University of Regina
Department of Mathematics and Statistics

MATH431/831 – Differential Geometry – Winter 2014

Homework Assignment No. 2

1. Consider the helix, as defined on page 2, Ch. 2 (Curves in Space) of the notes.
   (a) Calculate the torsion of the helix at an arbitrary point. **Hint.** Use the parametrization by arc length and the formula[1] $B' = -\tau N$.
   (b) Show that the normal line at an arbitrary point on the helix (that is, the straight line through that point which contains the vector[2] $N(s)$) intersects the z axis, and the angle between the two lines is equal to $\frac{\pi}{2}$.
   (c) Show that the angle between the tangent line at any point of the helix and the z axis (or, if you prefer, between the vectors T and e_3) is independent of the point. In the terminology of Section 2.3 of the notes, we say that the helix is a curve of constant slope with respect to the vector e_3.
2. Let $\alpha : (a, b) \to \mathbb{R}^3$ be a curve parametrized by arc length such that $\alpha''(s) \neq 0$, for all s in (a, b) (see the assumption mentioned on the first page of Ch. 2 of the notes).
   (a) Show that if the trace of α is contained in a plane, then the vectors $T(s)$ and $N(s)$ are parallel to that plane. **Hint.** Consider a vector v perpendicular to the plane and a point P in the plane. For any point $\alpha(s)$ on the curve, the vector $\alpha(s) - P$ is parallel to the plane, so it's perpendicular to v. You only need to show that both $\alpha'(s)$ and $\alpha''(s)$ are perpendicular to v (keyword: dot product).
   (b) Show that the trace of α is contained in a plane if and only if $\tau(s) = 0$, for all s.
3. (a) If $\kappa > 0$ and τ are two given numbers, determine all curves in space with *constant* curvature κ and *constant* torsion τ. **Hint.** It is sufficient to find one such curve (why?). Use question 1 above.
   (b) Let $\alpha : (a, b) \to \mathbb{R}^3$ be a curve with constant curvature $\kappa > 0$ and torsion equal to 0 at any point. Show that the trace α is a piece of a of a circle of radius $1/\kappa$.
   (c) Show that the trace of the curve
   $$\alpha(t) = \left(\frac{4}{5} \cos t, 1 - \sin t, -\frac{3}{5} \cos t \right)$$
   is a circle. Find the radius of the circle.
4. Finish the proof of Theorem 2.1.4. That is, compute $\alpha'''(t)$, calculate $\alpha' \times \alpha''$ and $(\alpha' \times \alpha'') \cdot \alpha'''$, and prove the two formulas.
5. (**Not to be marked.**) This is an example intended to show what kind of phenomena can happen if we neglect the assumption made in the notes, Ch. 2, first page. Consider the curve $\alpha : \mathbb{R} \to \mathbb{R}^3$ given by
   $$\alpha(t) = \begin{cases} (t, 0, e^{-\frac{1}{t^2}}), \text{ if } t > 0 \\ (0, 0, 0), \text{ if } t = 0 \\ (t, e^{-\frac{1}{t^2}}, 0), \text{ if } t < 0. \end{cases}$$
   (a) Show that the curve α is differentiable (that is, all its components are of class C^∞) and regular. Consequently, the vector $T(t)$ is defined for any t.
   (b) Show that $\kappa(0) = 0$, consequently $N(0)$ (as well as $B(0)$ and $\tau(0)$) is not defined.
   (c) Show that it is not possible to extend N and B by continuity at $t = 0$ (they both have a jump at 0).

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[1] Let $e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$ be the canonical basis of $\mathbb{R}^3$. The cross product of two vectors $v = (v_1, v_2, v_3) = v_1 e_1 + v_2 e_2 + v_3 e_3$ and $w = (w_1, w_2, w_3) = w_1 e_1 + w_2 e_2 + w_3 e_3$ is given by

$$v \times w = \begin{vmatrix} e_1 & e_2 & e_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

[2] General Formula: the line through the point $P = (x_0, y_0, z_0)$ which contains the vector $v = (v_1, v_2, v_3)$ has equations $\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}$.