DIFFERENTIAL GEOMETRY OF CURVES AND SURFACES

4. Lengths and Areas on a Surface

An important instrument in calculating distances and areas is the so called first fundamental form of the surface S at a point P. This is nothing but the restriction of the scalar product of $\mathbb{R}^3$ to the vector subspace T_PS. Before getting to the actual definition, we need a bit of linear algebra.

4.1 Quadratic forms (part I).[1] Let V be a two dimensional vector space. A *symmetric bilinear form* on V is a function $B : V \times V \to \mathbb{R}$ with the following properties:

SBF1. $B(v, w) = B(w, v)$, for all v, w in V
SBF2. $B(av_1 + bv_2, w) = aB(v_1, w) + bB(v_2, w)$, for all v_1, v_2, w in V and all a, b in $\mathbb{R}$.

The corresponding *quadratic form* is $Q : V \to \mathbb{R}$ given by

$$Q(v) := B(v, v)$$

for all v in V. It will become clear immediately why do we call it "quadratic" (that is, "of degree two"). Let's take a basis e_1, e_2 of V. We can write $v = v_1e_1 + v_2e_2$, $w = w_1e_1 + w_2e_2$ and we have

$$B(v, w) = [v]A[w]^T$$

where

$$[v] = (v_1\ v_2), [w] = (w_1\ w_2), [w]^T = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

The entries a, b, c of the symmetric matrix A are as follows:

$$a = B(e_1, e_1),\ b = B(e_1, e_2),\ c = B(e_2, e_2). \tag{1}$$

They are called the *coefficients* of the quadratic form Q. As about Q, for any $w = w_1e_1+w_2e_2$ in V we have

$$Q(w) = aw_1^2 + 2bw_1w_2 + cw_2^2 \tag{2}$$

which is indeed a degree two polynomial in w_1 and w_2. We note that if Q is given, we can recover B by the formula

$$B(v, w) = \frac{1}{2}(Q(v + w) - Q(v) - Q(w)),$$

which you may want to check.

We say that Q is a *positive definite* quadratic form if

$$Q(w) > 0, \text{ for all } w \text{ in } V, w \neq 0.$$

Equivalently,

$$aw_1^2 + 2bw_1w_2 + cw_2^2 > 0$$

for any two numbers w_1, w_2, not both of them equal to 0. In turn, this is equivalent to the fact that the quadratic polynomial $ax^2 + 2bx + c$ is positive for any x. We have proved the following result.

Lemma 4.1.1. *The quadratic form given by (2) is positive definite if and only if*

$$a > 0 \text{ and } b^2 - ac < 0.$$

[1] More about quadratic forms will be needed (and discussed) in the next chapters.

4.2. The first fundamental form. Let S be a regular surface in $\mathbb{R}^3$ and P a point on S. Recall that the tangent plane T_PS is a two dimensional vector subspace of $\mathbb{R}^3$.

Definition 4.2.1. The *first fundamental form* of S at P is the function denoted I_P, from T_PS to $\mathbb{R}$, given by

$$\mathrm{I}_P(v) := v \cdot v,$$

for any v in $T_P(S)$.

One can easily see that I_P is a quadratic form on the vector space T_PS, namely the one corresponding to the symmetric bilinear form $v \cdot w$, for $v, w \in T_PS$. It is obviously positive definite. As we will see immediately, it can be used to compute lengths and areas on S. First we would like to express it in terms of a local parametrization around P. So let $\varphi : U \to S$ be a local parametrization and $Q = (u, v)$ in U with $\varphi(Q) = P$. The vectors $\varphi'_u(Q)$ and $\varphi'_v(Q)$ are a basis of T_PS. By equation (1), the coefficients of I_P are as follows:

$$E := \varphi'_u(Q) \cdot \varphi'_u(Q),\ F := \varphi'_u(Q) \cdot \varphi'_v(Q),\ G := \varphi'_v(Q) \cdot \varphi'_v(Q).$$

If we let $Q = (u, v)$ vary in U, the formulas above give three differentiable functions, E, F, and G on U. Let's finish this discussion by recording the formula

$$\mathrm{I}_P(w_1\varphi'_u(Q) + w_2\varphi'_v(Q)) = w_1^2E(Q) + 2w_1w_2F(Q) + w_2^2G(Q),$$

for any w_1, w_2 in $\mathbb{R}$ (that is, for any vector $w_1\varphi'_u(Q) + w_2\varphi'_v(Q)$ in T_PS). Shortly, we write

$$\boxed{\mathrm{I}_p(w_1, w_2) = Ew_1^2 + 2Fw_1w_2 + Gw_2^2} \tag{3}$$

In the following we will show some calculations of the first fundamental form.

Example. We consider the cylinder represented in Figure 1, call it C. A local parametrization of it is given by

$$\varphi(u, v) = (\cos u, \sin u, v)$$

where $0 < u < 2\pi$, and v is in $\mathbb{R}$. Let's take the point $P = \varphi(u, v)$ on C. The plane T_PC is

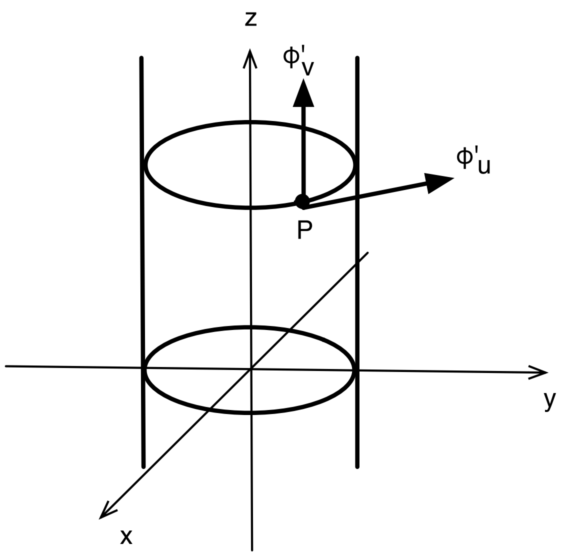

FIGURE 1. The cylinder and its tangent plane at the point P.

spanned by

$$\varphi'_u(u,v) = (-\sin u, \cos u, 0), \text{ and } \varphi'_v(u,v) = (0,0,1).$$

The coefficients of the first fundamental form are

$$E = \varphi'_u \cdot \varphi'_u = 1, \; F = \varphi'_u \cdot \varphi'_v = 0, \; G = \varphi'_v \cdot \varphi'_v = 1.$$

So if we fix (u,v) and we take w in $T_{\varphi(u,v)}S$ of the form

$$w = w_1\varphi'_u(u,v) + w_2\varphi'_v(u,v),$$

then we have

$$\mathrm{I}_P(w) = w_1^2 + w_2^2. \tag{4}$$

4.3. Lengths and areas. Let $\varphi : U \to S$ be a local parametrization of the surface S. We will give two formulas, one for the length of a curve, the other for the area of a "piece" of a surface, under the hypothesis that the curve and the piece of surface are contained in $\varphi(U)$. An important idea is that both formulas involve just the first fundamental form $\mathrm{I}_{\varphi(u,v)}$, where (u,v) is in U.

First let $\alpha : (a,b) \to \mathbb{R}^3$ be a parametrised curve, with $\alpha(t)$ in S, for all t. Take c, d such that $a < c < d < b$. Then, according to Definition 1.3.2, we have

$$\ell(\alpha|_{[c,d]}) = \int_c^d \|\alpha'(t)\|dt = \int_c^d \sqrt{\mathrm{I}_{\alpha(t)}(\alpha'(t))}dt.$$

We now turn to the area of a piece of a surface, which is contained in the image of the local parametrization $\varphi : U \to S$. First let us fix $Q_0 = (u_0, v_0)$ a point in U, take Δu and Δv two small numbers and let ΔR be the rectangle represented in Figure 2. What's the area of $\varphi(\Delta R)$? The latter can be approximated by the parallelogram whose sides have length equal to

$$\ell(u \mapsto \varphi(u, v_0); u_0 \le u \le u_0 + \Delta u)$$

respectively

$$\ell(v \mapsto \varphi(u_0, v); v_0 \le v \le v_0 + \Delta v)$$

and the angle between the two sides equal to the angle between $\varphi'_u(Q_0)$ and $\varphi'_v(Q_0)$. The first length is

$$\int_{u_0}^{u_0+\Delta u} \|\varphi'_u(\omega, v_0)\|d\omega$$

and is approximately equal to

$$\|\varphi'_u(Q_0)\|\Delta u = \|(\Delta u)\varphi'_u(Q_0)\|$$

Similarly, the second length is approximately equal to

$$\|\varphi'_v(Q_0)\|\Delta v = \|(\Delta v)\varphi'_v(Q_0)\|.$$

Consequently, the area of $\varphi(\delta R)$ can be approximated by the area of the parallelogram determined by the vectors $(\Delta u)\varphi'_u(Q_0)$ and $(\Delta v)\varphi'_v(Q_0)$. The latter is just the length of the scalar product of the two vectors. We have proved

$$\begin{aligned}\mathrm{Area}(\varphi(R)) &\approx \|(\Delta u)\varphi'_u(Q_0) \times (\Delta v)\varphi'_v(Q_0)\| = \|\varphi'_u(Q_0) \times \varphi'_v(Q_0)\|\Delta u \Delta v \\ &= \|\varphi'_u(Q_0) \times \varphi'_v(Q_0)\|\mathrm{Area}(\Delta R).\end{aligned}$$

This suggests[2] the following definition:

[2]See the definition of the double integral from vector calculus (for instance in Stewart's textbook).

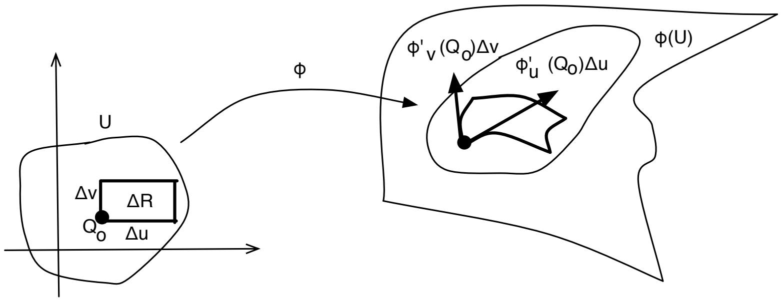

FIGURE 2. The area of $\varphi(\Delta R)$ is approximately equal to the area of the parallelogram determined by $\varphi'_u(Q_0)\Delta u$ and $\varphi'_v(Q_0)\Delta v$

Definition 4.3.1. If R is a compact[3] subset of U, then the *area* of $\varphi(R)$ is

$$\text{Area}(\varphi(R)) = \int\int_R \|\varphi'_u \times \varphi'_v\| dA.$$

The problem with this definition is that it is possible to cover a region of a surface by two parametrizations. More precisely, it is possible to have two parametrizations (U, φ) and $(\tilde{U}, \tilde{\varphi})$ such that

$$\varphi(R) = \tilde{\varphi}(\tilde{R}),$$

where R and $\tilde{R}$ are compact subsets of U, respectively $\tilde{U}$. One can show that if this is the case, then

$$\int\int_R \|\varphi'_u \times \varphi'_v\| dA = \int\int_{\tilde{R}} \|\tilde{\varphi}'_u \times \tilde{\varphi}'_v\| dA.$$

To justify this, we take the change of parameter map $\varphi^{-1} \circ \tilde{\varphi} = (h_1, h_2)$ and write

$$\tilde{\varphi}(\tilde{u}, \tilde{v}) = \varphi(h_1(\tilde{u}, \tilde{v}), h_2(\tilde{u}, \tilde{v})).$$

This implies

$$\tilde{\varphi}'_{\tilde{u}} = \frac{\partial \varphi}{\partial u}\frac{\partial h_1}{\partial \tilde{u}} + \frac{\partial \varphi}{\partial v}\frac{\partial h_2}{\partial \tilde{v}} = \frac{\partial h_1}{\partial \tilde{u}}\varphi'_u + \frac{\partial h_2}{\partial \tilde{v}}\varphi'_v.$$

We get a similar formula for $\tilde{\varphi}'_{\tilde{v}}$, and then we deduce that

$$\tilde{\varphi}'_{\tilde{u}} \times \tilde{\varphi}'_{\tilde{v}} = \det(J(h_1, h_2))\varphi'_u \times \varphi'_v,$$

where $J(h_1, h_2)$ is the Jacobi (2×2) matrix of (h_1, h_2). We note that (h_1, h_2) maps $\tilde{R}$ to R and we use the change of variables formulas for double integrals. For a more detailed justification, see for instance [dC], page 97. So the definition of the area above is independent of the choice of parametrization.

We can express area in terms of the first fundamental form (3) as follows. We use the general formula

$$\|a \times b\|^2 + (a \cdot b)^2 = \|a\|^2\|b\|^2,$$

where a, b are two arbitrary vectors in $\mathbb{R}^3$ (prove this!). For $a = \varphi'_u$ and $b = \varphi'_v$ we obtain

$$\|\varphi'_u \times \varphi'_v\| = \sqrt{EG - F^2}.$$

Note that $EG - F^2 > 0$, by Lemma 4.1.1. We obtain the following formula for the area.

$$\text{Area}(\varphi(R)) = \int\int_R \sqrt{EG - F^2} dA.$$

[3]That is, U is bounded and closed.

Exercise. Find the area of the torus (see the end of Section 3.1 and Homework no. 3, question 2).

4.4. Isometries and conformal maps. Let's start with the following example. Which is the first fundamental form of the xy coordinate plane? We saw in Example 1, Section 3.1 that a parametrization of that plane is

$$\varphi(u, v) = (u, v, 0),$$

where (u, v) is in $\mathbb{R}^2$. So at any point Q in $\mathbb{R}^2$ we have

$$\varphi'_u(Q) = (1, 0, 0), \ \varphi'_v(Q) = (0, 1, 0).$$

Thus for any P in the plane, the coefficients of I_P are as follows

$$E = \|(1, 0, 0)\|^2 = 1, \ F = (1, 0, 0) \cdot (0, 1, 0) = 0, \ G = \|(0, 0, 1)\|^2 = 1.$$

This implies

$$\mathrm{I}_P(w_1, w_2) = w_1^2 + w_2^2,$$

for any w_1, w_2 in $\mathbb{R}$. Now if we compare this with the first fundamental form of the cylinder given by (4), we see the same expression. In other words, the plane and the cylinder can be parametrized in such a way that the first fundamental forms are the same. The explanation is that, if we denote the plane by Π, then there exists a differentiable map $f : \Pi \to C$ whose differential

$$d(f)_P : T_P\Pi \to T_{f(P)}C$$

"preserves" the first fundamental forms I_P of Π at P, respectively $I_{f(P)}$ of C at $f(P)$. More precisely, we have

$$\mathrm{I}_{f(P)}(d(f)_P(w)) = \mathrm{I}_P(w), \tag{5}$$

for all w in $T_P\Pi$. The map f can be easily seen if we look at the parametrization of C given in Section 4.1. It is

$$f(x, y, 0) = (\cos x, \sin x, y),$$

for all $(x, y, 0)$ in Π.

Exercise. Check Equation (5) above.

More generally, we have the following definition.

Definition 4.4.1. (a) A differentiable map f between the regular surfaces S_1 and S_2 is a <u>*local isometry*</u> if for any point P in S and any vector w in T_PS_1 we have

$$\mathrm{I}_{f(P)}(d(f)_P(w)) = \mathrm{I}_P(w),$$

where I_P, $\mathrm{I}_{f(P)}$ are the first fundamental forms of S_1 at P, respectively of S_2 at $f(P)$.

(b) A differentiable map $f : S_1 \to S_2$ is an <u>*isometry*</u> if it is a local isometry and bijective.

(c) If there exists an isometry from S_1 to S_2 we say that the surfaces S_1 and S_2 are <u>*isometric*</u>.

For instance, the map $f : \Pi \to C$ described above is a local isometry, but *not* an isometry (because it's not injective). In fact, we can show that Π and C are not isometric (that is, there exists no isometry from Π to C).

Remarks. 1. One can easily see that if f is a local isometry, then it is a local diffeomorphism, that is, for any P in S_1 there exists an open neighborhood V_1 of P in S_1 and an open neighborhood V_2 of $f(P)$ in S_2 such that f is a diffeomorphism from V_1 to V_2.

2. A local isometry preserves the lengths of curves. More precisely, let $f : S_1 \to S_2$ be a local isometry. If α is a curve whose trace is on S_1, then $\beta(t) = f(\alpha(t))$ is a curve on S_2 with the property that

$$\ell(\alpha|_{[c,d]}) = \ell(\beta|_{[c,d]}).$$

Another example of a pair of locally isometric surfaces is the helicoid and the catenoid (this will be a homework exercise). Examples of isometric surfaces can be obtained by taking a sheet of paper and bending it into various shapes (the bent sheet is isometric to the initial one). A more concrete example is

$$f(x, y) = (\cos y, \sin y, x)$$

which is an isometry between $\mathbb{R} \times (0, 2\pi)$ and the cylinder C without the vertical line $(1, 0, z)$, z in $\mathbb{R}$.

We now define a new notion, namely conformal maps between surfaces.

Definition 4.4.2. A differentiable map f between the surfaces S_1 and S_2 is called a *conformal map* if for any point P in S and any vector w in $T_P S_1$ we have

$$\mathrm{I}_{f(P)}(d(f)_P(w)) = \lambda(P)\mathrm{I}_P(w), \tag{6}$$

where $\lambda : S_1 \to \mathbb{R}$ is a differentiable function with $\lambda(P) > 0$ for all P in S_1.

A conformal map does not preserve lengths, like an isometry (see Remark 2 above), but it preserves angles. This is what the following result say.

Proposition 4.4.3. *If $f : S_1 \to S_2$ is a conformal map, then for any point P in S_1 and any two vectors $w, \tilde{w}$ in $T_P S_1$, we have*

$$\frac{d(f)_P(w) \cdot d(f)_P(\tilde{w})}{\|d(f)_P(w)\|\|d(f)P(\tilde{w})\|} = \frac{w \cdot \tilde{w}}{\|w\|\|\tilde{w}\|}.$$

Consequently, the angle between the vectors w and $\tilde{w}$ is the same as the angle between their images under $d(f)_P$.

Proof. The result follows immediately from the following equation.

$$d(f)_P(w) \cdot d(f)_P(\tilde{w}) = \lambda(P) w \cdot \tilde{w}.$$

In turn, this follows from

$$\|d(f)_P(w)\|^2 = \lambda(P)\|w\|^2,$$

(which is a straightforward consequence of (6)), and the equations:

$$2d(f)_P(w) \cdot d(f)_P(\tilde{w}) = \|d(f)_P(w) + d(f)_P(\tilde{w})\|^2 - \|d(f)_P(w)\|^2 - \|d(f)_P(\tilde{w})\|^2,$$
$$2w \cdot \tilde{w} = \|w + \tilde{w}\|^2 - \|w\|^2 - \|\tilde{w}\|^2.$$

□

Exercise. Show that the stereographic projection Φ from the xy plane to the sphere S^2 (see Section 2.3 of the notes) is a conformal map.

References

[dC] M. P. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice Hall 1976

[Gr-Abb-Sa] A. Gray, E. Abbena, and S. Salamon, *Modern Differential Geometry with MATHEMATICA*, 3rd edition