University of Regina
Department of Mathematics and Statistics

MATH431/831 – Differential Geometry – Winter 2014

Homework Assignment No. 1

Solutions

1. Find the curvature of the ellipse at an arbitrary point (see the notes, Section 1.2, Example 3) and check that if $a > b$ then the ellipse is more curved at $(a, 0)$ than at $(0, b)$.

   Solution. We use $x(t) = a\cos t, y(t) = b\sin t$, and obtain

   $$\kappa_p(t) = \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{\frac{3}{2}}}.$$

   At $(a, 0)$ we have $t = 0$ and the curvature is

   $$\kappa_p(0) = \frac{a}{b^2}.$$

   At $(0, b)$ we have $t = \frac{\pi}{2}$ and the curvature is

   $$\kappa_p(\frac{\pi}{2}) = \frac{b}{a^2}.$$

   It is obvious that $a > b$ implies

   $$\frac{a}{b^2} > \frac{b}{a^2}.$$

2. In the context of Section 1.5 of the notes, check that the osculating circle given by Definition 1.5.2 satisfies the conditions (i),(ii), and (iii).

   Solution. (i) The distance between $(x(t), y(t))$ and $(\bar{x}(t), \bar{y}(t))$ is

   $$\|(\bar{x}(t), \bar{y}(t)) - (x(t), y(t))\| = \|\frac{1}{\kappa_p(t)\|\alpha'(t)\|}(-y'(t), x'(t))\| = \frac{1}{|\kappa_p(t)|}.$$

   So $(x(t), y(t))$ is on the osculating circle.

   (ii) The vector with tail at $(x(t), y(t))$ and tip at $(\bar{x}(t), \bar{y}(t))$ is

   $$(\bar{x}(t), \bar{y}(t)) - (x(t), y(t)) = \frac{1}{\kappa_p(t)\|\alpha'(t)\|}(-y'(t), x'(t)),$$

   which is perpendicular to $\alpha'(t) = (x'(t), y'(t))$ (the scalar product of them is 0). Consequently, the straight line perpendicular to that vector through $(x(t), y(t))$ is tangent to both the circle and the curve.

   (iii) The curvature of the circle of radius $\frac{1}{|\kappa_p(t)|}$ is

   $$\pm\frac{1}{\frac{1}{|\kappa_p(t)|}} = \pm\kappa_p(t).$$

3. Show that the evolute of the ellipse
$$\alpha(t) = (a\cos t, b\sin t)$$
is the curve of equation[1]
$$\bar{\alpha}(t) = \left(\frac{(a^2-b^2)\cos^3 t}{a}, \frac{(b^2-a^2)\sin^3 t}{b}\right).$$

Solution. This is a simple application of equation (4) (page 15) in the notes. We also refer to question 1 above. We obtain
$$\begin{aligned}\bar{x}(t) =& a\cos t + \frac{1}{\frac{ab}{(a^2\sin^2 t+b^2\cos^2 t)^{\frac{3}{2}}}\cdot\sqrt{a^2\sin^2 t + b^2\cos^2 t}}(-b\cos t)\\ & a\cos t - \frac{(a^2\sin^2 t + b^2\cos^2 t)b\cos t}{ab}.\end{aligned}$$
After a short calculation this gives indeed
$$\frac{(a^2-b^2)\cos^3 t}{a}.$$
Similar calculations can be done to obtain
$$\bar{y}(t) = \frac{(b^2-a^2)\sin^3 t}{b}.$$

4. (a) In the context of Section 1.6 of the notes, find a parametrization of the cycloid (as $\alpha(t) = (x(t), y(t))$) taking the radius of the circle to be 1. **Hint**You may want to assume that the circle rolls along the x axis and the initial position of the point on the circle is the origin O (see Figure 1). Choose the parameter t as the angle between the line segments $C\alpha(t)$ and CA.
   (b) Find all points t with $\alpha'(t) = 0$ (these are called *singular* points).
   (c) Determine the length of the piece of the cycloid which corresponds to a complete (that is, of 360^0) rotation of the circle.
   (d) Find the limits of the slope of the tangent line to the cycloid at $\alpha(t)$ as $t \to 2\pi$, $t < 2\pi$, respectively $t \to 2\pi$, $t > 2\pi$.

Solution.
(a) The crucial observation is that the (straight) line segment OA and the (circular) line segment $\alpha(t)A$ have the same length, which is t. Consequently, the coordinates of $\alpha(t)$ are
$$x(t) = t - \sin t, y(t) = 1 - \cos t.$$
The trace of the cycloid can be seen in Figure **??**.
(b) There is some interesting looking points on that curve, namely at $t = 2k\pi$. We only note that these are the singular points of the curve. So there is no tangent vector at any of those points.

[1]See Section 1.5 of the notes for a figure.

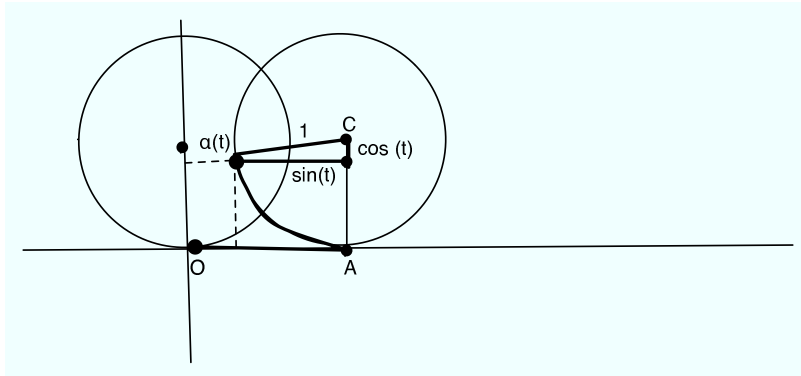

FIGURE 1. We choose t to be the angle between the lines $C\alpha(t)$ and CA (which is vertical).

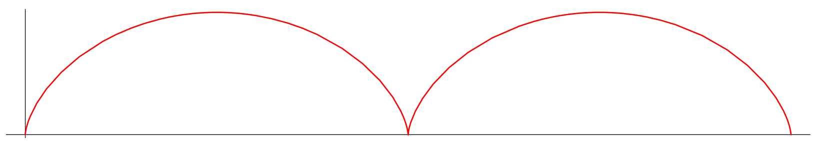

FIGURE 2. This is the trace of the cycloid, more precisely, the trajectory of the point on the circle after two complete rotations.

(c) The required length of the piece of the cycloid is

$$\int_0^{2\pi} \sqrt{(1-\cos t)^2 + \sin^2 t}dt = \int_0^{2\pi} \sqrt{2 - 2\cos t}dt =$$
$$= \int_0^{2\pi} \sqrt{4\sin^2 \frac{t}{2}}dt = \int_0^{2\pi} 2\sin\frac{t}{2}dt = -4\cos\frac{t}{2}|_0^{2\pi}$$
$$= 8,$$

where we have used that $1 - \cos t = 2\sin^2 \frac{t}{2}$.

(d) The slope of the tangent line is

$$\frac{y'(t)}{x'(t)} = \frac{\sin t}{1-\cos t}$$

for any $t \neq 2k\pi$, k integer. We use L'Hôpital's rule to deduce

$$\lim_{t\to 2\pi, t<2\pi} \frac{y'(t)}{x'(t)} = \lim_{t\to 2\pi, t<2\pi} \frac{\cos t}{\sin t} = \lim_{t\to 2\pi, t<2\pi} \tan t = -\infty.$$

In the same way, the limit of the slope from the right is ∞. This corresponds to the picture in Figure **??** (that is, the tangent lines from both sides approach the vertical line through $\alpha(2\pi)$).

5. In the context of Section 1.6 of the notes, find a parametrization of the cardioid (as $\alpha(t) = (x(t), y(t))$) taking the radius of the two circles to be 1. Then find the curvature of the cardioid at an arbitrary point.

Solution. See Figure **??**. We need to find the coordinates of $\alpha(t)$. We denote by t the angle between the line segments OA and CA, which is the same as the angle between AC and $\alpha(t)C$. In the triangle $DC\alpha(t)$ the angle D is right; the angle C is $t - (\frac{\pi}{2} - t) = 2t - \frac{\pi}{2}$. So

$$\|CD\| = \cos(2t - \frac{\pi}{2}) = \sin(2t) \quad \|\alpha(t)D\| = \sin(2t - \frac{\pi}{2}) = -\cos(2t)$$

The y coordinate of $\alpha(t)$ is

$$\|CB\| - \|CD\| = 2\sin t - \sin 2t = 2\sin t(1 - \cos t).$$

The x coordinate of $\alpha(t)$ is

$$\|\alpha(t)D\| + \|OB\| = -\cos(2t) + (2\cos t - 1) = -2\cos^2 t + 1 + (2\cos t - 1) = 2\cos t(1 - \cos t)$$

To summarize, the cardioid is the trace of

$$\alpha(t) = (2\cos t(1 - \cos t), 2\sin t(1 - \cos t)).$$

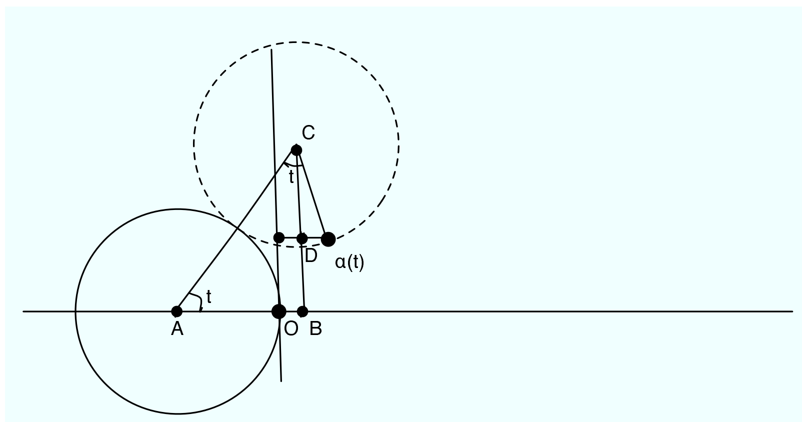

Figure 3

To find the curvature we need

$$x'(t) = 2(-\sin t + 2\cos t \sin t) = 2(-\sin t + \sin 2t), x''(t) = -2\cos t + 4\cos 2t$$

$$y'(t) = 2\cos t - 2\cos 2t, y''(t) = -2\sin t + 4\sin 2t.$$

So

$$\begin{aligned} x'y'' - x''y' &= 4(-\sin t + \sin 2t)(-\sin t + 2\sin 2t) - 4(\cos t - \cos 2t)(-\cos t + 2\cos 2t) \\ &= 4(\sin^2 t + \cos^2 t) + 8(\sin^2 2t + \cos^2 2t) + 12(-\sin t \sin 2t - \cos t \cos 2t) = 12(1 - \cos t), \end{aligned}$$

where we have used the formula $\cos a \cos b + \sin a \sin b = \cos(a - b)$. Also

$$(x'(t))^2 + (y'(t))^2 = 8 - 8\sin t \sin 2t - 8\cos t \cos 2t = 8(1 - \cos t).$$

The curvature is

$$\kappa_p(t) = \frac{12(1 - \cos t)}{(8(1 - \cos t))^{\frac{3}{2}}} = \frac{3}{4\sqrt{2}(1 - \cos t)^{\frac{1}{2}}}.$$

6. Find a parametrization by arc length of the catenary (see Section 1.6 of the notes). For simplicity, take $a = 1$.

Solution. Since $x(t) = t, y(t) = \cosh t$, we have
$$x'(t) = 1, y'(t) = \sinh t,$$
so
$$s(t) = \int_0^t \sqrt{1 + \sinh^2(u)}du = \int_0^t \cosh(u)du = \sinh u|_0^t = \sinh t.$$
We solve the equation
$$s = \cosh(t),$$
which gives
$$t = \operatorname{arcsinh}(s).$$
The desired parametrization is
$$\bar{\alpha}(s) = \alpha(\operatorname{arcsinh}(s)) = (\operatorname{arcsinh}(s), \cosh(\operatorname{arcsinh}(s))).$$
Because
$$\cosh v = \sqrt{1 + \sinh^2 v},$$
for any number v, we have
$$\cosh(\operatorname{arcsinh}(s)) = \sqrt{1 + s^2}.$$
Thus the parametrization by arc length is
$$\bar{\alpha}(s) = (\operatorname{arcsinh}(s), \sqrt{1 + s^2}).$$

7. Show that for any point P on the tractrix (see Section 1.6 of the notes) the length of the line segment between P and the y axis is constant (independent of t).

Solution. Recall from the notes that
$$\alpha(t) = (\sin t, \cos t + \log(\tan \frac{t}{2}).$$
To write the equation of the tangent line, we need to determine
$$\alpha'(t) = (\cos t, -\sin t + \frac{1}{\tan \frac{t}{2}} \cdot \frac{1}{\cos^2 \frac{t}{2}} \cdot \frac{1}{2}) = (\cos t, -\sin t + \frac{1}{\sin t}) = (\cos t, \frac{\cos^2 t}{\sin t}).$$
The tangent line goes through $\alpha(t)$ and has the direction given by $\alpha'(t)$. So the slope of the line is
$$\frac{\frac{\cos^2 t}{\sin t}}{\cos t} = \operatorname{cotan}(t).$$
The equation of the tangent line is
$$y - \cos t - \log(\tan \frac{t}{2}) = \operatorname{cotan}(t)(x - \sin t).$$
To find the intersection with the y axis, we make $x = 0$ and obtain
$$y = \log(\tan \frac{t}{2}).$$

The distance between $\alpha(t)$ and the point P on the y axis of coordinates $(0, \log(\tan \frac{t}{2})$ is

$$\sqrt{\sin^2 t + \cos^2 t} = 1,$$

which is indeed independent of t.

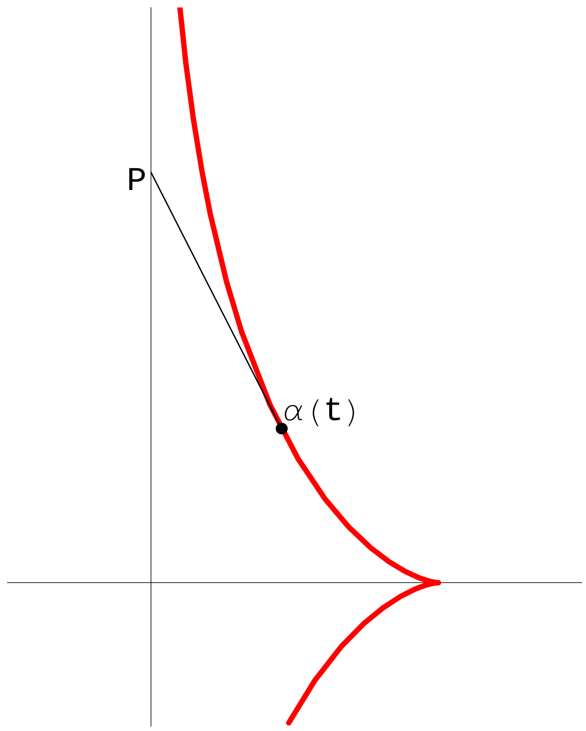

FIGURE 4. The point P is the intersection of the tangent line at $\alpha(t)$ with the y axis.