University of Regina
Department of Mathematics and Statistics

MATH431/831 – Differential Geometry – Winter 2014

Homework Assignment No. 3 - Solutions

1. Consider the map

$$\varphi(u,v) = (\cos u \sin v, \sin u \sin v, \cos v)$$

where $0 < u < 2\pi$, $0 < v < \pi$.

(a) Show that $\varphi(u,v)$ is on S^2 for all (u,v) (see also Figure 1). Which is the image of φ (in other words, which portion of the sphere is covered by all $\varphi(u,v)$)?

(b) Show that φ gives a local parametrization of the sphere S^2 (you may omit checking that $\varphi^{-1} : \varphi(U) \to U$ is continuous).

(c) Represent the coordinate curves on the sphere.

(d) Find another local parametrization on S^2 of the same kind which, together with the one given here, cover the whole sphere.

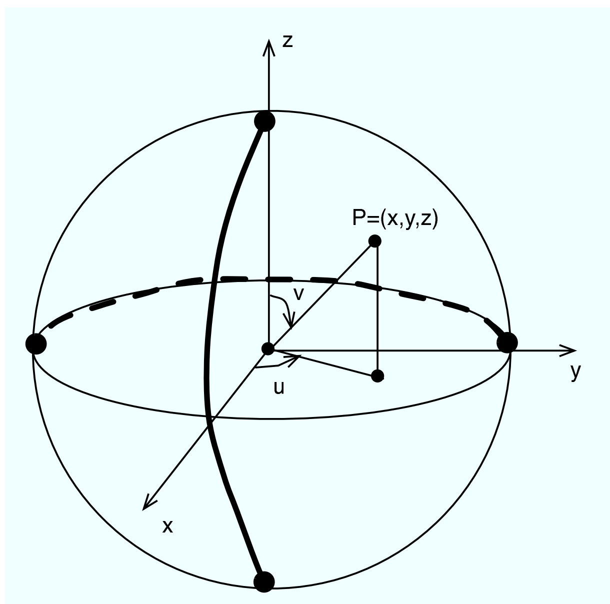

FIGURE 1. The image of φ is S^2 without the half circle represented in the xz plane. The image of $\tilde{\varphi}$ is S^2 without the (dotted) half circle represented in the xy plane.

Solution. (a) The only points not included in the image of φ are those with $u = 0$ or $v = 0$ or $v = \pi$. Those are exactly the points on the half circle contained in the xy plane represented in Figure 1.

(b) Only condition 2 needs to be checked. We have

$$\varphi'_u = (-\sin u \sin v, \cos u \sin v, 0), \ \varphi'_v = (\cos u \cos v, \sin u \cos v, -\sin v).$$

These vectors are linearly independent if and only if one of the determinants

$$\begin{vmatrix} -\sin u \sin v & \cos u \cos v \\ \cos u \sin v & \sin u \cos v \end{vmatrix} \quad \begin{vmatrix} \cos u \sin v & \sin u \cos v \\ 0 & -\sin v \end{vmatrix} \quad \begin{vmatrix} -\sin u \sin v & \cos u \cos v \\ 0 & -\sin v \end{vmatrix}$$

is different from zero. They are as follows:

$$-\sin v \cos v, \ \cos u \sin^2 v, \ \sin u \sin^2 v.$$

It's easy to check that for any u, v with $0 < u < 2\pi$, $0 < v < \pi$, at least one of the three numbers from above is non zero.

(c) If we fix $v = v_0$ we obtain the curve $u \mapsto \varphi(u, v_0)$, with $0 < u < 2\pi$. This is a circle (without a point) on the sphere which is parallel to the equator.

If we fix $u = u_0$ we obtain the curve $v \mapsto \varphi(u_0, v)$, with $0 < v < \pi$. This is a half circle on a sphere (a meridian).

(d) We will find another parametrization, whose image is the sphere without the dotted circle. That is

$$\tilde{\varphi}(u, v) = (-\cos u \sin v, \cos v, \sin u \sin v),$$

where again $0 < u < 2\pi$, $0 < v < \pi$.

2. `Consider the map`

$$\varphi(u, v) = ((a + r \cos u) \cos v, (a + r \cos u) \sin v, r \sin u)$$

`where` $0 < u < 2\pi$, $0 < v < 2\pi$.

(a) `Show that` $\varphi(u, v)$ `is on the torus` T `defined in section 3.1 of the notes (see also Figure 2). Which is the image of` φ `(in other words, which portion of the sphere is covered by all` $\varphi(u, v)$`)?`

(b) `Show that` φ `gives a local parametrization of the torus` T `(you may omit again checking that` $\varphi^{-1} : \varphi(U) \to U$ `is continuous).`

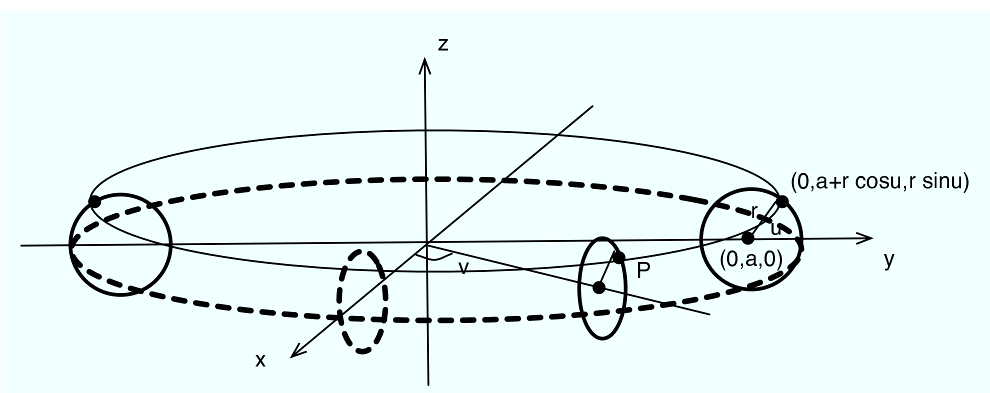

FIGURE 2. The points on T which are not in the image of φ are the dotted circles.

Solution. (a) One can easily see that the components of $\varphi(u, v)$ are exactly the coordinates of the point P in Figure 2. The only points not included in the image of φ are those with $u = 0$ and $v = 0$. These are the two dotted circles in Figure 2.

(b) Like before, the only difficult thing to check is that the vectors φ'_u and φ'_v are linearly independent. We have

$$\varphi'_u = (-r \sin u \cos v, -r \sin u \sin v, r \cos u)$$

and

$$\varphi'_v = (-(a + r \cos u) \sin v, (a + r \cos u) \cos v, 0).$$

We have

$$\begin{vmatrix} -r \sin u \cos v & -(a + r \cos u) \sin v \\ -r \sin u \sin v & (a + r \cos u) \cos v \end{vmatrix} = -r(a + r \cos u) \sin u$$

and

$$\begin{vmatrix} -r \sin u \sin v & (a + r \cos u) \cos v \\ r \cos u & 0 \end{vmatrix} = -r(a + r \cos u) \cos u \cos v$$

and

$$\begin{vmatrix} -r \sin u \cos v & -(a + r \cos u) \sin v \\ r \cos u & 0 \end{vmatrix} = r(a + r \cos u) \cos u \sin v$$

3. Let two points $p(t)$ and $q(t)$ move with the same speed, p starting from $(0, 0, 0)$ and moving along the z axis and q starting at $(a, 0, 0)$, $a \neq 0$, and moving parallel to the y axis. The motions go in both senses, that is, t can be positive or negative. Show that the line through $p(t)$ and $q(t)$ describes the set in $\mathbb{R}^3$ given by

$$y(x - a) + zx = 0.$$

Then show that this is a regular surface.

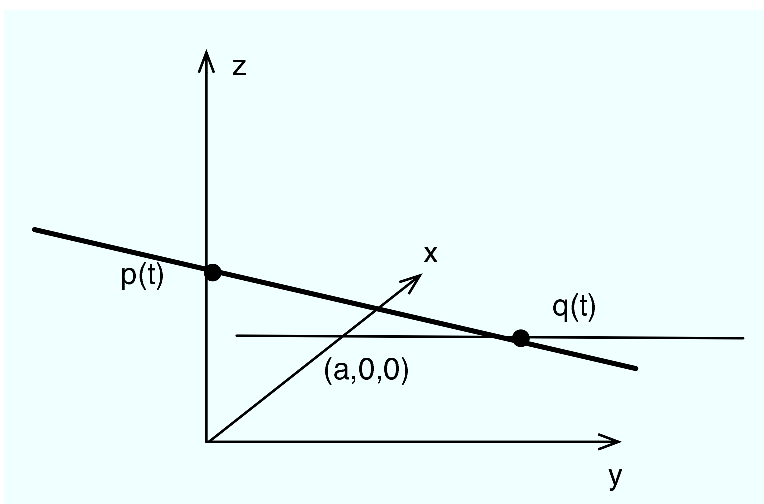

FIGURE 3. We are interested in all straight lines determined by $p(t)$ and $q(t)$, where t is in $\mathbb{R}$.

Solution. See Figure 3. The equation of the straight line determined by $p(t) = (0, 0, ct)$ and $q(t) = (a, ct, 0)$ is

$$\frac{x - 0}{a - 0} = \frac{y - 0}{ct - 0} = \frac{z - ct}{0 - ct},$$

so

$$\frac{x}{a} = \frac{y}{ct} = -\frac{z - ct}{ct}.$$

The idea is to eliminate t from these equations. From the first equation one obtains

$$ct = \frac{ay}{x}.$$

We replace in the second equation and obtain

$$\frac{y}{\frac{ay}{x}} = -\frac{z}{\frac{ay}{x}} + 1.$$

This gives

$$y(x - a) + zx = 0$$

which is the desired equation.

We consider the function

$$f(x, y, z) = y(x - a) + zx.$$

We have $f'_x = y + z$, $f'_y = x - a$ and $f'_z = x$. The latter two expressions cannot be simultaneously 0, so $f^{-1}(0)$ is a regular surface.

4. (a) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function and c a number such that for any P in $f^{-1}(c)$ at least one of $f'_x(P)$, $f'_y(P)$, and $f'_z(P)$ is non-zero. Consider the surface $S = f^{-1}(0)$. Show that the equation of the tangent plane to S at a point P is

$$xf'_x(P) + yf'_y(P) + zf'_z(P) = 0.$$

Hint. You may want to use the following general fact: the equation of a plane through the origin which is perpendicular to a given vector (a, b, c) is $ax+by+cz = 0$.

(b) Let S be the graph of the differentiable function $g : \mathbb{R}^2 \to \mathbb{R}$. Show that the equation of the tangent plane at a given point $P = (x_0, y_0, g(x_0, y_0))$ to S is

$$z = g_x(x_0, y_0)x + g_y(x_0, y_0)y.$$

Solution. (a) Let $\alpha'(0)$ be a tangent vector at P, where $\alpha : (-\epsilon, \epsilon) \to \mathbb{R}^3$ is a curve with $\alpha(t)$ in S, for all t and $\alpha(0) = P$. Set

$$\alpha(t) = (x(t), y(t), z(t)).$$

We have

$$f(x(t), y(t), z(t)) = 0,$$

for all t in $(-\epsilon, \epsilon)$. We differentiate (take derivative at 0) and obtain

$$f'_x(P)x'(0) + f'_y(P)y'(0) + f'_z(P)z'(0) = 0,$$

so the vector $(f'_x(P), f'_y(P), f'_z(P))$ is perpendicular to $\alpha'(0) = (x'(0), y'(0), z'(0))$ (their dot product is 0). We have shown that $(f'_x(P), f'_y(P), f'_z(P))$ is perpendicular to any vector in the tangent plane, so it is perpendicular to the plane itself. We use the hint and obtain the desired equation.

(b) Take $f(x, y, z) := g(x, y) - z$ and use point (a).

5. Construct a diffeomorphism (that is, a map f which is differentiable, bijective, and its inverse f^{-1} is differentiable) between the paraboloid of equation $z = x^2 + y^2$ and a plane, for instance the xy coordinate plane.
   Solution. Denote the paraboloid by P and the plane by Π. The map $f : P \to \Pi$ given by $f(x, y, z) = (x, y, 0)$ is such a diffeomorphism. Its inverse is $f^{-1} : \Pi \to P$, $f^{-1}(x, y, 0) = (x, y, x^2 + y^2)$. The two maps are both differentiable, as consequence of example 3, page 13, chapter 3 of the notes.
   Note. We can replace $x^2 + y^2$ by any function of two variables $g(x, y)$. So any graph of a function of two variables is a surface diffeomorphic to a plane.
6. **(Not to be marked)** Prove that if a regular surface S meets a plane Π in a single point P, then this plane coincides with the affine tangent plane to S at P. **Note.** By definition, the *affine tangent plane* to S at P is $P + T_P S$, which is an affine two-dimensional subspace of $\mathbb{R}^3$. For instance, if (U, φ) is a local parametrization and $\varphi(Q) = P$, then the affine tangent plane is the plane through P which is parallel to $\varphi'_u(Q)$ and $\varphi'_v(Q)$.
   Solution. See Figure 4.

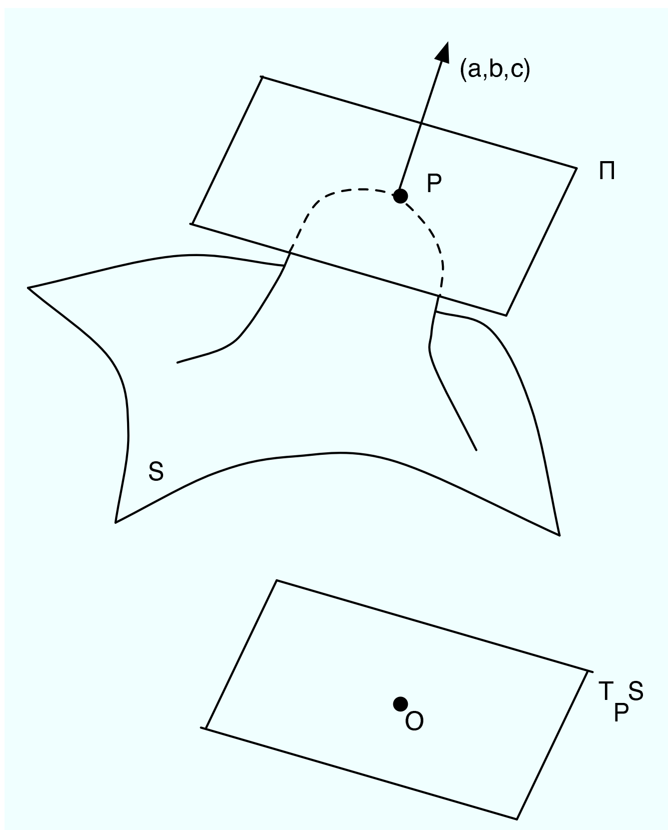

FIGURE 4. We are proving that the plane Π is the same as $P + T_P S$.

We consider a parametrization (U, φ) of S such that $\varphi(Q) = P$ for some $Q = (u_0, v_0)$ in U. Write

$$\varphi(u, v) = (x(u, v), y(u, v), z(u, v))$$

for all $(u.v)$ in U. Assume that the equation of the plane Π is

$$(ax + by + cz + d = 0.$$

The fact that the plane Π has a unique intersection point with S is the same as saying that the function $U \to \mathbb{R}$

$$(u, v) \mapsto ax(u, v) + by(u, v) + cz(u, v) + d$$

has a unique zero at Q. We deduce (see the lemma below) that the partial derivatives of $ax(u, v) + by(u, v) + cz(u, v) + d$ at (u_0, v_0) are zero. This means

$$ax'_u(Q) + by'_u(Q) + bz'_u(Q) = 0, \; ax'_v(Q) + by'_v(Q) + bz'_v(Q) = 0.$$

This means that the vector (a, b, c) is perpendicular to the plane spanned by $\varphi'_u(Q)$ and $\varphi'_v(Q)$, which is the tangent plane. So both the affine tangent plane and the plane Π are perpendicular to (a, b, c) and go through P: they must be equal.

Lemma. *Let $g : U \to \mathbb{R}$ be a differentiable function, where U is an open subset of $\mathbb{R}^2$. If g has a unique zero at $Q = (u_0, v_0)$, then*

$$\frac{\partial g}{\partial u}(Q) = 0 \text{ and } \frac{\partial g}{\partial v}(Q) = 0.$$

Proof. Assume that one of the partial derivatives, for instance $\frac{\partial g}{\partial v}(Q)$, is non zero. Consider the function $G : U \to \mathbb{R}^2$, $G(u, v) = (u, g(u, v))$. The Jacobian matrix of G at Q is

$$(JG)(Q) = \begin{pmatrix} 1 & g'_u(Q) \\ 0 & g'_v(Q) \end{pmatrix},$$

which is nonsingular. Consequently G is a local diffeomeorphism around Q, that is, there exists open neighborhoods W of Q in U and V of

$$G(Q) = (u_0, 0)$$

in $\mathbb{R}^2$ such that G is a diffeomorphism from W to V. In V there exists at least one (actually infinitely many) point $(u_1, 0)$ with $u_1 \neq u_0$. The point $G^{-1}(u_1, 0)$ is in $W \subset U$, it is different from $G^{-1}(u_0, 0) = Q$ and it satisfies

$$G(G^{-1}(u_1, 0)) = (u_1, 0),$$

which implies

$$g(G^{-1}(u_1, 0)) = 0.$$

This contradicts that Q is the only zero point of g.

Remark. The result in the lemma is not true for functions of one variable. If a function $x \mapsto g(x)$ has a unique zero at x_0, then its derivative doesn't have to vanish at x_0 (it's very simple to find a counterexample!)