University of Regina
Department of Mathematics and Statistics

MATH431/831 – Differential Geometry – Winter 2014

Homework Assignment No. 4

1. Find the area of the torus (see the end of Section 3.1). **Hint.** Use the parametrization given in Homework 3, Question 2. Because that parametrization does not cover the whole torus, you will have to compute first the area corresponding to the square $\epsilon \le u, v \le 2\pi - \epsilon$, then make $\epsilon \to 0$.
2. Show that the area of the graph of $g : \mathbb{R}^2 \to \mathbb{R}$ over the compact region R in the xy plane is given by
$$\int\int_R \sqrt{1 + (g'_x)^2 + (g'_y)^2}\, dA.$$
3. Let S, $\tilde{S}$ be two surfaces, (U, φ) and $(U, \tilde{\varphi})$ local parametrizations of S, respectively $\tilde{S}$ (with the same domain U) and denote by E, F, G, respectively $\tilde{E}, \tilde{F}, \tilde{G}$ the coefficients of the corresponding first fundamental forms (they are functions from U to $\mathbb{R}$). Show that if
$$E(Q) = \tilde{E}(Q),\ F(Q) = \tilde{F}(Q),\ G(Q) = \tilde{G}(Q),$$
for any Q in U, then the map
$$f := \tilde{\varphi} \circ \varphi^{-1} : \varphi(U) \to \tilde{S}$$
is a local isometry. **Hint.** First show that $d(f)_P(\varphi'_u(Q)) = \tilde{\varphi}'_u(Q)$ and $d(f)_P(\varphi'_v(Q)) = \tilde{\varphi}'_v(Q)$, for any Q in U.
4. Consider the example at the beginning of Section 4.4. Show that $f : \Pi \to C$ given by $f(x, y, 0) = (\cos x, \sin x, y)$ is a local isometry. **Hint.** One can do that by describing $d(f)_P : T_P\Pi \to T_{f(P)}C$ explicitly and checking equation (5), as indicated in the notes. You are invited to do it differently (and more economically!), namely by using question no. 3. Warning: the standard parametrizations of Π and C are not defined on the same open subset U of $\mathbb{R}^2$, like in question 3. However, you can use the same *method.*
5. Consider the helix described in Section 2.1 of the notes, and make $a := 1$, $b := 1$. Through each of its points consider the straight line parallel to the horizontal plane which intersects the z axis. The surface generated by these lines is called the *helicoid* (see Figure 1). It has the global parametrization
$$\varphi_1(u, v) = (u \cos v, u \sin v, v)$$
where u and v are in $\mathbb{R}$ (this follows easily from the equation of the helix given in chapter 2). We also consider the *catenoid*, which is generated by rotating the catenary $x = \cosh z$ situated in the xz plane around the z axis (see Figure 2). It has a local parametrization
$$\varphi_2(u, v) = (\cosh u \cos v, \cosh u \sin v, u)$$
where u is in $\mathbb{R}$ and $0 < v < 2\pi$. Show that the piece of helicoid between the plane $z = 0$ and $z = 2\pi$ and the catenoid are locally isometric. More precisely, show that such a local isometry is given by
$$f(\varphi_1(\sinh u, v)) = \varphi_2(u, v),$$
for all $\varphi_1(u, v)$ on the helicoid. **Hint.** Use the result mentioned in question no. 3 for $\varphi(u, v) := \varphi_1(\sinh u, v)$ and $\tilde{\varphi}(u, v) := \varphi_2(u, v)$.

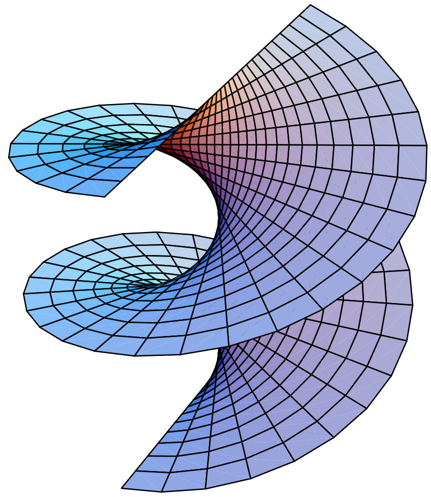

FIGURE 1. The helicoid.

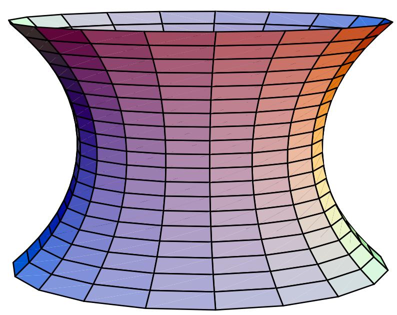

FIGURE 2. The catenoid.