University of Regina
Department of Mathematics and Statistics

MATH431/831 – Differential Geometry – Winter 2014

Homework Assignment No. 2 - Solutions

1. Consider the helix, as defined on page 2, Ch. 2 (Curves in Space) of the notes.
   (a) Calculate the torsion of the helix at an arbitrary point. **Hint.** Use the parametrization by arc length and the formula $B' = -\tau N$.
   (b) Show that the normal line at an arbitrary point on the helix (that is, the straight line through that point which contains the vector[1] $N(s)$) intersects the z axis, and the angle between the two lines is equal to $\frac{\pi}{2}$.
   (c) Show that the angle between the tangent line at any point of the helix and the z axis (or, if you prefer, between the vectors T and e_3) is independent of the point. In the terminology of section 2.3 of the notes, we say that the helix is a curve of constant slope with respect to the vector e_3.

Solution. (a) We have

$$\bar{\alpha}(s) = (a\cos\frac{s}{c}, a\sin\frac{s}{c}, \frac{bs}{c}),$$

thus

$$T(s) = \bar{\alpha}'(s) = (-\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c}),$$

$$N(s) = \frac{1}{\kappa(s)}\bar{\alpha}''(s) = (-\cos\frac{s}{c}, -\sin\frac{s}{c}, 0).$$

We deduce that

$$B(s) = T(s)\times N(s) = \begin{vmatrix} e_1 & e_2 & e_3 \\ -\frac{a}{c}\sin\frac{s}{c} & \frac{a}{c}\cos\frac{s}{c} & \frac{b}{c} \\ -\cos\frac{s}{c} & -\sin\frac{s}{c} & 0 \end{vmatrix} = (\frac{b}{c}\sin\frac{s}{c}, -\frac{b}{c}\cos\frac{s}{c}, \frac{a}{c}).$$

Consequently,

$$B'(s) = (\frac{b}{c^2}\cos\frac{s}{c}, \frac{b}{c^2}\sin\frac{s}{c}, 0).$$

From $B' = -\tau N$ we deduce

$$\tau(s) = \frac{b}{c^2} \text{ (independent of } s).$$

(b) The normal line is parallel to $N(s)$ and goes through $\bar{\alpha}(s)$. It equations are

$$\frac{x - a\cos\frac{s}{c}}{\cos\frac{s}{c}} = \frac{y - a\sin\frac{s}{c}}{\sin\frac{s}{c}},\ z = \frac{bs}{c}.$$

They can be written as

$$y = x\tan\frac{s}{c},\ z = \frac{bs}{c}.$$

This line intersects the z axis at the point $(0, 0, \frac{bs}{c})$. The latter point has the same z coordinate as $\bar{\alpha}(s)$, consequently the (normal) line determined by it and $\bar{\alpha}(s)$ is perpendicular to the z axis.

(c) The angle θ between T and e_3 is determined by

$$\cos\theta = T\cdot e_3 = \frac{b}{c},$$

[1] General Formula: the line through the point $P = (x_0, y_0, z_0)$ which contains the vector $v = (v_1, v_2, v_3)$ has equations $\frac{x-x_0}{v_1} = \frac{y-y_0}{v_2} = \frac{z-z_0}{v_3}$.

which is indeed independent of s.

2. Let $\alpha : (a,b) \to \mathbb{R}^3$ be a curve parametrized by arc length such that $\alpha''(s) \neq 0$, for all s in (a,b) (see the assumption mentioned on the first page of chapter 2 of the notes).
   (a) Show that if the trace of α is contained in a plane, then the vectors $T(s)$ and $N(s)$ are parallel to that plane. **Hint.** Consider a vector v perpendicular to the plane and a point P in the plane. For any point $\alpha(s)$ on the curve, the vector $\alpha(s)-P$ is parallel to the plane, so it's perpendicular to v. You only need to show that both $\alpha'(s)$ and $\alpha''(s)$ are perpendicular to v (keyword: dot product).
   (b) Show that the trace of α is contained in a plane if and only if $\tau(s) = 0$, for all s.

   Solution. (a) We have

$$(\alpha(s) - P) \cdot v = 0 \Rightarrow \alpha'(s) \cdot v = 0 \to \alpha''(s) \cdot v = 0,$$

for all s. Thus $T(s)$ and $N(s)$ are perpendicular to v, hence parallel to the plane of the curve.

(b) " $\Rightarrow$ " Assume that the curve is contained in a plane. The vectors $T(s)$ and $N(s)$ are parallel to the plane (see point (a)), hence $B(s)$ is constant (it's one of the unit vectors perpendicular to the plane). This implies $B'(s) = 0$, hence $\tau(s) = 0$ for all s.

" $\Leftarrow$ " Assume that $\tau(s) = 0$ for all s. This implies $B'(s) = 0$, thus $B(s) = B_0$ is constant. On the other hand, if we fix s_0, we observe that

$$\frac{d}{ds}(\alpha(s) - \alpha(s_0)) \cdot B(s) = \alpha'(s)B(s) + (\alpha(s) - \alpha(s_0))B'(s) = 0,$$

thus

$$(\alpha(s) - \alpha(s_0)) \cdot B_0 = 0.$$

This implies that $\alpha(s)$ is in the plane which contains $\alpha(s_0)$ and is perpendicular to B_0.

3. (a) If $\kappa > 0$ and τ are two given numbers, determine all curves in space with *constant* curvature κ and *constant* torsion τ. **Hint.** It is sufficient to find one such curve (why?). Use question 1.
   (b) Let $\alpha : (a,b) \to \mathbb{R}^3$ be a curve with constant curvature $\kappa > 0$ and torsion equal to 0 at any point. Show that the trace α is a piece of a of a circle of radius $1/\kappa$.
   (c) Show that the trace of the curve

$$\alpha(t) = (\frac{4}{5}\cos t, 1 - \sin t, -\frac{3}{5}\cos t)$$

   is a circle. Find the radius of the circle.

   Solution. (a) By looking at question 1, we see that any helix has both curvature and torsion constant. So we can guess that the *only* curves with curvature and torsion constant are the helices. We have to proceed systematically, as follows.

First we show that if $\kappa > 0$ and τ are two given numbers, then there exists a helix of curvature equal to κ and torsion equal to τ. This amounts to showing that there exists two numbers a and b such that

$$\frac{a}{a^2+b^2} = \kappa, \ \frac{b}{a^2+b^2} = \tau.$$

To find a and b we divide the two equations and obtain

$$a = b\frac{\kappa}{\tau}.$$

The second equation gives

$$\frac{b}{(b\frac{\kappa}{\tau})^2 + b^2} = \tau$$

thus

$$b = \frac{\tau}{\kappa^2 + \tau^2}.$$

Then we obtain

$$a = \frac{\kappa}{\kappa^2 + \tau^2}.$$

Now let α be the helix with a and b given by the previous two equations. If β is an arbitrary curve of curvature κ and torsion τ, then we have

$$\beta(s) = A\alpha(s) + X$$

where A is an orthogonal transformation with $Ae_3 = Ae_1 \times Ae_2$ and X is a constant vector in $\mathbb{R}^3$.

(c) We determine the curvature and the torsion of α. To this end we need

$$\alpha'(t) = (-\frac{4}{5}\sin t, -\cos t, \frac{3}{5}\sin t),\ \alpha''(t) = (-\frac{4}{5}\cos t, \sin t, \frac{3}{5}\cos t),\ \alpha'''(t) = (\frac{4}{5}\sin t, \cos t, -\frac{3}{5}\sin t)$$

then

$$\alpha'(t) \times \alpha''(t) = \begin{vmatrix} e_1 & e_2 & e_3 \\ -\frac{4}{5}\sin t & -\cos t & \frac{3}{5}\sin t \\ -\frac{4}{5}\cos t & \sin t & \frac{3}{5}\cos t \end{vmatrix} = -\frac{3}{5}e_1 - \frac{4}{5}e_3$$

and

$$(\alpha'(t) \times \alpha''(t)) \cdot \alpha'''(t) = 0.$$

The curvature is

$$\kappa(t) = \frac{\|(-\frac{3}{5}, 0, -\frac{4}{5})\|}{\|(-\frac{4}{5}\sin t, -\cos t, \frac{3}{5}\sin t)\|^3} = 1$$

and the torsion is

$$\tau(t) = 0.$$

Since these are both constant, the curve must be a helix with $a = 1$ and $b = 0$, which is a circle of radius 1.

4. `Finish the proof of Theorem 2.1.4. That is, compute` $\alpha'''(t)$`, calculate` $\alpha' \times \alpha''$ `and` $(\alpha' \times \alpha'') \cdot \alpha'''$`, and prove the two formulas.`

Solution. We continue the calculations from the notes. We have

$$\begin{aligned}
\alpha'''(t) &= \frac{d}{dt}\left(\frac{d^2s}{dt^2}T + \left(\frac{ds}{dt}\right)\kappa N\right) \\
&= \frac{d^3}{dt^3}T + \frac{d^2s}{dt^2}\cdot\frac{dT}{ds}\cdot\frac{ds}{dt} + 2\frac{ds}{dt}\cdot\frac{d^2s}{dt^2}\kappa N + \left(\frac{ds}{dt}\right)^2\cdot\frac{d\kappa}{dt}N + \left(\frac{ds}{dt}\right)^2\kappa\frac{dN}{ds}\cdot\frac{ds}{dt} \\
&= \frac{d^3s}{dt^3}T + 3\frac{d^2s}{dt^2}\cdot\frac{ds}{dt}\kappa N + \left(\frac{ds}{dt}\right)^2\cdot\frac{d\kappa}{dt}N + \left(\frac{ds}{dt}\right)^3\kappa(-\kappa T + \tau B) \\
&= \left(\frac{d^3s}{dt^3} - \kappa^2\left(\frac{ds}{dt}\right)^3\right)T + \left(3\frac{d^2s}{dt^2}\cdot\frac{ds}{dt}\kappa + \left(\frac{ds}{dt}\right)^2\cdot\frac{d\kappa}{dt}\right)N + \tau\left(\frac{ds}{dt}\right)^3\kappa B.
\end{aligned}$$

We deduce that

$$\alpha' \times \alpha'' = \left(\frac{ds}{dt}\right)^3\kappa T \times N = \left(\frac{ds}{dt}\right)^3\kappa B$$

which implies the formula for κ. Then

$$(\alpha' \times \alpha'') \cdot \alpha''' = \left(\frac{ds}{dt}\right)^3 \kappa B \cdot \alpha''' = \left(\frac{ds}{dt}\right)^6 \kappa^2 \tau = \tau \|\alpha' \times \alpha''\|^2$$

which implies the second formula.

5. (**Not to be marked.**) This is an example intended to show what kind of phenomena can happen if we neglect the assumption made in the notes, Ch. 2, first page. Consider the curve $\alpha : \mathbb{R} \to \mathbb{R}^3$ given by

$$\alpha(t) = \begin{cases} (t, 0, e^{-\frac{1}{t^2}}), & \text{if } t > 0 \\ (0,0,0), & \text{if } t = 0 \\ (t, e^{-\frac{1}{t^2}}, 0), & \text{if } t < 0. \end{cases}$$

(a) Show that the curve α is differentiable (that is, all its components are of class C^∞) and regular. Consequently, the vector $T(t)$ is defined for any t.
(b) Show that $\kappa(0) = 0$, consequently $N(0)$ (as well as $B(0)$ and $\tau(0)$) is not defined.
(c) Show that it is not possible to extend N and B by continuity at $t = 0$ (they both have a jump at 0).

Solution. (a) It is an exercise in calculus that the functions

$$f(t) = \begin{cases} e^{-\frac{1}{t^2}}, & \text{if } t > 0 \\ 0, & \text{if } t \leq 0 \end{cases}$$

and

$$g(t) = \begin{cases} e^{-\frac{1}{t^2}}, & \text{if } t > 0 \\ 0, & \text{if } t \leq 0 \end{cases}$$

are of class C^∞ and the derivatives of any order of f and g at 0 are equal to 0 (**Hint:** one shows recursively that f and g have derivatives from the left and from the right of any order at 0, and they are both equal to 0).

(b) To find $\kappa(0)$ we note that the first formula in Theorem 2.1.4 works in this case. More precisely, we have to do again what we did in the first part of the proof: reparametrize the curve by arc length, obtain $\beta(s) = \alpha(t(s))$, and note that we have again

$$\alpha''(t) = \frac{d^2 s}{dt^2} T + \left(\frac{ds}{dt}\right)^2 \frac{dT}{ds}. \tag{1}$$

The curvature at $t = 0$ is

$$\kappa(0) = \left\| \frac{dT}{ds}(s(0)) \right\|.$$

We have $\alpha''(0) = 0$ and

$$\frac{d^2 s}{dt^2}(0) = 0.$$

The last equation has to be justified: We have

$$\frac{ds}{dt} = \begin{cases} \sqrt{1 + \left(\frac{2}{t^3} e^{-\frac{1}{t^2}}\right)^2}, & \text{if } t \neq 0 \\ 0, & \text{if } t \neq 0 \end{cases}$$

Then

$$\frac{d^2 s}{dt^2}(0) = \lim_{t \to 0} \frac{d}{dt}\left(\frac{ds}{dt}\right) = 0$$

where one has to work a bit to justify the last equality (that is, compute the derivative of $\sqrt{1+\left(\frac{2}{t^3}e^{-\frac{1}{t^2}}\right)^2}$ and show that it approaches 0 it $t \to 0$).

Now coming back to (1), we deduce that

$$\frac{dT}{ds}(s(0)) = 0,$$

thus $\kappa(0) = 0$, as desired.

(c) First note that $T(0) = (1, 0, 0)$. As $t \to 0$, $t < 0$, the vector $N(t)$ approaches a vector in the xy plane which is perpendicular to $T(0)$. As $t \to 0$, $t > 0$, the vector $N(t)$ approaches a vector in the xz plane which is perpendicular to $T(0)$. The two limit vectors cannot be equal. Thus we cannot extend N at 0. Consequently, we cannot extend B at 0 as well.