University of Regina
Department of Mathematics and Statistics

MATH431/831 – Differential Geometry – Winter 2014

Homework Assignment No. 5

1. Consider again the hyperbolic paraboloid $\mathcal{H}$ (see Chapter 5, page 11), which is the surface of equation $z = y^2 - x^2$. Choose the obvious (global) parametrization of $\mathcal{H}$ and the corresponding unit normal vector field N (the Gauss map). Determine as follows:

(a) $T_O\mathcal{H}$, (b) $d(N)_O$, (c) k_1 and k_2, (d) two principal vectors at O, (e) H and K at O.

Hint. The hardest point is (b): to do it, take a curve α on $\mathcal{H}$ with $\alpha(0) = O$ and calculate $d(N)_O(\alpha'(0))$.

2. A *surface of revolution* is obtained by rotating the curve α in the xz plane described by

$$x = f(v), \quad z = g(v), \ a < v < b$$

around the z axis (see Figures 1 and 2). Call S the subset of $\mathbb{R}^3$ obtained in this way.

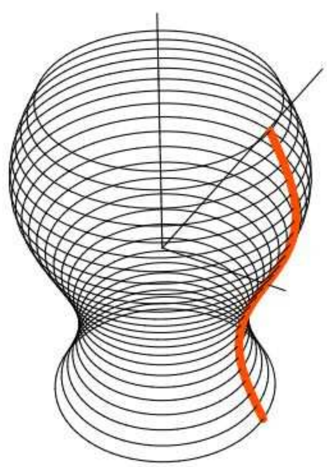

Figure 1. A surface of revolution.

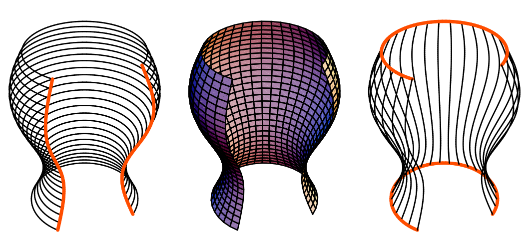

Figure 2. Parallels and meridians on a surface of revolution.

The curve in the xz plane is called the *generating curve*, the circles described by points of C after rotation are called *parallels* and the curves described by $v \mapsto \varphi(u_0, v)$, with $a < v < b$ are called *meridians*. A parametrization of S is given by

$$\varphi(u, v) = (f(v) \cos u, f(v) \sin u, g(v)),$$

where $0 < u < 2\pi$, and $a < v < b$. You don't need to justify this: just notice that the assumption

$$f(v) \neq 0 \text{ for any } v$$

is important, because otherwise we would have $f(v_0) = 0$ for some v_0, which would imply that $\varphi(u, v_0) = (0, 0, v_0)$, for all u, so φ wouldn't be injective. Also note that this is a *local* parametrization, since not all points on S are in the image of φ. To simplify the calculations, we will assume that the curve α is parametrized by arc length, that is

$$f'(v)^2 + g'(v)^2 = 1,$$

for all v.

(a) Compute E, F, G and e, f, g at any point $P = \varphi(u, v)$.

(b) Show that there is a principal vector tangent to the meridian through P, and another one tangent to the parallel through P.

(c) Show that the Gauss curvature of S at $P = \varphi(u, v)$ is

$$K = -\frac{f''(v)}{f(v)}.$$

3. Consider the surface S which is the graph of the function h of two variables. That is, S is described by $z = h(x, y)$. Show that the Gauss curvature of S at the point $(x, y, h(x, y))$ is

$$K = \frac{h''_{xx} h''_{yy} - (h''_{xy})^2}{(1 + (h'_x)^2 + (h'_y)^2)^2}.$$