DIFFERENTIAL GEOMETRY OF CURVES AND SURFACES

3. Regular Surfaces

3.1. The definition of a regular surface. Examples. The notion of surface we are going to deal with in our course can be intuitively understood as the object obtained by a potter full of phantasy who takes several pieces of clay, flatten them on a table, then models of each of them arbitrarily strange looking pots, and finally glue them together and flatten the possible edges and corners. The resulting object is, formally speaking, a subset of the three dimensional space $\mathbb{R}^3$. Here is the rigorous definition of a surface.

Definition 3.1.1. A subset $S \subset \mathbb{R}^3$ is a[1] *regular surface* if for any point P in S one can find an open subspace $U \subset \mathbb{R}^2$ and a map $\varphi : U \to S$ of the form

$$\varphi(u, v) = (x(u, v), y(u, v), z(u, v)),$$

for $(u, v) \in U$ with the following properties:

1. The function φ is differentiable, in the sense that the components $x(u, v)$, $y(u, v)$ and $z(u, v)$ have partial derivatives of any order.
2. For any Q in U, the differential map $d(\varphi)_Q : \mathbb{R}^2 \to \mathbb{R}^3$ is (linear and) injective.
3. There exists an open subspace $V \subset \mathbb{R}^3$ which contains P such that $\varphi(U) = V \cap S$ and moreover, $\varphi : U \to V \cap S$ is a homeomorphism[2].

The pair (U, φ) is called a *local parametrization*, or a *chart*, or a *local coordinate system* around P. Conditions 1 and 2 say that (U, φ) is a *regular patch*. See also Figure 1.

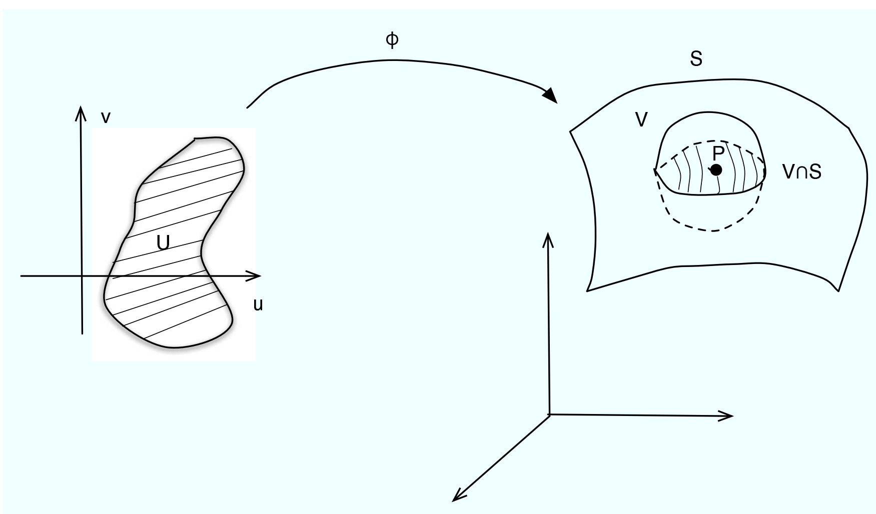

Figure 1. The pair (U, φ) is a local parametrization around P on the surface S.

Now let's try to understand the definition. In a less precise, but (hopefully!) more intuitive formulation, condition 3 is saying that any point P is contained in one of the strange looking pots made by the potter to be glued together; the map φ is just the "continuous" deformation (done by the potter's hands) of the *flat* piece of clay U into $V \cap S$. Condition 1 is just meant to let us do *differential* geometry, that is, study these objects by using derivatives (in the

[1]Terminology: in this course, whenever we say *surface* we actually mean *regular surface*.

[2]That is, it is continuous, bijective, and its inverse $\varphi^{-1} : V \cap S \to U$ is continuous as well (here $V \cap S$ is equipped with the topology of subspace of $\mathbb{R}^3$).

same way as we only considered *differentiable* curves in the previous two chapters of the course).

But what's hiding behind condition 2? Answer: it guarantees that there is a tangent plane to the surface at any point. Here are the detailed explanations. First let us recall from Vector Calculus that the differential of $\varphi : U \to \mathbb{R}^3$,

$$\varphi(u, v) = (x(u, v), y(u, v), z(u, v))$$

at a point Q is the linear map $d(\varphi)_Q : \mathbb{R}^2 \to \mathbb{R}^3$ whose matrix is the Jacobi matrix

$$(J\varphi)_Q := \begin{pmatrix} x'_u(Q) & x'_v(Q) \\ y'_u(Q) & y'_v(Q) \\ z'_u(Q) & z'_v(Q) \end{pmatrix}.$$

Here the entries of the matrix from above are

$$x'_u = \frac{\partial x}{\partial u}, x'_v = \frac{\partial x}{\partial v} \ldots \text{ etc.}$$

that is, the partial derivatives of the components x, y, z of φ. A result from Linear Algebra says that the map $d(\varphi)_Q$ is injective if and only if the rank of its matrix is equal to 2. In turn, this is equivalent to the fact that the two columns of the matrix are linearly independent vectors in $\mathbb{R}^3$. These two vectors have a natural geometric interpretation, as follows. Let the coordinates of Q be (u_0, v_0). We consider the *coordinate curve* $u \mapsto \varphi(u, v_0)$, which goes through $\varphi(Q)$. The tangent vector to this curve at u_0 is

$$\frac{d}{du}|_{u=u_0}\varphi(u, v_0) = \frac{\partial \varphi}{\partial u}(Q) = \left(\frac{\partial x}{\partial u}(Q), \frac{\partial y}{\partial u}(Q), \frac{\partial z}{\partial u}(Q)\right)$$

which is the first column of $J(\varphi)_Q$. Similarly, the tangent vector to the other coordinate curve, namely $v \mapsto \varphi(u_0, v)$, at v_0 is the other column of $J(\varphi)_Q$.

In conclusion, condition 2 in the definition is saying that for any Q in U, the tangent vectors $\varphi'_u(Q)$, $\varphi'_v(Q)$ to the coordinate curves (which are the same as the columns of the Jacobi matrix $J(\varphi)_Q$) are linearly independent.

In this way, we can define the tangent plane to S at $\varphi(Q)$ as the plane through $\varphi(Q)$ which is parallel to $\varphi'_u(Q)$ and $\varphi'_v(Q)$. We will get back to the notion of tangent plane later on.

Examples of surfaces. 1. Let S be the horizontal coordinate plane, which consists of all points $(x, y, 0)$ with x, y in $\mathbb{R}$. We claim that S is a surface, in the sense of Definition 3.1.1. To see that, we take U to be the whole $\mathbb{R}^2$ and $\varphi : \mathbb{R}^2 \to S$ given by

$$\varphi(u, v) = (u, v, 0),$$

for all (u, v) in $\mathbb{R}^2$. It s obvious that any point $P = (x, y, 0)$ of S is in the image of φ (we could say that (U, φ) is a *global* parametrization). Now let's check the conditions 1, 2, and 3 above. First, al three components of φ are differentiable (i.e. they have partial derivatives of any order). Second, the Jacobi matrix of φ at a point $Q = (u, v)$ is

$$(J\varphi)_Q := \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Its columns are linearly independent. Third, we take $V = \mathbb{R}^3$, we note that $V \cap S = S$, and we see that $\varphi : \mathbb{R}^2 \to S$ is a homeomorphism (it is obviously continuous, bijective, and its inverse is given by $\varphi^{-1}(x, y, 0) = (x, y)$, which is also continuous, as restriction of the projection map $\mathbb{R}^3 \to \mathbb{R}^2$, $(x, y, z) \mapsto (x, y)$ to S). See also Figure 2.

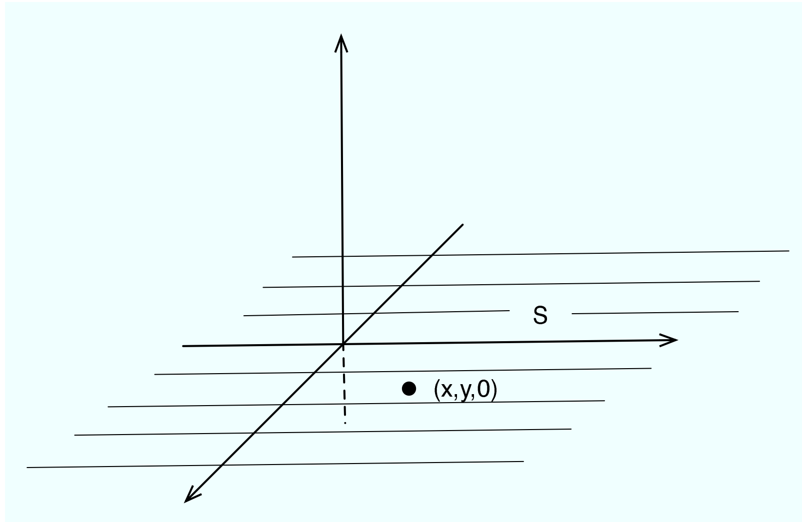

FIGURE 2. The horizontal coordinate plane is an example of a surface (probably the simplest one).

2. In the same way as before, if U is any open subset of the horizontal coordinate plane (for instance the inside of a circle), then U is a surface. See Figure 3.

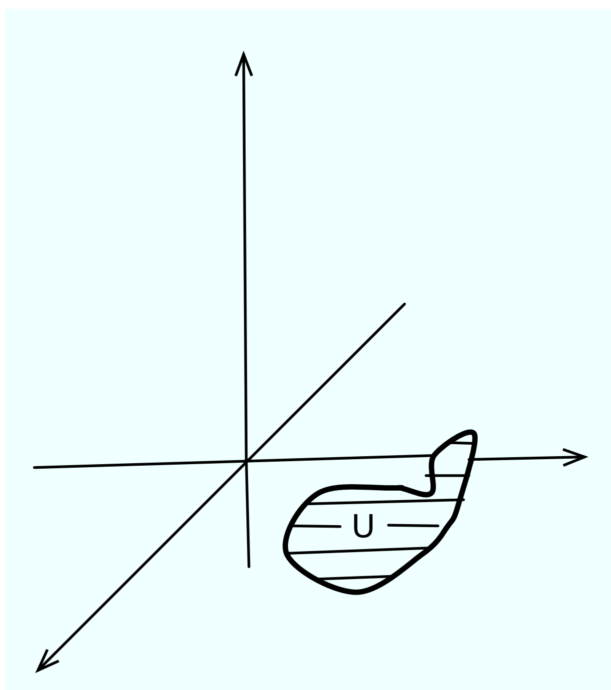

FIGURE 3. A thin piece of clay on a table, not touched by the potter's hands.

3. The usual notation for the sphere in $\mathbb{R}^3$ with centre at O and radius 1 is S^2. It consists of points (x, y, z) in $\mathbb{R}^3$ with distance to the origin O equal to 1. Equivalently, they must satisfy

$$x^2 + y^2 + z^2 = 1.$$

We will show that S^2 is a regular surface. To this end we construct some local parametrizations whose images cover S^2.

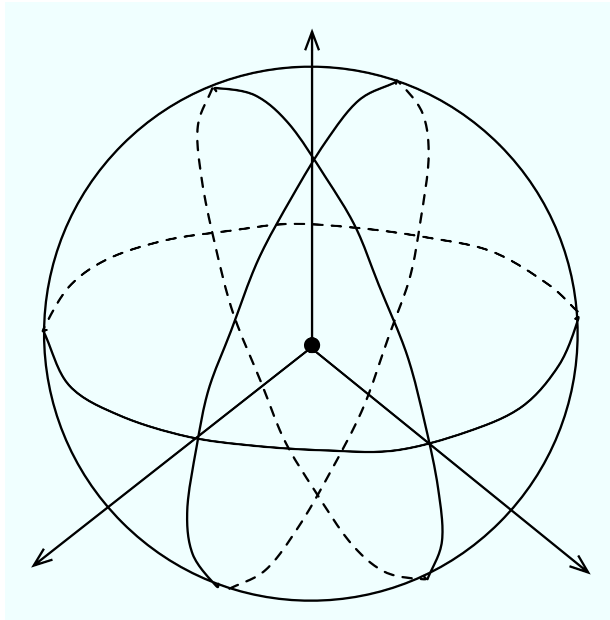

FIGURE 4. The sphere is covered by six semi-spheres, each of them being the image of a local parametrization of the form (U, φ).

Consider the semi-sphere S^+_{xy} which consists of points (x, y, z) in S^2 with $z > 0$. Let π_1 denote the orthogonal projection of S^+_{xy} to the horizontal coordinate plane. The image of π_1 is the open disk D_{xy} bounded by the circle of radius 1 and centre 0; it consists of all points (x, y) in the horizontal plane with

$$x^2 + y^2 < 1.$$

One can easily see that π_1 is bijective and its inverse, call it φ_1, is

$$\varphi_1(x, y) = (x, y, \sqrt{1 - x^2 - y^2}).$$

Exercise. Show that $\varphi_1 : D_{xy} \to S^+_{xy}$ is a local parametrization, that is, it satisfies conditions 1, 2, and 3 in Definition 3.1.1.

Similarly one can consider five other local parametrizations (see Figure 4). Finally, we only need to note that each point P in S^2 belongs to one of the six semi-spheres. So S^2 is a regular surface.

Remark 1. The following question concerning Definition 3.1.1 can be raised: is condition 2 not implied by conditions 1 and 3? Here are two examples which show that the answer is "no, it's not implied". Probably the simplest one is $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$,

$$\varphi(u, v) = (u^3, v^3, 0).$$

This is another "parametrization" of the horizontal coordinate plane (see Example 1 above). It satisfies conditions 1 and 2 (easy to check this!), but not condition 3. Indeed, we have

$$\varphi'_u(u, v) = (3u^2, 0, 0) \text{ and } \varphi'_v(u, v) = (0, 3v^2, 0),$$

for any $(u,v) \in \mathbb{R}^2$. This implies that at $(0,0)$ we have

$$\varphi'_u(0,0) = \varphi'_v(0,0) = (0,0,0)$$

so these two vectors cannot be linearly independent. Note that in this case, the inconvenience can be fixed: we can so to say reparametrize $\varphi(\mathbb{R}^2)$ and make a surface out of it (like we did in Example 1).

A slightly more complicated (but more instructive) example[3] is $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$, $\varphi(u,v) = (u^3, v^3, uv)$. The image of φ (that is, the potential surface parametrized by φ) is represented in Figure 5. Again, it's simple to check that φ satisfies conditions 1 and 3, and again we have

$$\varphi'_u(0,0) = \varphi'_v(0,0) = (0,0,0),$$

that is, condition 2 is not satisfied. But this time we have no chance to reparametrize $\varphi(\mathbb{R}^2)$ and make a regular surface out of it: the figure shows that we cannot remove the inconvenient of not having a tangent plane at the point $(0,0,0)$.

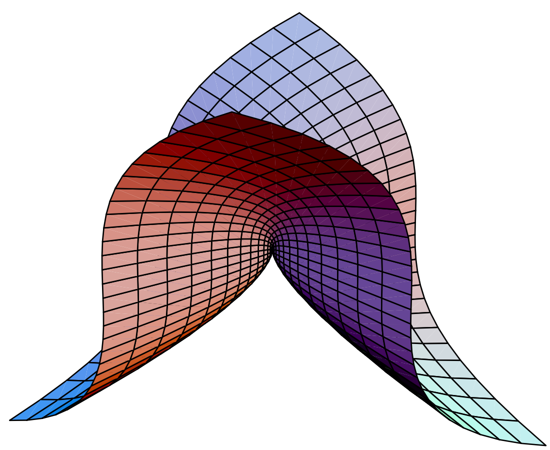

FIGURE 5. The image of $\varphi(u,v) = (u^3, v^3, uv)$, which is not a *regular* surface.

Remark 2. In the same spirit as above, one can ask if the requirement that $\varphi^{-1} : V \cap S \to U$ is continuous at point 3 does not follow from the other conditions. The following example is showing that this is not the case, so we always have to check this, too. Take

$$\varphi(u,v) = (\sin u, \sin 2u, v)$$

where $0 < u < 2\pi$ and $0 < v < 1$ (otherwise expressed, (u,v) is in $(0,2\pi) \times (0,1)$). Its image, call it S, can be seen in Figure 6. One can check that all conditions in Definition 3.1.1 are satisfied, except that $\varphi^{-1} : S \to (0,2\pi) \times (0,1)$ is not continuous (at points of the form $(0,0,v)$). One needs to be a a bit advanced in point set topology to understand the last thing. However, the lesson is that the "open" ribbon bent like in the figure is not a regular surface.

[3]See [Gr-Abb-Sa], page 291.

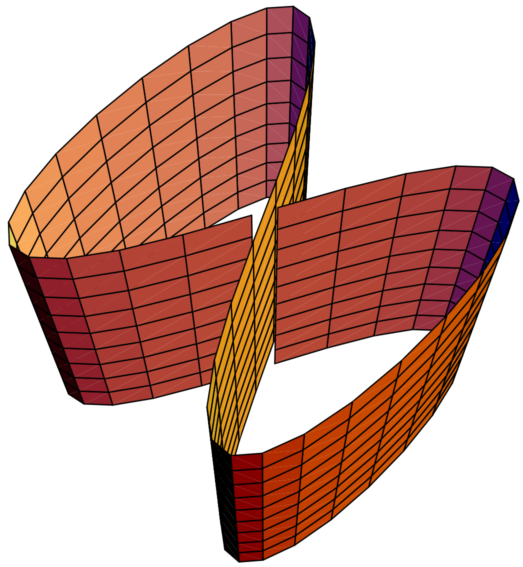

FIGURE 6. The image of $\varphi(u, v) = (\sin u, \sin 2u, v)$, where $0 < u < 2\pi$, $0 < v < 1$, which is not a regular surface.

Whoever took Vector Calculus knows that "graphs of functions of two variables are surfaces". In fact those graphs are just a special type of regular level surfaces of functions of three variables, which are discussed below.

Theorem 3.1.2. *Let W be an open subset of $\mathbb{R}^3$, $g : W \to \mathbb{R}$ a differentiable function and c a number. We denote by $g^{-1}(c)$ the set*[4] *of all points P in W with $g(P) = c$. Assume that for any P in $g^{-1}(c)$ at least one of the partial derivatives $g'_x(P)$, $g'_y(P)$, and $g'_z(P)$ is different from*[5] *0. Then $g^{-1}(c)$ is a regular surface.*

Note. If the assumptions above are satisfied, we say that $g^{-1}(c)$ is a regular level of g.

Proof. Without loss of generality we may assume that $c = 0$ (try to figure out why). We take P in $g^{-1}(0)$ and we check Definition 3.1.1. In other words, we construct U, φ and V. By the hypothesis of the theorem, we know that $g'_x(P) \neq 0$, or $g'_y(P) \neq 0$ or $g'_z(P) \neq 0$. Let's say, for instance, that $g'_z(P) \neq 0$. We consider the function $F : W \to \mathbb{R}^3$ given by

$$F(x, y, z) = (x, y, g(x, y, z))$$

for all (x, y, z) in W. The Jacobi matrix of F at P is

$$(JF)_P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ g'_x(P) & g'_y(P) & g'_z(P) \end{pmatrix}.$$

Its determinant is $g'_z(P)$, which is different from 0. From the Inverse Function Theorem we deduce that there exists an open subspace V of W, which contains P, and an open subspace V_1 of $\mathbb{R}^3$, which contains $F(P)$, such that F restricted to V is a diffeomorphism from V to V_1. By this we mean that F, regarded as a map from V to V_1, is differentiable, bijective,

[4] This is called the pre-image or the level of c.

[5] Equivalently, the gradient of g at P is e nonzero vector.

and its inverse F^{-1} from V_1 to V is differentiable. We note that $F(V \cap g^{-1}(0))$ consists of points in V_1 which are of the form $(x, y, 0)$; we identify the set of points of the latter form with $\mathbb{R}^2$ and so we have

$$F(V \cap g^{-1}(0)) = V_1 \cap \mathbb{R}^2 = U$$

where U is open in $\mathbb{R}^2$. The last equation implies

$$F^{-1}(U) = V \cap g^{-1}(0).$$

The pair consisting of U and F^{-1} restricted to U is a local parametrization of $g^{-1}(0)$ around P (you may want to check carefully conditions 1,2, and 3 in Definition 3.1.1). $\square$

Corollary 3.1.3. *If U is an open subset of $\mathbb{R}^2$ and $h : U \to \mathbb{R}$ is a differentiable function, then the set S of $\mathbb{R}^3$ which consists of all points $(x, y, h(x, y))$ with (x, y) in U is a surface.*

Note. The set S is called the graph of h.

Proof. Let $U \times \mathbb{R}$ denote the set of all triples (x, y, z) with (x, y) in U and z in $\mathbb{R}$. We consider the function $g : U \times \mathbb{R} \to \mathbb{R}$ given by

$$g(x, y, z) = z - f(x, y).$$

One can easily see that

$$S = g^{-1}(0).$$

We want to apply Theorem 3.1.3. To this end, we show that at any P in $g^{-1}(0)$ at least one of the partial derivatives of g is non-zero. But one can see that

$$g'_z(x, y, z) = 1 \neq 0$$

at any point (x, y, z). By Theorem 3.1.3, the graph S is a surface. $\square$

Examples. 1. The sphere S^2 is, as we saw already, the set of all points (x, y, z) in $\mathbb{R}^3$ with

$$x^2 + y^2 + z^2 = 1.$$

Take $g : \mathbb{R}^3 \to \mathbb{R}$, $g(x, y, z) = x^2 + y^2 + z^2$. Then S^2 is just $g^{-1}(1)$.

Exercise. Show that for any $P = (x, y, z)$ in S^2, at least one of the numbers $g'_x(P), g'_y(P)$ and $g'_z(P)$ is different from 0.

From Theorem 3.1.2 we deduce that S^2 is a regular surface (more precisely, we reprove this fact, see example 3 above).

2. More generally, we can take the *ellipsoid*, which is the set of all points in $\mathbb{R}^3$ with

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

The best way to visualize it is by noting that its intersections with each of the coordinated planes is an ellipse (for instance, the intersection with the xy plane has equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$). We can prove that the ellipsoid is a regular surface in exactly the same way as above for the sphere S^2. See also Figure 7.

3. The *hyperboloid with one sheet* has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

We can also prove that it is a regular surface in the same way as the in the previous two examples. See also Figure 8 (try to understand the intersections with the coordinate planes).

4. The *torus* is obtained by rotating a circle around a straight line which is contained in the plane of the circle and has no intersection point with the circle. It is not difficult to

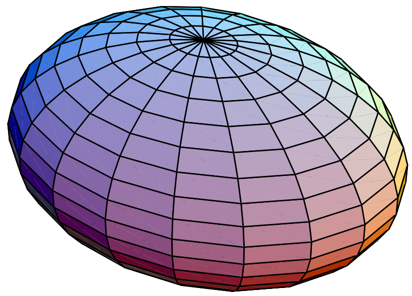

FIGURE 7. The ellipsoid (figure out where the coordinate axes are).

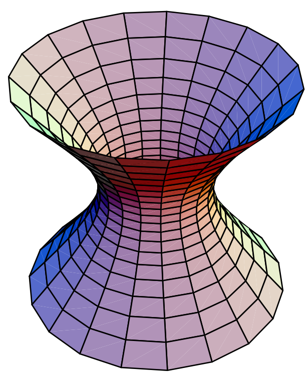

FIGURE 8. The hyperboloid with one sheet (again, figure out where the coordinate axes are).

figure out what the resulting surface looks like (a doughnut, see also Figure 9). Denote by T the resulting set.

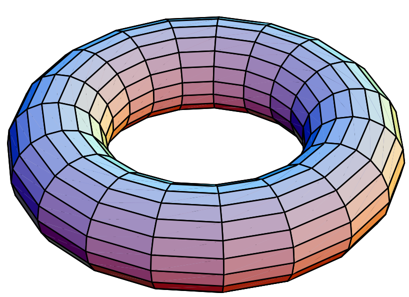

FIGURE 9. The torus.

Let's show that it is a regular surface. Intuitively, it looks very likely that Definition 3.1.1, is satisfied. However, it's simpler to use Theorem 3.1.2.

Exercise. Choose the coordinate system such that the rotation line is the z axis and the rotated circle is in the yz plane (see Figure 10).

(a) Show that a point $P = (x, y, z)$ is on T if and only if

$$(\sqrt{x^2 + y^2} - a)^2 + z^2 = r^2. \tag{1}$$

Hint. Let $(0, y_0, z_0)$ be the point on the original (vertical) circle which gives rise to P by rotation. Show that we have $x^2 + y^2 + (z - z_0)^2 = y_0^2$ and $z = z_0$. Use the fact that $(0, y_0, z_0)$ is on the original circle to deduce an equation involving y_0 and z_0. Combine the three equations you have obtained to deduce equation 1.

(b) Use Theorem 3.1.2 to deduce that T is a regular surface.

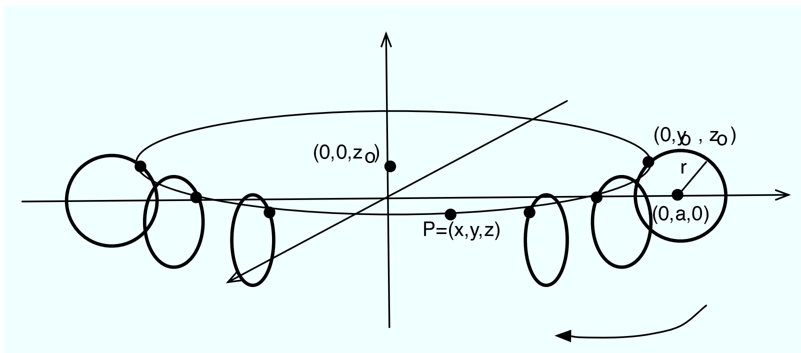

FIGURE 10. The torus which arises by rotating the "original" circle in the yz plane about the z axis.

3.2. Differentiable maps and functions.[6] Let S be a regular surface. Very often in this course we will deal with functions f from S to $\mathbb{R}$. A simple example is when we fix a point P_0 in $\mathbb{R}^3$ and we define the function f which assigns to any P on S the number $f(P)$ given by the squared distance from P_0 and P. We will actually need to deal with derivatives of such functions, so we would like to trace a general framework where to refer whenever we use these. This is the context of differentiable functions. Here is the definition.

Definition 3.2.1. We say that a function $f : S \to \mathbb{R}$ is *differentiable* if for any P in S there exists a local parametrization (U, φ) with P in $\varphi(U)$ such that the function $f \circ \varphi : U \to \mathbb{R}$ given by

$$(f \circ \varphi)(u, v) = f(\varphi(u, v)), \text{ for all } (u, v) \in U$$

is differentiable (see also Figure 3).

It is important to note that the function $f \circ \varphi$ in the previous definition is defined on U, which is a subset of $\mathbb{R}^2$; in other words, $f \circ \varphi$ is a function of two variables, and we know exactly what differentiability of such functions means (they must have partial derivatives with respect to u and v).

If we look carefully at the definition, we see that one requests the existence of *one* local coordinate system around each given point. On the other hand, around a given point there exists in general more than one parametrization (think of the sphere, see Figure 4). It is natural to raise the following question: is it not possible to have a parametrization (U, φ) with $f \circ \varphi$ differentiable and another parametrization (U', φ') with $f \circ \varphi'$ *not* differentiable? In other words, does the definition of differentiability depend on the choice of the parametrization? The answer is "no, it doesn't depend". To be able to prove this, we need the following theorem (see also Figure 12).

[6]The following concerns terminology: a *function* takes values in $\mathbb{R}$ whereas the range of a *map* is *not* $\mathbb{R}$.

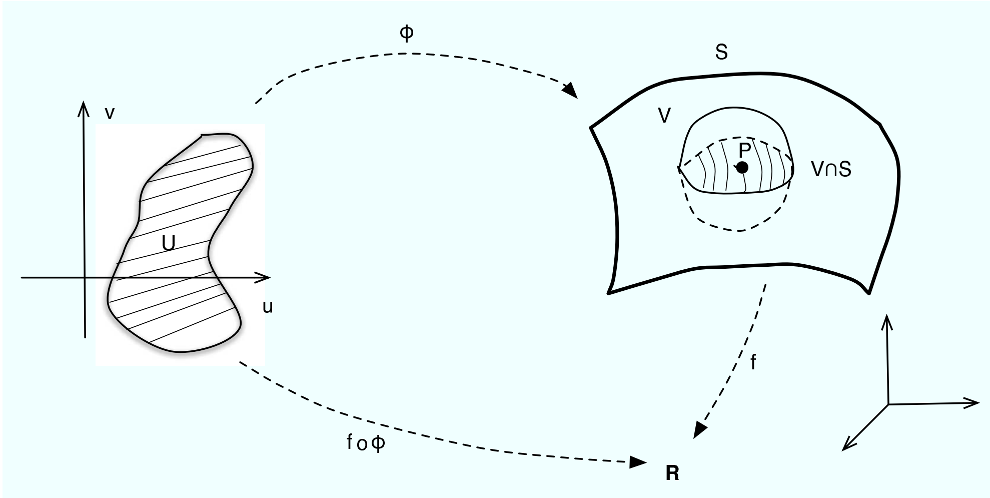

FIGURE 11. The dotted arrows are: φ, f and the composition $f \circ \varphi$.

Theorem 3.2.2. (Change of Parameters) *On the regular surface S one considers two local parametrizations (U, φ) and $(\tilde{U}, \tilde{\varphi})$ with the property that the intersection*

$$W := \varphi(U) \cap \tilde{\varphi}(\tilde{U})$$

is non-empty. Then the map

$$\varphi^{-1} \circ \tilde{\varphi} : \tilde{\varphi}^{-1}(W) \to \varphi^{-1}(W)$$

is differentiable.

Proof's Idea. We take Q in $\tilde{\varphi}^{-1}(W)$ and show that $\varphi^{-1} \circ \tilde{\varphi}$ is differentiable on an open neighborhood of Q in $\tilde{\varphi}^{-1}(W)$. The main point of the proof is that there exists a differentiable function $F : U \times \mathbb{R} \to \mathbb{R}^3$ such that:

- F restricted to $U \times \{0\}$ is φ
- F is a local diffeomorphism at $(\varphi^{-1}(\tilde{\varphi}(Q)), 0)$

We deduce that on an open neighborhood of Q in $\tilde{\varphi}^{-1}(W)$ we have $\varphi^{-1} \circ \tilde{\varphi} = F^{-1} \circ \tilde{\varphi}$, which is differentiable. The details can be found in [dC, page 70] (see also [Gr-Abb-Sa, page 300]).

Note. The map $\varphi^{-1} \circ \tilde{\varphi} : \tilde{\varphi}^{-1}(W) \to \varphi^{-1}(W)$ is called *change of coordinates*. It is a map from an open subset in $\mathbb{R}^2$ to an open subset in $\mathbb{R}^2$, and we know what its differentiability means (both its components have partial derivatives). If we interchange φ and $\tilde{\varphi}$, we deduce that the map

$$\tilde{\varphi}^{-1} \circ \varphi : \varphi^{-1}(W) \to \tilde{\varphi}^{-1}(W)$$

is differentiable as well. But this is just the inverse of $\varphi^{-1} \circ \tilde{\varphi}$, thus the latter map is actually a diffeomorphism (which means that it is differentiable, bijective, and its inverse is differentiable).

We are now ready to return to Definition 3.2.1 and the discussion following it. The following is a corollary of Theorem 3.2.2.

Corollary 3.2.3. *If $f : S \to \mathbb{R}$ is differentiable, then* for any *local parametrization (U, φ) of S the function $f \circ \varphi : U \to \mathbb{R}$ is differentiable.*

Proof. We take Q an arbitrary point in U and show that there exists an open subset N in $\mathbb{R}^2$ which contains Q and is contained in U such that the function $f \circ \varphi$ restricted to N is

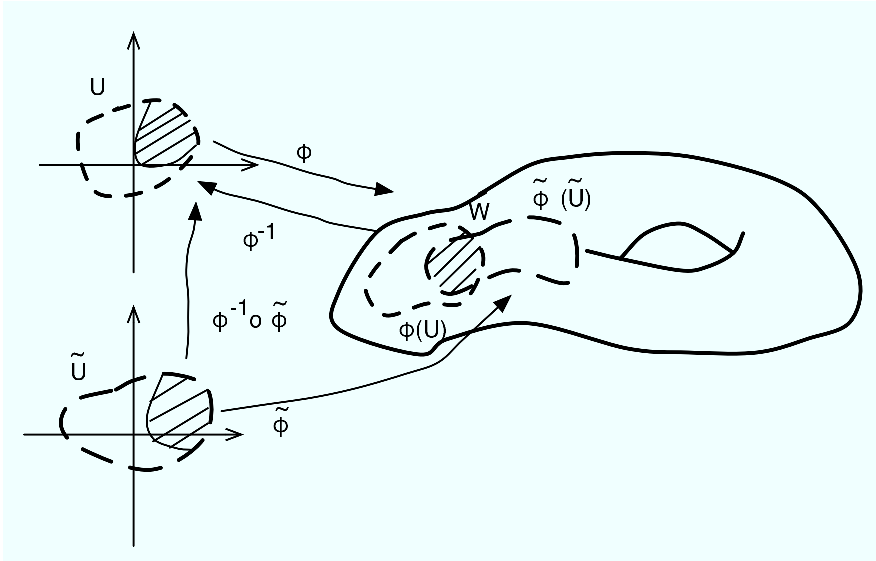

FIGURE 12. The shaded regions are as follows: the one on the surface is the intersection $\varphi(U) \cap \tilde{\varphi}(\tilde{U})$, denoted W; the one in U is $\varphi^{-1}(W)$; the one in $\tilde{U}$ is $\tilde{\varphi}^{-1}(W)$.

differentiable. To this end, we consider the point

$$P := \varphi(Q)$$

in S and we use Definition 3.2.1: there exists a local parametrization $(\tilde{U}, \tilde{\varphi})$ such that P is in $\tilde{\varphi}(\tilde{U})$ and the map

$$f \circ \tilde{\varphi} : \tilde{U} \to \mathbb{R}$$

is differentiable. The intersection

$$W := \varphi(U) \cap \tilde{\varphi}(\tilde{U})$$

is nonempty, since P is an element of it. We take $N := \varphi^{-1}(W)$. On N the function $f \circ \varphi$ can be written as

$$f \circ \varphi = f \circ \tilde{\varphi} \circ \tilde{\varphi}^{-1} \circ \varphi.$$

Since both $\tilde{\varphi}^{-1} \circ \varphi$ and $f \circ \tilde{\varphi}$ are differentiable, their composition is differentiable as well. Hence $f \circ \varphi$ restricted to N is differentiable, QED. $\square$

Examples of differentiable functions on surfaces. 1. Let $F : \mathbb{R}^3 \to \mathbb{R}$ be an arbitrary differentiable function and let S be a regular surface. Then the function $f : S \to \mathbb{R}$ which is the restriction of F to S is differentiable (in the sense of Definition 3.2.1). Indeed, if (U, φ) is any parametrization of S, then the function $f \circ \varphi : U \to \mathbb{R}$ is the same as $F \circ \varphi$, which is differentiable (as composition of two differentiable functions).

In the following we take two standard examples of differentiable functions from $\mathbb{R}^3$ to $\mathbb{R}$: the height and the distance function. Restricted to any surface, they give differentiable functions.

2. *Height functions.* We consider a plane through the origin. The *height* of the point P with respect to the plane is, up to a possible minus sign, the length of the line segment PQ, which is perpendicular to the plane. We choose v a unit vector perpendicular to the plane (see Figure 13).

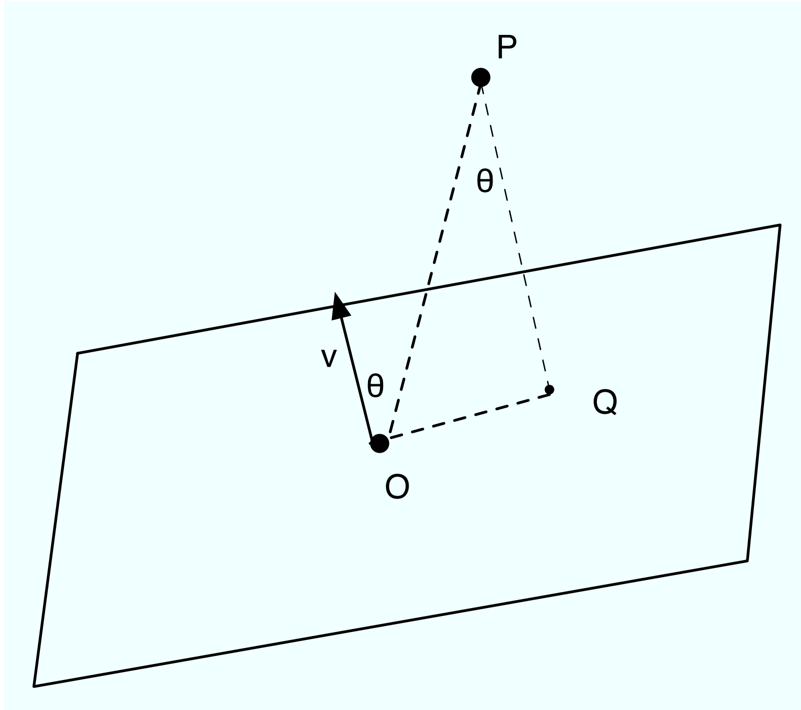

FIGURE 13. The height of P with respect to the plane in the figure is $v \cdot P$.

Then the height of P with respect to the plane is

$$\|PQ\| = \|OP\| \cos\theta = \vec{PQ} \cdot v.$$

If we identify the point P with the vector $\vec{OP}$, we can describe the height function $h_v : \mathbb{R}^3 \to \mathbb{R}$ by the formula

$$h_v(P) = P \cdot v,$$

for any $P \in \mathbb{R}^3$. If we take only P in the surface S, we obtain the height function $h_v : S \to \mathbb{R}$. For instance, the height function with respect to e_3 restricted to the sphere S^2 is differentiable and has a minimum at the South pole $S = (0, 0, -1)$ and a maximum at the North pole $N = (0, 0, 1)$.

3. Distance functions. We fix a point P_0 and we consider the function

$$d(P) = \|P - P_0\|^2,$$

for all $P \in \mathbb{R}^3$. This function is differentiable. Its restriction to any surface S is a differentiable function on S.

Very often we will be interested in maps from a surface to another and their differentials. For instance, the antipodal map a from S^2 to S^2, given by $a(x, y, z) = (-x, -y, -z)$ or the map f from S^2 to the ellipse $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ given by

$$f(x, y, z) = (ax, by, cz).$$

Definition 3.2.4. Let S_1 and S_2 be two surfaces and f a map from S_1 to S_2. We say that f is *differentiable* if for any P in S_1 there exists local parametrizations (U_1, φ_1) on S_1 and (U_2, φ_2) on S_2 with P in $\varphi_1(U_1)$ and $f(P)$ in $\varphi_2(U_2)$ such that the map

$$\varphi_2^{-1} \circ f \circ \varphi_1 : \varphi_1^{-1}(\varphi_1(U_1) \cap f^{-1}(\varphi_2(U_2))) \to U_2$$

is differentiable.

The following is another corollary of Theorem 3.2.2.

Corollary 3.2.5. *If $f : S_1 \to S_2$ is a differentiable map, then for any point P in S_1 and any two local parametrizations (U_1, φ_1), (U_2, φ_2) on S_1, respectively S_2 with P in $\varphi_1(U_1)$ and $f(P)$ in $\varphi_2(U_2)$, the map*

$$\varphi_2^{-1} \circ f \circ \varphi_1 : \varphi_1^{-1}(\varphi_1(U_1) \cap f^{-1}(\varphi_2(U_2))) \to U_2$$

is differentiable.

Examples of differentiable maps between surfaces.

1. If $f : S_1 \to S_2$ and $g : S_2 \to S_3$ are differentiable, then the composition $g \circ f : S_1 \to S_3$ is differentiable.

2. If S is a surface and (U, φ) a local parametrization, then both $\varphi : U \to S$ and $\varphi^{-1} : \varphi(U) \to U$ are differentiable (recall that U is a surface, see Section 3.1, Example 2).

3. If $F : \mathbb{R}^3 \to \mathbb{R}^3$ is a differentiable map and S_1, S_2 are two surfaces with $F(S_1) \subset S_2$, then the restriction of F to S_1 is a differentiable map between S_1 and S_2 (the proof is a consequence of 1 and 2 above).

4. The antipodal map $a : S^2 \to S^2$, $a(x, y, z) = (-x, -y, -z)$ is differentiable (from 3).

5. The map f from S^2 to the ellipse $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, given by $f(x, y, z) = (ax, by, cz)$, is differentiable (again from 3).

3.3. Tangent vectors, the tangent plane, and the differential of a map.

One possibility to define the tangent plane to a surface at a point P has already been mentioned in Section 3.1: one could choose a local parametrization (U, φ) with $P = \varphi(Q)$, where Q is in U and then consider the plane determined by $\varphi'_u(Q)$ and $\varphi'_v(Q)$. The inconvenient of this definition is that it depends on the choice of the parametrization around P (we recall that in general there exists more than one parametrization around a point on a surface). This is why we prefer to use a quite different idea. First we define the tangent vectors at P, as follows.

Definition 3.3.1. (i) A <u>*tangent vector*</u> to the regular surface S at its point P is a vector of the form $\alpha'(0)$, where α is a curve $\alpha : (-\epsilon, \epsilon) \to \mathbb{R}^3$ such that

- $\alpha(t)$ is in S, for all t
- $\alpha(0) = P$.

  (ii) The <u>*tangent plane*</u> to S at P is the set of all tangent vectors to S at P. It is denoted $T_P(S)$.

We will prove now that $T_P(S)$ is really a plane, that is, a two dimensional vector subspace of $\mathbb{R}^2$. Also that for any local parametrization (U, φ) around P the space T_PS coincides with the plane determined by $\varphi'_u(Q)$ and $\varphi'_v(Q)$ (where $\varphi(Q) = P$).

Theorem 3.3.2. *Let S be a regular surface, (U, φ) a local parametrization, $Q \in U$, and $P := \varphi(Q)$. Then the space of all tangent vectors at P to S coincides with the image of the injective linear map*

$$d(\varphi)_Q : \mathbb{R}^2 \to \mathbb{R}^3.$$

This space is the same as the two dimensional vector space spanned by the vectors $\varphi'_u(Q)$ and $\varphi'_v(Q)$.

Proof. We have to show that

$$T_PS = d(\varphi)_Q(\mathbb{R}^2).$$

$\underline{T_PS \subset d(\varphi)_Q(\mathbb{R}^2)}$: Let α be a curve like in Definition 3.3.1. The vector $\alpha'(0)$ is in T_PS. We need to show that it is in the image of $d(\varphi)_Q$. We consider the curve β in U given by

$$\beta(t) = \varphi^{-1}(\alpha(t)),$$

where t in $(-\epsilon, \epsilon)$, like in Figure 14. The curve β is differentiable (to justify this we use the argument in the proof of Theorem 3.2.2: we can extend φ to a differentiable map $F : U \times \mathbb{R} \to \mathbb{R}^3$ which is a local diffeomorphism at Q, so we have $\varphi^{-1}(\alpha(t)) = F^{-1}(\alpha(t))$, for all t sufficiently close to 0). We have $\alpha(t) = \varphi(\beta(t))$ for all t, which implies that

$$\alpha'(0) = \frac{d}{dt}|_0\varphi(\beta(t)) = d(\varphi)_{\beta(0)}(\frac{d}{dt}|_0\beta(t)) = d(\varphi)_Q(\beta'(0))$$

which is in the image of $d(\varphi)_Q$.

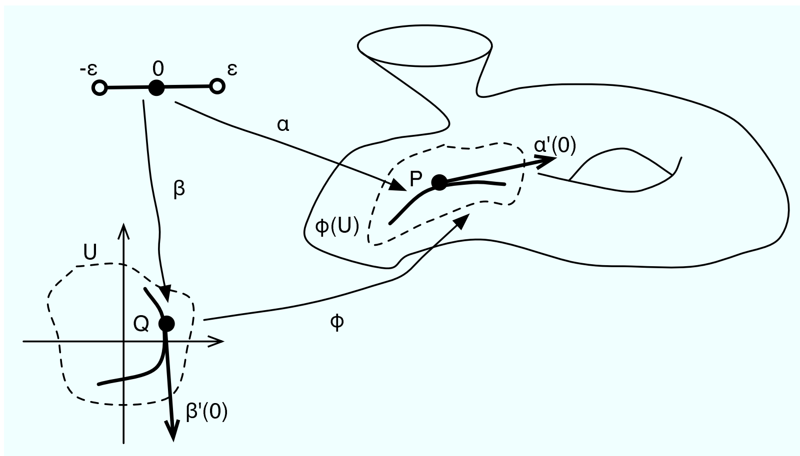

FIGURE 14. The curves α and β are related by $\alpha(t) = \varphi(\beta(t))$.

$\underline{d(\varphi)_Q(\mathbb{R}^2) \subset T_PS}$: Let a be a vector in $\mathbb{R}^2$, and take $w := d(\varphi)_Q(a)$. We need to show that w is in T_PS. To this end, we take the curve

$$\alpha(t) = \varphi(Q + ta),$$

where t is in an interval of the form $(-\epsilon, \epsilon)$, with ϵ sufficiently small. We have

$$\alpha'(0) = d(\varphi)_Q(a) = w.$$

The proof is finished. $\square$

The following lemma will be needed later:

Lemma 3.3.3. *Let (U, φ) be a local parametrization on the surface S, and $\alpha'(0)$ a vector in T_PS like in Definition 3.3.1, where $P = \varphi(Q)$, with $Q \in U$. Write*

$$\varphi^{-1}(\alpha(t)) = (u(t), v(t)),$$

for all t in $(-\epsilon, \epsilon)$. Then the coordinates of $\alpha'(0)$ with respect to the basis $\varphi'_u(Q)$ and $\varphi'_v(Q)$ are $u'(0)$ and $v'(0)$.

Proof. We have

$$\alpha(t) = \varphi(u(t), v(t))$$

which implies

$$\alpha'(0) = \frac{d}{dt}|_0\varphi(u(t), v(t)) = \frac{\partial\varphi}{\partial u}(Q)\frac{d}{dt}|_0u(t) + \frac{\partial\varphi}{\partial v}(Q)\frac{d}{dt}|_0v(t) = u'(0)\varphi'_u(Q) + v'(0)\varphi'_v(Q).$$

Here we have used the chain rule for functions of two variables. $\square$

The next notion we will define is the differential of a map between two surfaces.

Definition 3.3.4. Let S_1 and S_2 are two surfaces, $f : S_1 \to S_2$ a differentiable map (in the sense of Definition 3.2.4), and P a point in S_1. The *differential* of f at P is the map

$$d(f)_P : T_P S_1 \to T_{f(P)} S_2$$

defined as follows: take w in $T_P S_1$ and α a curve like in Definition 3.3.1 with $w = \alpha'(0)$; then

$$d(f)_P(w) = \frac{d}{dt}|_0(f(\alpha(t))). \tag{2}$$

The problem with the definition is that there exists several curves on S_1 through P with the same tangent vector. More specifically, if $w = \alpha_1'(0) = \alpha_2'(0)$, are the vectors $d/dt|_0 f(\alpha_1(t))$ and $d/dt|_0(\alpha_2(t))$ equal? The positive answer is given by the following theorem.

Theorem 3.3.5. *(a) The vector $d/dt|_0 f(\alpha(t))$ in Definition 3.3.4 is the same for any curve α in S_1 with $\alpha'(0) = a$. Thus the map $d(f)_P : T_P S_1 \to T_{f(P)} S_2$ is well defined.*

(b) The map $d(f)_P : T_P S_1 \to T_{f(P)} S_2$ is linear.

Proof. Let us consider local parametrizations (U, φ) and $(\tilde{U}, \tilde{\varphi})$ of S_1 around P, respectively of S_2 around $f(P)$. We know that the map

$$\tilde{\varphi}^{-1} \circ f \circ \varphi : \varphi^{-1}(\varphi(U) \cap f^{-1}(\tilde{\varphi}(\tilde{U})) \to \tilde{U}$$

is a differentiable map between two open subsets in $\mathbb{R}^2$. Let us denote its components by h_1 and h_2, that is

$$\tilde{\varphi}^{-1} \circ f \circ \varphi(u, v) = (h_1(u, v), h_2(u, v)),$$

for all (u, v) in the domain.

Like in Lemma 3.3.3, we denote

$$\varphi^{-1}(\alpha(t)) = (u(t), v(t))$$

and deduce that we have

$$w = \alpha'(0) = u'(0)\varphi_u'(Q) + v'(0)\varphi_v'(Q),$$

where $Q = \varphi^{-1}(P)$. We consider the curve γ given by

$$\gamma(t) := f(\alpha(t))$$

and we are interested in the tangent vector $\gamma'(0)$. We are planning to use again Lemma 3.3.3. To this end we will need to find the components of $\tilde{\varphi}^{-1}(\gamma(t))$. We can write

$$\tilde{\varphi}^{-1}(\gamma(t)) = \tilde{\varphi}^{-1} \circ \gamma(t) = \tilde{\varphi}^{-1} \circ f \circ \alpha(t) = \tilde{\varphi}^{-1} \circ f \circ \varphi \circ \varphi^{-1} \circ \alpha(t) = (h_1(u(t), v(t)), h_2(u(t), v(t)).$$

Take $\tilde{Q} := \tilde{\varphi}^{-1}(f(P))$. The coordinates of $\gamma'(0)$ with respect to the basis $(\tilde{\varphi}_u'(\tilde{Q}), \tilde{\varphi}_v'(\tilde{Q}))$ are

$$\frac{d}{dt}|_0 h_1(u(t), v(t)) = (h_1)_u'(Q)u'(0) + (h_1)_v'(Q)v'(0)$$

and

$$\frac{d}{dt}|_0 h_2(u(t), v(t)) = (h_2)_u'(Q)u'(0) + (h_2)_v'(Q)v'(0).$$

In other words, we have

$$\gamma'(0) = [(h_1)_u'(Q)u'(0) + (h_1)_v'(Q)v'(0)]\tilde{\varphi}_u'(\tilde{Q}) + [(h_2)_u'(Q)u'(0) + (h_2)_v'(Q)v'(0)]\tilde{\varphi}_u'(\tilde{Q}).$$

The last equation is decisive for the proof, as it shows as follows:

- The vector $\gamma'(0)$ depends only on w, that is, on its coordinates with respect to the basis $(\varphi'_u(Q), \varphi'_v(Q))$ (so it does not depend on the curve α). This shows that the map $d(f)_Q$ is well defined.
- The map $d(f)_P : T_P S_1 \to T_{f(P)} S_2$ is linear. Its matrix with respect to the bases $(\varphi'_u(Q), \varphi'_v(Q))$ and $(\tilde{\varphi}'_u(\tilde{Q}), \tilde{\varphi}'_v(\tilde{Q}))$ is
$$\begin{pmatrix} (h_1)'_u(Q) & (h_1)'_v(Q) \\ (h_2)'_u(Q) & (h_2)'_v(Q) \end{pmatrix}$$
which is just the Jacobi matrix $J(\tilde{\varphi}^{-1} \circ f \circ \varphi)_Q$.

$\square$

We can do similar things for differentiable functions $f : S \to \mathbb{R}$, as follows.

Definition 3.3.6. Let S be a regular surface, $f : S \to \mathbb{R}$ be a differentiable function (in the sense of Definition 3.2.1) and P a point in S_1. The *differential* of f at P is the function

$$d(f)_P : T_P S \to \mathbb{R}$$

defined as follows: take w in $T_P S$ and α a curve like in Definition 3.3.3 with $w = \alpha'(0)$; then

$$d(f)_P(w) = \frac{d}{dt}|_0 (f(\alpha(t))). \tag{3}$$

Again we can show that $d(f)_P(w)$ is independent of the choice of the curve α with $\alpha'(0) = w$, so the function $d(f)_P$ is well defined. Moreover, the function is linear.

Remarks. (a) One can easily show that if $F : \mathbb{R}^3 \to \mathbb{R}$ is differentiable, S is a surface, and $f : S \to \mathbb{R}$ is the restriction of F to S, then the function $d(f)_P : T_P S \to \mathbb{R}$ is the restriction of $d(F)_P : \mathbb{R}^3 \to \mathbb{R}$ to $T_P S$.

(b) Similarly, take S_1, S_2 two surfaces, $G : \mathbb{R}^3 \to \mathbb{R}^3$ a differentiable map with $G(S_1) \subset S_2$ and $g : S_1 \to S_2$ the restriction of G to S_1. If P is a point in S_1, then $d(G)_P(T_P S_1) \subset T_{g(P)} S_2$ and the differential map $d(g)_P : T_P S_1 \to T_{g(P)} S_2$ is the same as the restriction of $d(G)_P : \mathbb{R}^3 \to \mathbb{R}^3$ to $T_P(S_1)$.

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