University of Regina
Department of Mathematics and Statistics

MATH431/831 – Differential Geometry – Winter 2014

Homework Assignment No. 3

1. Consider the map
$$\varphi(u, v) = (\cos u \sin v, \sin u \sin v, \cos v)$$
where $0 < u < 2\pi$, $0 < v < \pi$.
   (a) Show that $\varphi(u, v)$ is on S^2 for all (u, v) (see also Figure 1). Which is the image of φ (in other words, which portion of the sphere is covered by all $\varphi(u, v)$)?
   (b) Show that φ gives a local parametrization of the sphere S^2 (you may omit checking that $\varphi^{-1} : \varphi(U) \to U$ is continuous).
   (c) Represent the coordinate curves on the sphere.
   (d) Find another local parametrization on S^2 of the same kind which, together with the one given here, cover the whole sphere.
2. Consider the map
$$\varphi(u, v) = ((a + r \cos u) \cos v, (a + r \cos u) \sin v, r \sin u)$$
where $0 < u < 2\pi$, $0 < v < 2\pi$.
   (a) Show that $\varphi(u, v)$ is on the torus T defined in Section 3.1 of the notes (see also Figure 2). Which is the image of φ (in other words, which portion of the sphere is covered by all $\varphi(u, v)$)?
   (b) Show that φ gives a local parametrization of the torus T (you may omit again checking that $\varphi^{-1} : \varphi(U) \to U$ is continuous).
3. Let two points $p(t)$ and $q(t)$ move with the same speed, p starting from $(0, 0, 0)$ and moving along the z axis and q starting at $(a, 0, 0)$, $a \neq 0$, and moving parallel to the y axis. The motions go in both senses, that is, t can be positive or negative. Show that the line through $p(t)$ and $q(t)$ describes the set in $\mathbb{R}^3$ given by
$$y(x - a) + zx = 0.$$
Then show that this is a regular surface.
4. (a) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function and c a number such that for any P in $f^{-1}(c)$ at least one of $f'_x(P)$, $f'_y(P)$, and $f'_z(P)$ is non-zero. Consider the surface $S = f^{-1}(0)$. Show that the equation of the tangent plane to S at a point P is
$$x f'_x(P) + y f'_y(P) + z f'_z(P) = 0.$$
Hint. You may want to use the following general fact: the equation of a plane through the origin which is perpendicular to a given vector (a, b, c) is $ax + by + cz = 0$.
   (b) Let S be the graph of the differentiable function $g : \mathbb{R}^2 \to \mathbb{R}$. Show that the equation of the tangent plane at a given point $P = (x_0, y_0, g(x_0, y_0))$ to S is
$$z = g_x(x_0, y_0)x + g_y(x_0, y_0)y.$$
5. Construct a diffeomorphism (that is, a map f which is differentiable, bijective, and its inverse f^{-1} is differentiable) between the paraboloid of equation $z = x^2 + y^2$ and a plane, for instance the xy coordinate plane.
6. **(Not to be marked).** Prove that if a regular surface S meets a plane π in a single point P, then this plane coincides with the affine tangent plane to S at P. **Note.** By definition, the *affine tangent plane* to S at P is $P + T_P S$, which is an affine two-dimensional subspace of $\mathbb{R}^3$. For instance, if (U, φ) is a local parametrization and $\varphi(Q) = P$, then the affine tangent plane is the plane through P which is parallel to $\varphi'_u(Q)$ and $\varphi'_v(Q)$.

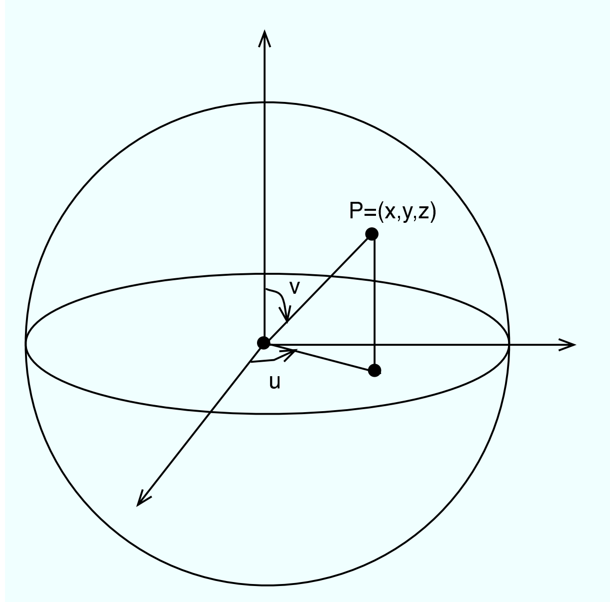

FIGURE 1. A local parametrization of S^2.

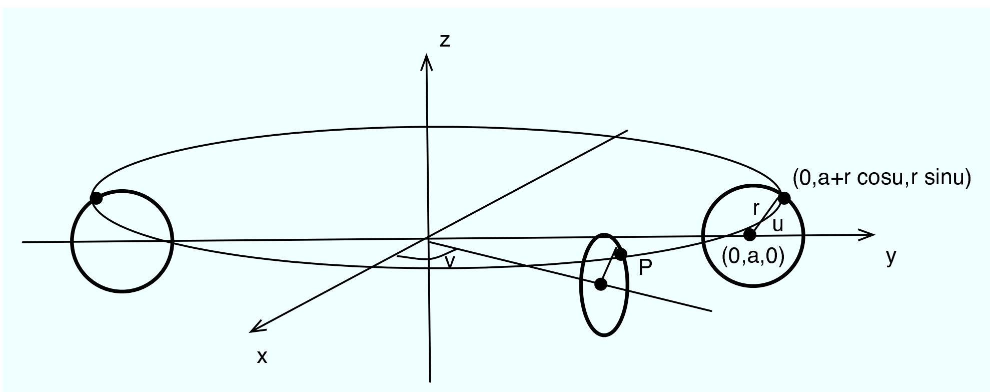

FIGURE 2. A local parametrization of the torus T: which are the coordinates of P?