that the normal curvature of this curve is

$$\kappa_n = \alpha''(0) \cdot N(P) = (\kappa n) \cdot N(P) = \kappa(n \cdot N(P)) = \pm\kappa.$$

Let’s record the result:

For any v in T_PS with $\|v\| = 1$, the number $\mathrm{II}_P(v)$ equals the curvature of the normal section along v at the point P, up to a possible negative sign.

Informally speaking, $\mathrm{II}_P(v)$ is telling you how curved will be your road if you go in the direction indicated by v.

[6] It is the right moment to understand why did we take the negative sign in Definition 5.3.2.

[7] In Ch. 2, the normal vector to a curve at a point is denoted by N; right now, N already has a meaning, so we need to choose a different notation, which will be n.

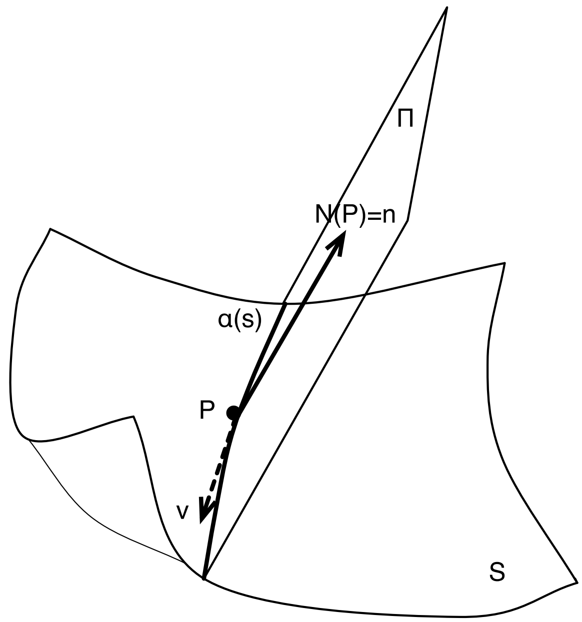

FIGURE 6. The normal section along v is the curve α (note that the latter is both on the surface S and contained in the plane Π).

We note that sometimes we can get equality between κ and κ_n by choosing the Gauss map appropriately. For instance, if the surface lies on one side of the (affine) tangent plane, we choose N to point towards this side. The normal vector $n = \frac{1}{\kappa}\alpha''(0)$ to the normal section points towards the concave side of the curve, thus it equals $N(P)$ (compare to the situation represented in Figure 6, see also Example 2 below).

Examples. 1. *The plane* Π *revisited.* Let Π be an arbitrary plane, P a point on it and v a vector in $T_P\Pi$ (by the way, this is the plane parallel to Π going through the origin). To compute $\mathrm{II}_P(v)$, we consider the normal section along v. This is a straight line. We know from Chapter 1 that its curvature is 0. We deduce that

$$\mathrm{II}_P(v) = 0, \text{ for all } v \text{ in } T_P\Pi.$$

We actually knew this, see Example 1 on page 2.

2. *The sphere* S^2 *revisited.* See Figure 8.

Again, to determine $\mathrm{II}_P(v)$, we find the normal section along v. This is a circle with centre at O. The curvature of the latter is 1. Because of the choice of N, we have $N = n$ (not its negative!), thus

$$\mathrm{II}_P(v) = 1.$$

This was also known, because

$$\mathrm{II}_P(v) = -d(N)_P(v) \cdot v = v \cdot v = 1.$$

3. *The cylinder revisited.* Look again at Example 3, page 3. At the end of that discussion we distinguished two types of vectors, denoted v and w. We can easily compute $\mathrm{II}_P(v)$ and $\mathrm{II}_P(w)$, because the corresponding normal sections are a straight line, respectively a circle. But what can we do to find II_P of an *arbitrary* tangent unit vector? The normal section is an ellipse, and in principle we can compute its curvature. However, there exists a much simpler method, which will be described in what follows.

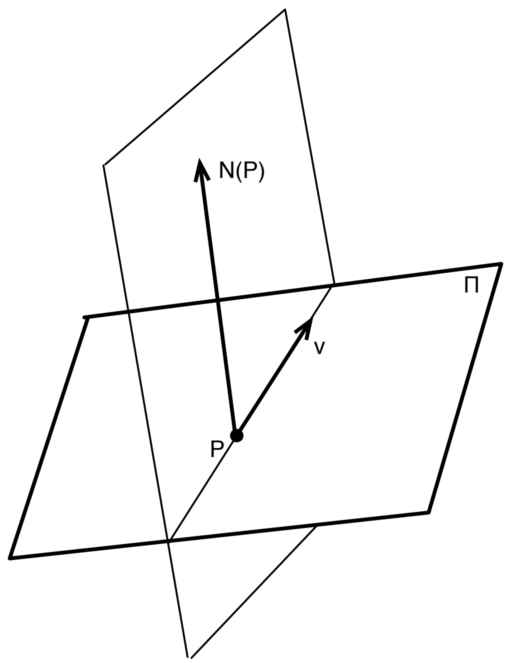

FIGURE 7. The normal section along v is the straight line through P which is parallel to v.

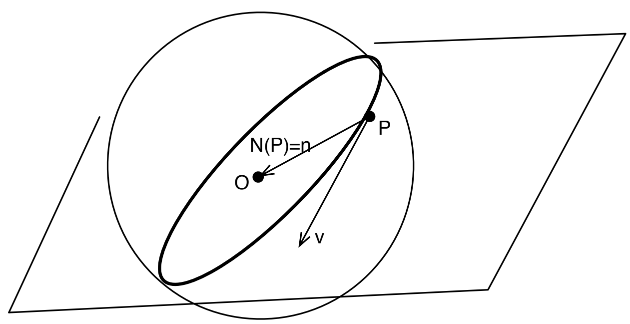

FIGURE 8. The normal section along v is a big circle on the sphere.

By Theorem 5.3.1, the map $-d(N)_P : T_PS \to T_PS$ is self-adjoint. By Theorem 5.2.3, it is important to consider the numbers $-d(N)_P(v) \cdot v = \mathrm{II}_P(v)$, where v is in T_PS, $\|v\| = 1$. More precisely, let us consider the extrema of the second fundamental form on the unit circle in T_PS, as follows

$$k_1 := \max\{\mathrm{II}_P(v) \mid v \text{ in } T_PS,\ \|v\| = 1\},$$
$$k_2 := \min\{\mathrm{II}_P(v) \mid v \text{ in } T_PS,\ \|v\| = 1\}.$$

Definition 5.3.5. The numbers k_1 and k_2 defined above are the *principal curvatures* of S at P. (Note that they are the eigenvalues of $-d(N)_P : T_PS \to T_PS$.) A vector e in T_PS with the property

$$d(N)_P(e) = -k_1 e \text{ or } d(N)_P(e) = -k_2 e$$

is called a *principal vector*.

We know by Theorem 5.2.3 that there exist two vectors e_1, e_2 in $T_P S$ such that

$$d(N)_P(e_1) = -k_1 e_1, \; d(N)_P(e_2) = -k_2 e_2.$$

Moreover, $\|e_1\| = \|e_2\| = 1$ and e_1 is perpendicular to e_2. As a direct consequence of Lemma 5.2.1, we deduce as follows.

Theorem 5.3.6. *If v is in $T_P S$ with $\|v\| = 1$, having the form*

$$v = (\cos\theta) e_1 + (\sin\theta) e_2$$

(see also Figure 9), then

$$\mathrm{II}_P(v) = k_1 \cos^2\theta + k_2 \sin^2\theta.$$

The last equation is known as *Euler's formula.*

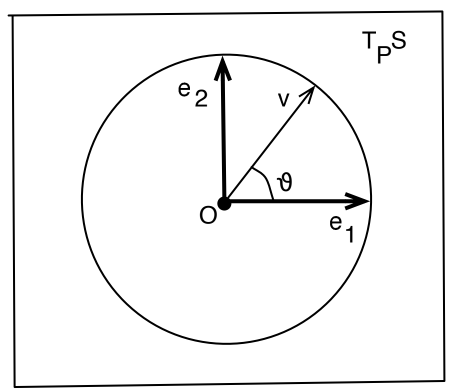

FIGURE 9. We have $v = (\cos\theta)e_1 + (\sin\theta)e_2$.

Examples. 1. If we look again at the examples on page 3, we see that

- at any point P of the plane we have $k_1 = k_2 = 0$ and any unit tangent vector is a principal vector
- at any point P of the sphere S^2 we have $k_1 = k_2 = 1$ and any unit tangent vector is a principal vector
- at any point P of the cylinder we have $k_1 = 1$ and $k_2 = 0$ (see also Figure 10). The principal vectors are e_1 and e_2. We can use Euler's formula to compute $\mathrm{II}_P(v)$ for any v in $T_P S$ with $\|v\| = 1$. More precisely, for v like in Figure 9, we have $\mathrm{II}_P(v) = \cos^2\theta$.

2. Let us consider the parabolic hyperboloid, which is the graph of the function $h(x, y) = y^2 - x^2$. In Figure 11, the two parabolas in the right hand side picture have a third significance: they are the normal sections along the principal directions through O. Their normal vectors at O are $N(O)$, respectively $-N(O)$ (compare with the discussion on the top of page 8).

Exercise. If $\mathcal{H}$ denotes the hyperboloid, determine $T_O\mathcal{H}$, $d(N)_O$, II_O, the principal curvatures at O, and two principal vectors at O.

The notions defined below are relevant for the study of surfaces, as we will see later on.

Definition 5.3.7. If k_1 and k_2 are the principal curvatures of S at P, then

- the *Gauss curvature* of S at P is

$$K(P) = k_1 k_2$$

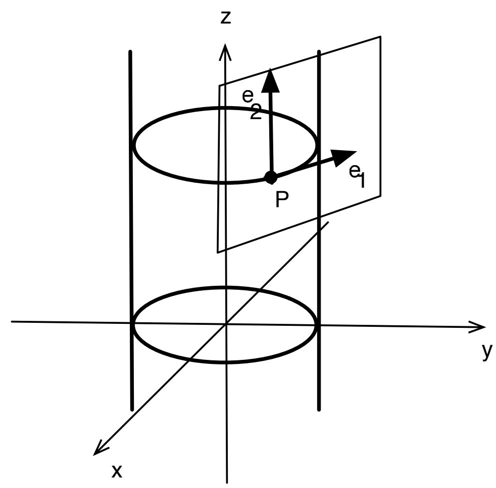

FIGURE 10. The principal vectors at P are e_1 and e_2.

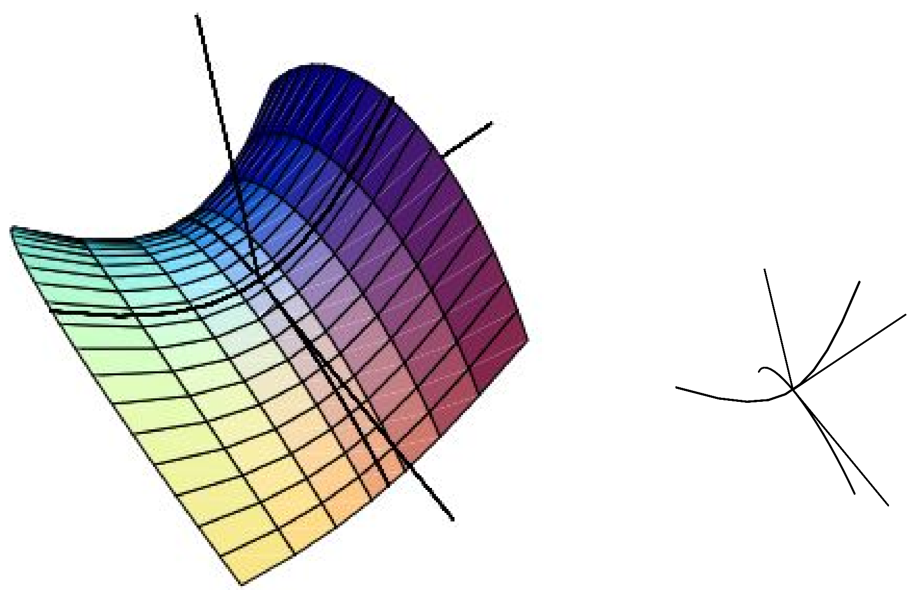

FIGURE 11. The hyperbolic paraboloid has the parametrization $\varphi(u,v) = (u, v, v^2 - u^2)$. The two parabolas in the right hand side image are the coordinate curves through O (which is their intersection point); they also represent the intersections of the surface with the coordinate planes xz and yz. Namely, the lower parabola is in the xz plane, the upper one in the yz plane. The origin is a "saddle" point.

- the *mean curvature* of S at P is

$$H(P) = \frac{1}{2}(k_1 + k_2).$$

Alternatively, we can express these in terms of the linear endomorphism $d(N)_P$ of T_PS as follows:

$$K(P) = \det(d(N)_P), \quad H(P) = -\mathrm{trace}(d(N)_P).$$

5.4. The second fundamental form in local coordinates. We have already mentioned that if (U, φ) is a parametrization of a surface, then $\varphi(U)$ is orientable via the Gauss map

$$(N \circ \varphi)(Q) := \frac{1}{\|\varphi'_u(Q) \times \varphi'_v(Q)\|}\varphi'_u(Q) \times \varphi'_v(Q), \tag{2}$$

for Q in U. The main goal of this section is to give formulas for the quantities defined in the previous section in terms of φ.

Consider the differential map

$$d(N)_P : T_PS \to T_PS,$$

where $P = \varphi(Q)$, $Q = (u_0, v_0)$ in U. We can easily determine the coefficients of the second fundamental form II_P with respect to the basis $\varphi'_u(Q), \varphi'_v(Q)$. Let us denote them by e, f, and g. By Section 4.1, Equation (1), they actually are

$$\begin{aligned} e &= -d(N)_P(\varphi'_u(Q)) \cdot \varphi'_u(Q) = (N \circ \varphi)(Q) \cdot \varphi''_{uu}(Q) \\ f &= -d(N)_P(\varphi'_u(Q)) \cdot \varphi'_v(Q) = -d(N)_P(\varphi'_v(Q)) \cdot \varphi'_u(Q) = (N \circ \varphi)(Q) \cdot \varphi''_{uv}(Q) \\ g &= -d(N)_P(\varphi'_v(Q)) \cdot \varphi'_v(Q) = (N \circ \varphi)(Q) \cdot \varphi''_{vv}(Q) \end{aligned} \tag{3}$$

where we have used the product rule and the fact that $(N\circ\varphi)\cdot\varphi'_u = 0$ and $(N\circ\varphi)\cdot\varphi'_v = 0$ (see similar computations in the proof of Theorem 5.3.1 above). Combined with (2), the previous three equations give e, f, g *in terms of* φ *and its partial derivatives* *of order one and two.*

Next we determine the matrix of the linear map $d(N)_P$. This is

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where the numbers a_{ij} are determined by

$$\begin{aligned} d(N)_P(\varphi'_u(Q)) &= a_{11}\varphi'_u(Q) + a_{21}\varphi'_v(Q) \\ d(N)_P(\varphi'_v(Q)) &= a_{12}\varphi'_u(Q) + a_{22}\varphi'_v(Q) \end{aligned}$$

The previous two equations combined with (3) give as follows:

$$\begin{aligned} -f &= (a_{11}\varphi'_u(Q) + a_{21}\varphi'_v(Q)) \cdot \varphi'_v(Q) = a_{11}F + a_{21}G \\ -f &= (a_{12}\varphi'_u(Q) + a_{22}\varphi'_u(Q)) \cdot \varphi'_u(Q) = a_{12}E + a_{22}F \\ -e &= (a_{11}\varphi'_u(Q) + a_{21}\varphi'_v(Q)) \cdot \varphi'_u(Q) = a_{11}E + a_{21}F \\ -g &= (a_{12}\varphi'_u(Q) + a_{22}\varphi'_u(Q)) \cdot \varphi'_v(Q) = a_{12}F + a_{22}G \end{aligned}$$

In matrix notation, the last four equations give

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix} = -\begin{pmatrix} e & f \\ f & g \end{pmatrix} \Rightarrow \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = -\begin{pmatrix} e & f \\ f & g \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}.$$

Because

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = \frac{1}{EG - F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}$$

we deduce

$$a_{11} = \frac{fF - eG}{EG - F^2}, \; a_{12} = \frac{gF - fG}{EG - F^2}$$
$$a_{21} = \frac{eF - fE}{EG - F^2}, \; a_{22} = \frac{fF - gE}{EG - F^2}$$

These are called the *Weingarten equations*.

We are also interested in the Gauss, respectively mean curvature (see Definition 5.3.7). They are

$$K = \det(d(N)_P) = a_{11}a_{22} - a_{12}a_{21} \quad H = -\mathrm{trace}(d(N)_P) = -(a_{11} + a_{22}).$$

After making the calculations we obtain

$$K = \frac{eg - f^2}{EG - F^2} \tag{4}$$
$$H = \frac{eG - 2fF + gE}{2(EG - F^2)}.$$

Finally, we can determine the principal curvatures. These are the eigenvalues of $-d(N)_P$, that is, the eigenvalues of the matrix $-(a_{ij})_{1 \le i,j \le 3}$. They are the roots of the polynomial

$$\det\left(\lambda I + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) = \lambda^2 - 2\lambda H + K.$$

We obtain

$$k_1 = H + \sqrt{H^2 - K}, \; k_2 = H - \sqrt{H^2 - K}.$$

Historical note. The notions and results presented in this chapter were discovered during the centuries in a totally different order. I would like to give here a short guideline concerning this. It has always been clear that in order to measure the "curvature" of a surface at a point one needs to measure the curvature of various normal sections. This led naturally to the notion of normal curvature in a prescribed tangent direction, like in Figure 6. A remarkable result was found by Euler in 1760: there exists two *perpendicular* directions e_1, e_2 in the tangent plane such that the normal curvature in the direction of e_1 is minimal and the normal curvature in if the direction of e_2 is maximal; moreover, the normal curvature in an arbitrary direction v is given by what we call now Euler's formula (see Theorem 5.3.6). A few years later, around 1776, Meusnier realized that we can also use *arbitrary* curves $\alpha : (-\epsilon, \epsilon) \to S$ (parametrized by arc length) with $\alpha(0) = P$. More precisely, he showed that of κ is the curvature of α at P and ϕ the angle between the $N(P)$ and $\alpha''(0)$, then the number $\kappa \cos\phi$ is the same for all curves α with the same tangent vector at P, being equal to the normal curvature of S along $\alpha'(0)$. Fundamentally new ideas were brought by Gauss, in his paper entitled *Disquisitiones generales circa superficies curvas*, published in 1827. Among the main contributions of that work we mention as follows:

- he defined the "Gauss map" and the second fundamental form and noted that the results discovered by his predecessors can be proved by using these instruments
- he proved that the "Gauss curvature" $K(P)$ depends only on the coefficients E, F, G of the *first* fundamental form (this is a striking result, if we compare it with Equation (4))
- he discovered a formula for the sum of the angles of a triangle on a surface.

The last two items will be discussed in the remaining of our course.

For more historical details, one can read Chapter 3 of Spivak's monograph [Sp].

References

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