Equivariant cohomology of real flag manifolds

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Abstract. Let $P = G/K$ be a semisimple non-compact Riemannian symmetric space, where $G = I_0(P)$ and $K = G_p$ is the stabilizer of $p \in P$. Let $X$ be an orbit of the (isotropy) representation of $K$ on $T_p(P)$ ($X$ is called a real flag manifold). Let $K_0 \subset K$ be the stabilizer of a maximal flat, totally geodesic submanifold of $P$ which contains $p$. We show that if all the simple root multiplicities of $G/K$ are at least 2 then $K_0$ is connected and the action of $K_0$ on $X$ is equivariantly formal. In the case when the multiplicities are equal and at least 2, we will give a purely geometric proof of a formula of Hsiang, Palais and Terng concerning $H^*(X)$. In particular, this gives a conceptually new proof of Borel’s formula for the cohomology ring of an adjoint orbit of a compact Lie group.

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1. Introduction

Let $G/K$ be a non-compact symmetric space, where $G$ is a non-compact connected semisimple Lie group and $K \subset G$ a maximal compact subgroup. Then $K$ is connected [He, Thm. 1.1, Ch. VI] and there exists a Lie group automorphism $\tau$ of $G$ which is involutive and whose fixed point set is $G^\tau = K$. The involutive automorphism $d(\tau)_e$ of $\mathfrak{g} = \text{Lie}(G)$ induces the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k}$ (the same as $\text{Lie}(K)$) and $\mathfrak{p}$ are the $(+1)$-, respectively $(-1)$-eigenspaces of $(d\tau)_e$. Since $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, the space $\mathfrak{p}$ is $\text{Ad}_G(K) := \text{Ad}(K)$-invariant. The orbits of the action of $Ad(K)$ on $\mathfrak{p}$ are called real flag manifolds, or $s$-orbits. The restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{p}$ is an $\text{Ad}(K)$-invariant inner product on $\mathfrak{p}$, which we denote by $\langle \cdot , \cdot \rangle$.

Fix $\mathfrak{a} \subset \mathfrak{p}$ a maximal abelian subspace. Recall that the roots of the symmetric space $G/K$ are linear functions $\alpha : \mathfrak{a} \to \mathbb{R}$ with the property that the space

$$\mathfrak{g}_\alpha := \{ z \in \mathfrak{g} : [x, z] = \alpha(x)z \text{ for all } x \in \mathfrak{a} \}$$

is non-zero. The set $\Pi$ of all roots is a root system in $(\mathfrak{a}^*, \langle \cdot , \cdot \rangle)$. Pick $\Delta \subset \Pi$ a simple root system and let $\Pi^+ \subset \Pi$ be the corresponding set of positive roots. For any $\alpha \in \Pi^+$ we have

$$\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} = \mathfrak{k}_\alpha + \mathfrak{p}_\alpha,$$

where $\mathfrak{k}_\alpha = (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}) \cap \mathfrak{k}$ and $\mathfrak{p}_\alpha = (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}) \cap \mathfrak{p}$. We have the direct decompositions

$$\mathfrak{p} = \mathfrak{a} + \sum_{\alpha \in \Pi^+} \mathfrak{p}_\alpha, \quad \mathfrak{k} = \mathfrak{k}_0 + \sum_{\alpha \in \Pi^+} \mathfrak{k}_\alpha,$$

where $\mathfrak{k}_0$ denotes the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. The multiplicity of a root $\alpha \in \Pi^+$ is

$$m_\alpha = \dim \mathfrak{k}_\alpha + \dim \mathfrak{k}_{2\alpha}.$$
We note that this definition is slightly different from the standard one (see e.g. [Lo, Ch. VI, section 4]) which says that the multiplicity of \( \alpha \) is just \( \dim_k \alpha \).

Now \( k_0 \) is the Lie algebra of the Lie group \( K_0 := C_K(a) \) as well as of \( K'_0 := N_K(a) \). One can see that \( K_0 \) is a normal subgroup of \( K'_0 \); the Weyl group of the symmetric space \( G/K \) is

\[
W = K'_0/K_0.
\]

It can be realized geometrically as the (finite) subgroup of \( O(a,\langle , \rangle) \) generated by the reflections about the hyperplanes \( \ker \alpha, \alpha \in \Pi^+ \).

Take \( x_0 \in a \) and let \( X = Ad(K)x_0 \) be the corresponding flag manifold. The goal of our paper is to describe the cohomology, always with coefficients in \( \mathbb{R} \), of \( X \). The first main result concerns the action of \( K_0 \) on \( X \).

**Theorem 1.1.** If the symmetric space \( G/K \) has all root multiplicities \( m_\alpha, \alpha \in \Delta \), strictly greater than 1 then:

(a) \( K_0 \) is connected;

(b) the action of \( K_0 \) on \( X = Ad(K)x_0 \) is equivariantly formal, in the sense that

\[
H^*_{K_0}(X) \cong H^*(X) \otimes H^*_{K_0}(pt)
\]

by an isomorphism of \( H^*_{K_0}(pt) \)-modules;

(c) we have the isomorphisms of \( \mathbb{R} \)-vector spaces

\[
H^*(X) \cong \sum_{w \in W} H^*-d_w(w.x_0), \quad H^*_{K_0}(X) \cong \sum_{w \in W} H^*-d_w(K_0)(w.x_0).
\]

Here

\[
d_w = \sum m_\alpha
\]

where the sum runs after all \( \alpha \in \Pi^+ \) such that \( \alpha/2 \notin \Pi^+ \) and the line segment \( [x_0, w.x_0] \) crosses the hyperplane \( \ker \alpha \).

**Remark.** Let \( U \) be the (compact) Lie subgroup of \( G^c \) whose Lie algebra is \( \mathfrak{t} \oplus i\mathfrak{p} \). Then the manifold \( X = Ad(K)x_0 \) is the “real locus” [Go-Ho], [Bi-Gu-Ho] of an anti-symplectic involution on the adjoint orbit \( Ad(U)x_0 \) (see e.g. [Du, section 5]). The natural action of the torus \( T := \exp(i\mathfrak{a}) \) on this orbit is Hamiltonian. In this way, \( X \) fits into the more general framework of [Go-Ho] and [Bi-Gu-Ho]. But these papers investigate \( X \) from the perspective of the action of \( T_\mathbb{R} = T \cap K = T \cap K_0 \), whereas we are interested here in the action on \( X \) of a group which may be larger than \( T_\mathbb{R} \), namely \( K_0 \).

In the second part of our paper we will deal with the ring structure of the usual cohomology of \( X \), under the supplementary assumption that the symmetric space has all root multiplicities equal. By [He, Ch. X, Table VI], their common value can be only 2, 4 or 8. An important ingredient is the action of \( W = K'_0/K_0 \) on \( X \) given by

\[
hK_0.\ Ad(k)x_0 = Ad(k)Ad(h^{-1})x_0,
\]

for any \( h \in K'_0 \) and \( k \in K \). By functoriality, this induces an action of \( W \) on \( H^*(X) \). We also note that \( W \) acts in a natural way on \( a^* \).
Theorem 1.2. Assume that $G/K$ is an irreducible non-compact symmetric space whose simple root multiplicities are equal to the same number, call it $m$, which is at least 2. Take $X = Ad(K)x_0$.

(i) If $x_0$ is a regular point of $a$, then there exists a canonical linear $W$-equivariant isomorphism $\Phi : a^* \to H^m(X)$. Its natural extension $\Phi : S(a^*) \to H^*(X)$ is a surjective ring homomorphism whose kernel is the ideal $\langle S(a^*)^W \rangle$ generated by all nonconstant $W$-invariant elements of $S(a^*)$. Consequently we have the $R$-algebra isomorphism $H^*(X) \cong S(a^*)/\langle S(a^*)^W \rangle$.

(ii) If $x_0$ is an arbitrary point in $a$, then we have the $R$-algebra isomorphism $H^*(X) \cong S(a^*)^{W_{x_0}}/\langle S(a^*)^W \rangle$, where $W_{x_0}$ is the $W$-stabilizer of $x_0$.

Remark. Any real flag manifold $X = Ad(K)x$ with the canonical embedding in $(p, \langle , \rangle)$ is an element of an isoparametric foliation [Pa-Te]. The topology of such manifolds, including their cohomology rings, has been investigated by Hsiang, Palais and Terng in [Hs-Pa-Te] (see also [Ma]). The formulas for $H^*(X)$ given by Theorem 1.2 have been proved by them in that paper. Even though we do use some of their ideas (originating in [Bo-Sa]), our proof is different: they rely on Borel’s formula [Bo] for the cohomology of a generic adjoint orbit of a compact Lie group, whereas we actually prove it.

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2. Symmetric spaces with multiplicities at least 2 and their $s$-orbits

Let $G/K$ be an arbitrary non-compact symmetric space, $x_0 \in a$ and $X = Ad(K)x_0$ the corresponding $s$-orbit. The latter is a submanifold of the Euclidean space $(p, \langle , \rangle)$. The Morse theory of height functions on $X$ will be an essential instrument. The following proposition summarizes results from [Bo-Sa] or [Hs-Pa-Te] (see also [Ma]).

Proposition 2.1. (i) If $a \in a$ is a general vector (i.e. not contained in any of the hyperplanes $\ker \alpha$, $\alpha \in \Pi^+$), then the height function $h_a(x) = \langle a, x \rangle$, $x \in X$ is a Morse function. Its critical set is the orbit $W_{x_0}$.

(ii) Assume that $a$ and $x_0$ are contained in the same Weyl chamber in $a$. Then the index of $h_a$ at the critical point $w.x_0$ is

$$d_w = \sum m_\alpha$$

where the sum runs after all $\alpha \in \Pi^+$ such that $\alpha/2 \notin \Pi^+$ and the line segment $[a, wx_0]$ crosses the hyperplane $\ker \alpha$.

In the next lemma we consider the situation when all root multiplicities are at least 2.

Lemma 2.2. Assume that the root multiplicities $m_\alpha$, $\alpha \in \Delta$, of the symmetric space $G/K$ are all strictly greater than 1. Then:
(i) for any general vector \( a \in \mathfrak{a} \), the height function \( h_a : X \to \mathbb{R} \) is \( \mathbb{Z} \)-perfect,
(ii) the space \( K_0 \) is connected,
(iii) if \( X = \text{Ad}(K)x_0 \), then the orbit \( W.x_0 \) is contained in the fixed point set \( X^{K_0} \).

**Proof.** (i) According to [Ko, Theorem 1.1.4], there exists a metric on \( X \) such that if two critical points \( x \) and \( y \) can be joined by a gradient line, then \( x = s_\gamma y \), where \( \gamma \in \Pi^+ \). By (2), the difference of the indices of \( x \) and \( y \) is different from \( \pm 1 \). Because the stable and unstable manifolds intersect transversally [Ko, Corollary 2.2.7], the Morse complex of \( h_a \) has all boundary operators identically zero, hence \( h_a \) is \( \mathbb{Z} \)-perfect.

(ii) Take \( a \in \mathfrak{a} \) a general vector. The height function \( h_a \) on \( \text{Ad}(K)a \) is \( \mathbb{Z} \)-perfect. From (2) we deduce that \( H_1(\text{Ad}(K)a, \mathbb{Z}) = 0 \), thus \( \text{Ad}(K)a \) is simply connected. On the other hand, the stabilizer \( C_K(a) \) is just \( K_0 \) (see e.g. [Bo-Sa]). Because \( K/K_0 \) is simply connected and \( K \) is connected, we deduce that \( K_0 \) is connected.

(iii) The height function \( h_a \) is \( \text{Ad}(K_0) \)-invariant, thus \( \text{Crit}(h_a) = W.x_0 \) is also \( \text{Ad}(K_0) \)-invariant. The result follows from the fact that \( K_0 \) is connected. \( \square \)

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Point (a) was proved in Lemma 2.2 (ii).

(b) According to [Gu-Gi-Ka, Proposition C.25] it is sufficient to show that \( H^\ast_{K_0}(X) \) is free as a \( H^\ast_{K_0}(\text{pt}) \)-module. In order to do that we consider the height function \( h_a : X \to \mathbb{R} \) corresponding to a general \( a \in \mathfrak{a} \). We use the same arguments as in the proof of Lemma 2.2, (i). The function \( h_a \) is a \( K_0 \)-invariant. By the same reasons as above, the \( K_0 \)-equivariant Morse complex [Au-Br, Sections 5 and 6] has all boundary operators identically zero. Thus \( H^\ast_{K_0}(X) \) is a free \( H^\ast_{K_0}(\text{pt}) \)-module (with a basis indexed by \( \text{Crit}(h_a) = W.x_0 \)).

(c) The space \( H_\ast(X) \) has a basis \( \{[X_{w,x_0}] : w \in W \} \), where \( X_{w,x_0} \) is some \( d_w \)-dimensional cycle in \( X \), \( w \in W \). The evaluation pairing \( H^\ast(X) \times H_\ast(X) \to \mathbb{R} \) is non-degenerate; consider the basis of \( H^\ast(X) \) dual to \( \{[X_{w,x_0}] : w \in W \} \), which gives one element of degree \( d_w \) for each \( w.x_0 \). The result follows. \( \square \)

3. **Cohomology of \( s \)-orbits of symmetric spaces with uniform multiplicities at least 2**

Throughout this section \( G/K \) is a non-compact irreducible symmetric space whose simple root multiplicities are all equal to \( m \), where \( m \geq 2 \); \( x_0 \in \mathfrak{a} \) is a regular element and

\[
X = \text{Ad}(K)x_0 \simeq K/K_0
\]

is the corresponding real flag manifold. There are three such symmetric spaces; their compact duals are (see e.g. [Hs-Pa-Te, Section 3]):

1. any connected simple compact Lie group \( K \); we have \( m = 2 \); the flag manifold is \( X = K/T \), where \( T \) is a maximal torus in \( K \);
2. \( SU(2n)/Sp(n) \) where \( m = 4 \); the flag manifold is \( X = Sp(n)/Sp(1)^{\times n} \);
3. \( E_6/F_4 \) where \( m = 8 \); the flag manifold is \( X = F_4/\text{Spin}(8) \).
Let \( \Delta = \{ \gamma_1, \ldots, \gamma_l \} \) be a simple root system of \( \Pi \). To each \( \gamma_j \) corresponds the distribution \( E_j \) on \( X \), defined as follows: its value at \( x_0 \) is
\[
E_j(x_0) = \langle \gamma_j, x_0 \rangle
\]
and \( E_j \) is \( K \)-invariant, i.e.
\[
E_j(Ad(k)x_0) = Ad(k)E_j(x_0),
\]
for all \( k \in K \).

A basis of \( H_m(X) \) can be obtained as follows: Assume that \( x_0 \) is in the (interior of the) Weyl chamber \( C \subset \mathfrak{a} \) which is bounded by the hyperplanes \( \ker \gamma_j, 1 \leq j \leq l \). The Weyl group \( W \) is generated by \( s_j \), which is the reflection of \( a \) about the wall \( \ker \gamma_j, 1 \leq j \leq l \). For each \( 1 \leq j \leq l \) we consider the Lie subalgebra \( k_0 + \mathfrak{g}_j \) of \( k_0 \); denote by \( K_j \) the corresponding connected subgroup of \( K \). It turns out that the orbit \( Ad(K_j)x_0 \) is a round \( m \)-dimensional metric sphere in \((\mathfrak{p}, \langle \, , \rangle)\). To any \( x = Ad(k)x_0 \in X \) we attach the round sphere
\[
S_j(x) = Ad(k)Ad(K_j)x_0.
\]
The spheres \( S_j \) are integral manifolds of the distribution \( E_j \). We denote by \([S_j]\) the homology class carried by any of the spheres \( S_j(x), x \in X \). It turns out that \( S_1(x_0), \ldots, S_l(x_0) \) are cycles of Bott-Samelson type (see [Bo-Sa], [Hs-Pa-Te]) for the index \( m \) critical points of the height function \( h_a \), thus \([S_1], \ldots, [S_l] \) is a basis of \( H_m(X) \).

The following result concerning the action of \( W \) on \( H_m(X) \) was proved in [Hs-Pa-Te, Corollary 6.10] (see also [Ma, Theorem 2.1.1]):

**Proposition 3.1.** We can choose an orientation of the spheres \( S_j, 1 \leq j \leq l \), such that the linear isomorphism \( \mathfrak{a} \rightarrow H_m(X) \) determined by
\[
\gamma_j^\vee := \frac{2\gamma_j}{\langle \gamma_j, \gamma_j \rangle} \mapsto [S_j],
\]
\( 1 \leq j \leq l \), is \( W \)-equivariant.

We need one more result concerning the action of \( W \) on \( H^*(X) \):

**Lemma 3.2.** Let \( x \in \mathfrak{a} \) be an arbitrary element, \( C = C_K(x) \) its centralizer in \( K \), and let
\[
p : X = K/K_0 \rightarrow Ad(K)x = K/C
\]
be the natural map induced by the inclusion \( K_0 \subset C \). Then the map \( p^* : H^*(Ad(K)x) \rightarrow H^*(X) \) is injective. Its image is
\[
p^*H^*(Ad(K)x) = H^*(X)^{W_x}
\]
where the right hand side denotes the set of all \( W_x \)-invariant elements of \( H^*(X) \). Here \( W_x \) denotes the \( W \)-stabilizer of \( x \). In particular, the only elements in \( H^*(X) \) which are \( W \)-invariant are those of degree 0, i.e.
\[
H^*(X)^W = H^0(X).
\]
Proof. The map $p : K/K_0 \to K/C$ is a fibre bundle. The fiber $C/K_0$ is an $s$-orbit of the symmetric space $C_G(x)/C_K(x)$. The latter has all root multiplicities equal to $m$, as they are all root multiplicities of some roots of $G/K$. By Theorem 1.1 (ii), $C/K_0$ can have non-vanishing cohomology groups only in dimensions which are multiples of $m$. The same can be said about the cohomology of the space $K/C$. Because $m \in \{2, 4, 8\}$, the spectral sequence of the bundle $p : K/K_0 \to K/C$ collapses, which implies that $p^*$ is injective.

The map $p$ is $W$-equivariant with respect to the actions of $W$ on $Ad(K)\times_0$, respectively $Ad(K)$ defined by (1). Thus if $w \in W_x$, then $w|_{Ad(K)_x}$ is the identity map, hence we have $p \circ w = p$. This implies the inclusion

$$p^*H^*(Ad(K)_x) \subset H^*(X)^W_x.$$ 

On the other hand, the action of $W$ on $X$ defined by (1) is free, as the $Ad(K)$ stabilizer of the general point $x_0$ reduces to $K_0$. Consequently we have

$$H^*(X)^W_x = H^*(X/W_x)$$

and

$$\chi(X/W_x) = \frac{\chi(X)}{|W_x|} = \frac{|W|}{|W_x|},$$

where $\chi$ denotes the Euler-Poincaré characteristic. It follows from Theorem 1.1 (c) that

$$\dim H^*(X)^W_x = \frac{|W|}{|W_x|} = \dim H^*(Ad(K)_x).$$

Now we use that $p^*$ is injective.

In order to prove the last statement of the lemma, we take $x = 0 \in a$. $\square$

Let us consider the Euler class $\tau_i = e(E_i) \in H^m(X)$, $1 \leq i \leq l$. We will prove that:

**Lemma 3.3.**

(i) The cohomology classes $\tau_i$, $1 \leq i \leq l$ are a basis of $H^m(X)$.

(ii) The linear isomorphism $\Phi : a^* \to H^m(X)$ determined by

$$\gamma_i \mapsto e(E_i),$$

$1 \leq i \leq l$, is $W$-equivariant.

**Proof.** By Proposition 3.1 we know that

$$s_{is}[S_j] = [S_j] - d_{ji}[S_i],$$

where

$$d_{ji} = 2 \frac{\langle \gamma_j^\vee, \gamma_i^\vee \rangle}{\langle \gamma_i^\vee, \gamma_i^\vee \rangle}.$$ 

Denote by $\langle \ , \ \rangle$ the evaluation pairing $H^m(M) \times H_m(M) \to \mathbb{R}$. Consider $\alpha_j \in H^m(M)$ such that $\langle \alpha_j, [S_i] \rangle = \delta_{ij}$, $1 \leq i, j \leq l$. Take the expansion

$$\tau_i = \sum_{j=1}^l t_{ij}[S_j],$$

where

$$t_{ij} = \frac{\langle \gamma_j^\vee, \gamma_i^\vee \rangle}{\langle \gamma_i^\vee, \gamma_i^\vee \rangle}.$$
The automorphism $s_i$ of $X$ maps the distribution $E_i$ onto itself and changes its orientation (since so does the antipodal map on an $m$-dimensional sphere). Thus

$$s_i^*(\tau_i) = -\tau_i.$$  

Consequently we have

$$t_{ij} = \langle \tau_i, [S] \rangle = \langle -s_i^*(\tau_i), [S] \rangle = -\langle \tau_i, s_i[S] \rangle = -\langle \tau_i, [S] - d_{ji}[S] \rangle = -t_{ij} + 2d_{ji}$$

which implies $t_{ij} = d_{ji}$. By Proposition 3.1, the matrix $(d_{ij})$ is the Cartan matrix of the root system dual to $\Pi$, hence it is non-singular. Consequently $\tau_i$, $1 \leq i \leq l$ is a basis of $H^m(X)$. Again by Proposition 3.1 we have

$$\langle s_j^*(\tau_i), [S] \rangle = \langle \tau_i, [S] - d_{kj}[S] \rangle = t_{ik} - d_{kj}t_{ij} = t_{ik} - t_{jk}d_{ji},$$

thus

$$s_j^*(\tau_i) = \tau_i - d_{ji}\tau_j.$$  

It remains to notice that $d_{ji}$ can also be expressed as

$$d_{ji} = 2\langle \gamma_i, \gamma_j \rangle.$$

We are now ready to prove Theorem 1.2:

**Proof of Theorem 1.2** (i) Consider the ring homomorphism $\Phi : S(\mathfrak{a}^*) \rightarrow H^*(X)$ induced by $\gamma_i \mapsto e(E_i)$, $1 \leq i \leq l$. By Lemma 3.3, $\Phi$ is $W$-equivariant and from Lemma 3.2 we deduce that $\langle S(\mathfrak{a}^*)_W \rangle \subset \ker \Phi$. By Lemma 3.4 (see below), it is sufficient to prove that

$$\Phi(\prod_{\alpha \in \Pi^+} \alpha) \neq 0.$$

To this end, we will describe explicitly $\Phi(\alpha)$, for $\alpha \in \Pi^+$. Write $\alpha = w.\gamma_j$, where $w \in W$. The latter is of the form $w = hK_0$, with $h \in K_0'$. The image of $S_j(x_0)$ by the automorphism $w$ of $X$ is

$$w(S_j(x_0)) = Ad(K_j)Ad(h^{-1})x_0 = Ad(h^{-1})Ad(hK_jh^{-1})x_0 = Ad(h^{-1})Ad(K_\alpha)x_0 = Ad(h^{-1})S_\alpha(x_0) = S_\alpha(Ad(h^{-1})x_0) = S_\alpha(w.x_0).$$

Here $K_\alpha$ is the connected subgroup of $K$ of Lie algebra $\mathfrak{k}_0 + \mathfrak{t}_\alpha$ and $S_\alpha(x_0) := Ad(K_\alpha)x_0$ is a round metric sphere through $x_0$; for any $x = Ad(k)x_0 \in X$ we have $S_\alpha(x) := Ad(k)S_\alpha(x_0)$, which is an integral manifold of

$$E_\alpha(x) = Ad(k)[\mathfrak{t}_\alpha, x_0].$$

It is worth mentioning in passing that the spheres $S_\alpha$ and the distributions $E_\alpha$ are the curvature spheres, respectively curvature distributions of the isoparametric submanifold $X \subset p$ (see the remark following Theorem 1.2 in the introduction). Thus the differential of $w$ satisfies $(dw)(E_j) = E_\alpha$, which implies

$$e(E_j) = w^*e(E_\alpha).$$
Consequently
\[ \Phi(\alpha) = \Phi(w \cdot \gamma_j) = w^{-1} \cdot \Phi(\gamma_j) = (w^{-1})^*(e(E_j)) = e(E_\alpha). \]

We deduce that
\[ \Phi(\prod_{\alpha \in \Pi^+} \alpha) = \prod_{\alpha \in \Pi^+} e(E_\alpha) = e\left( \sum_{\alpha \in \Pi^+} E_\alpha \right). \]

On the other hand,
\[ \sum_{\alpha \in \Pi^+} E_\alpha(x_0) = \sum_{\alpha \in \Pi^+} [\mathfrak{e}_\alpha, x_0] = [\mathfrak{e}, x_0] = T x_0 X \]
thus
\[ \sum_{\alpha \in \Pi^+} E_\alpha = TX. \]

It follows that
\[ \Phi(\prod_{\alpha \in \Pi^+} \alpha) = e(TX), \]
which is different from zero, as
\[ e(TX)([X]) = \chi(X) = |W|, \]
where \( \chi(X) \) is the Euler-Poincaré characteristic of \( X \).

(ii) We apply Lemma 3.2.

The following lemma has been used in the proof:

**Lemma 3.4.** ([Hi, Lemma 2.8]) Let \( I \) be a graded ideal of \( S(\mathfrak{a}^*) \) which is also a vector subspace and such that \( \langle S(\mathfrak{a}^*)_+ \rangle \subset I \). We have \( I = \langle S(\mathfrak{a}^*)_W \rangle \) if and only if
\[ \prod_{\alpha \in \Pi^+} \alpha \notin I. \]

A proof of this lemma can also be found in the appendix.

4. **Appendix: Proof of Lemma 3.4**

The goal of this appendix is to provide a proof of Lemma 3.4, which is stated without a proof in [Hi]. As mentioned in the introduction, the Weyl group \( W \) can be realized as the group of orthogonal transformations of \( \mathfrak{a} \) generated by the reflections \( s_\alpha, \alpha \in \Pi^+ \). In fact, if \( \{\gamma_1, \ldots, \gamma_l\} \) is a simple root system, then \( W \) is generated by \( s_i := s_{\gamma_i}, 1 \leq i \leq l \). Denote by \( w_0 \) the longest element of \( W \), where the length is measured with respect to the generating set \( \{s_1, \ldots, s_l\} \). We will use the notations
\[ \mathcal{S} := S(\mathfrak{a}^*), \quad I_W := \langle S(\mathfrak{a}^*)_W \rangle. \]

First of all we note that the action of \( W \) on the polynomial ring \( \mathcal{S} \) is given by
\[ (w.f)(x) = f(w^{-1}.x), \]
where \( w \in W, f \in \mathcal{S}, x \in \mathfrak{a} \). This action preserves the grading of \( \mathcal{S} \), hence the ideal \( I_W \) generated by the nonconstant \( W \)-invariant polynomials is also graded. The most prominent example of a polynomial which is not \( W \)-invariant is
\[ d = \prod_{\alpha \in \Pi^+} \alpha. \]
In fact $d$ is skew-invariant, in the sense that $w.d = (-1)^{l(w)}d$, for any $w \in W$.

If $\alpha \in \Pi^+$, we consider the operator $\Delta_\alpha : S \to S$ defined as follows:

$$\Delta_\alpha(f) = \frac{f - s_\alpha f}{\alpha},$$

$f \in S$. Note that $f - s_\alpha f$ vanishes on the space $\ker \alpha$, hence $\Delta_\alpha(f)$ is really a polynomial. The following result is straightforward:

**Lemma 4.1.** If $w \in W$, $\alpha \in \Pi^+$, $f, g \in S$, then we have:

(a) $\Delta_\alpha(fg) = \Delta_\alpha(f)g + s_\alpha(f)\Delta_\alpha(g)$;
(b) $\Delta_\alpha(I_W) \subset I_W$.

To any $w \in W$ we can associate the operator $\Delta_w : S \to S$, which has degree $-l(w)$, and is defined as follows: take $w = s_{i_1} \cdots s_{i_k}$ a reduced expression and put $\Delta_w = \Delta_{\gamma_1} \cdots \Delta_{\gamma_i}$. We note that $\Delta_w$ does not depend on the choice of the reduced expression (see e.g. [Hi, Proposition 2.6]). The operators obtained in this way have the following property (see [Hi, Lemma 3.1]):

$$\Delta_w \circ \Delta_w' = \begin{cases} \Delta_{ww'}, & \text{if } l(ww') = l(w) + l(w') \\ 0, & \text{otherwise} \end{cases}$$

(3)

A classical result which goes back to Chevalley, says that the ideal $I_W$ is generated by $l$ homogeneous polynomials, which are algebraically independent. Let $d_1, \cdots, d_l$ denote their degrees. It follows that the Poincaré polynomial of $S/I_W$ is:

$$P(S/I_W) = \sum_{k=0}^{\infty} (\dim S^k - \dim I_W^k) t^k = \prod_{j=1}^{l} (1 + t + \cdots + t^{d_j-1}).$$

Combined with the fact that $d_1 + \cdots + d_l = N + l$ (see for instance [Hu, Theorem 3.9]), this tells us that $I^k = S^k$, for $k \geq N + 1$. The same polynomial can be expressed as (see [Hu, Theorem 3.15]):

$$P(S/I_W) = \sum_{w \in W} t^{l(w)}.$$

We deduce that $\dim S^k - \dim I_W^k$ equals the number of $w \in W$ with $l(w) = k$, $0 \leq k \leq N$. The following result describes a direct complement of $I_W^k$ in $S^k$:

**Proposition 4.2.** For any $0 \leq k \leq N$, the elements $\Delta_w(d)$, $w \in W$, $l(w) = N - k$ are linearly independent and span a direct complement of $I_W^k$ in $S^k$.

**Proof.** The number of elements of $W$ of length $k$ equals the number of elements of length $N - k$, hence we only have to prove that the polynomials $\Delta_w(d)$, where $l(w) = N - k$ are linearly independent and their span intersected with $I_W$ is $\{0\}$. To this end, it is sufficient to show that if

$$\sum_{l(w)=N-k} \lambda_w \Delta_w(d) \in I_W^k,$$

then all $\lambda_w$ must vanish. Indeed, if we fix $v \in W$ with $l(v) = N - k$, then by (3), we have

$$\Delta_{w_0v^{-1}} \left( \sum_{l(w)=N-k} \lambda_w \Delta_w(d) \right) = \lambda_v.$$

The left hand side of this equation is in $I_W^0$, hence it must be 0.
We are ready to prove Lemma 3.4:

**Proof of Lemma 3.4** We prove by induction on \( k \) that \( I_W^k = I^k \), \( 0 \leq k \leq N \). Things are clear for \( k = N \): \( I_W^N \) equals \( I^N \) because \( I_W^N \subset I^N \neq S^N \) and the codimension of \( I_W^N \) in \( S^N \) is 1 (see Proposition 4.2). Now, from \( I^{k+1} = I_W^{k+1} \) we deduce that \( I^k = I_W^k \). Suppose that we have

\[
f := \sum_{l(w) = N - k} \lambda_w \Delta_w(d) \in I^k,
\]

where \( \lambda_w \in \mathbb{R} \), not all of them equal to 0. We will prove by induction on \( m \in \{0, \ldots, k \} \) the following claim

**Claim.** For any \( h_m \in S^m \) and any \( \alpha_1, \ldots, \alpha_m \in \Pi^+ \), we have

\[
h_m \Delta_{\alpha_1} \circ \ldots \circ \Delta_{\alpha_m}(f) \in I^k.
\]

For \( m = 0 \), this is trivial. Suppose it is true for a certain \( m \) and prove it for \( m + 1 \). If \( h_m \in S_m \), \( \alpha_1, \ldots, \alpha_m \in \Pi^+ \), \( h \) an arbitrary homogeneous polynomial of degree 1, and \( \alpha \) a positive root, then we have

\[
h h_m \Delta_{\alpha_1} \circ \ldots \circ \Delta_{\alpha_m}(f) \in I^{k+1} = I_W^{k+1},
\]

hence its image by \( \Delta_\alpha \) is in \( I_W^k \subset I^k \). We deduce that

\[
\Delta_\alpha(h_m \Delta_{\alpha_1} \circ \ldots \circ \Delta_{\alpha_m}(f)) + s_\alpha(h) \Delta_\alpha(h_m) \Delta_{\alpha_1} \circ \ldots \circ \Delta_{\alpha_m}(f) + s_\alpha(h) s_\alpha(h_m) \Delta_\alpha \circ \Delta_{\alpha_1} \circ \ldots \circ \Delta_{\alpha_m}(f)
\]

is in \( I^k \), consequently \( s_\alpha(h h_m) \Delta_\alpha \circ \Delta_{\alpha_1} \circ \ldots \circ \Delta_{\alpha_m}(f) \in I^k \). Since any \( h_{m+1} \in S^{m+1} \) is a linear combination of polynomials of the form \( s_\alpha(h h_m) \), the claim is proved.

We deduce that for any \( v \in W \) with \( l(v) = k \), and any \( h_k \in S^k \) we have that

\[
h_k \Delta_v(f) \in I^k.
\]

Fix now \( w \in W \) with \( l(w) = N - k \) and take \( v := w_0 w^{-1} \). Then \( \Delta_v(f) = \lambda_w \) by (1), hence \( \lambda_w h_k \in I^k \), for any \( h_k \in S^k \). But then \( \lambda_w \) must vanish, since \( I^k \neq S^k \) (if they were equal, from \( k \leq N \) we would deduce \( I^N = S^N \), which is false). We conclude that \( f = 0 \), which is a contradiction. This finishes the proof. \( \square \)
REFERENCES


