Social Studies 201
Notes for November 28, 2003

These notes contain a discussion of a test of a proportion for a large sample size.

B. Test of a proportion, large sample size –section 9.4, p. 622.

An hypothesis test for a population proportion can be conducted using the same principles of hypothesis testing as for a population mean. Recall from the notes of November 17 that a proportion is a special case of a mean. When considering the proportion of the population that takes on a particular characteristic, the variable takes on only two values, those without the characteristic and those with the characteristic, so the mean of this variable is equal to the proportion of the population with the characteristic.

Also recall that when a random sample is selected from a population with the proportion p of members having a particular characteristic, the sample proportions are normally distributed.

> **Sampling distribution of a sample proportion** $\hat{p}$. If random samples of size n are drawn from a population with a proportion p of the population having a particular characteristic, and if the sample sizes are large, then the sample proportions $\hat{p}$ are normally distributed with mean p and standard deviation $\sqrt{pq/n}$. That is
>
> $$\hat{p} \text{ is Nor}\left(p, \sqrt{\frac{pq}{n}}\right).$$

For this result, a large sample size means a sample size n greater than 5 divided by the smaller of p or $q = 1 - p$.

Using this distribution for the sample proportion $\hat{p}$, the number of standard deviations this sample proportion $\hat{p}$ is from the hypothesized proportion of p is

$$Z = \frac{\hat{p} - p}{\sqrt{\frac{pq}{n}}}.$$

When conducting an hypothesis test for p, if this Z-value is in the critical region, we reject H_0, but if it is not in the critical region there is insufficient evidence to reject the null hypothesis.

Using the above result, an hypothesis test can be constructed for a population proportion, using the same six steps as used for a test of a mean. The principles involved in an hypothesis test of a proportion are identical to those for testing a mean.

Example – election polls

In the 1999 Saskatchewan provincial election, the NDP received 38.73% of the of the total vote and the Saskatchewan party received 39.61% of the total vote. The CBC poll of 800 respondents, conducted about two weeks prior to the November 5, 2003 Saskatchewan provincial election reported that 42% of voters would vote NDP and 39% would vote for the Saskatchewan party. From these results, can you conclude that

1. Support for the NDP has increased? (0.10 level of significance).
2. Support for the Saskatchewan party has changed? (0.10 significance level).
3. Comment on the conclusions.

1. Test for support for the NDP

Let p be the true proportion of Saskatchewan voters who supported the NDP just before the November 5, 2003 election. The hypothesis test is as follows.

1. **Hypotheses**. The question is whether support for the NDP has increased. Since the null hypothesis must be an equality, it makes sense to hypothesize no change in support for the NDP and reject this hypothesis only if there is evidence of some increase in support. Thus the hypotheses are:

   $$\text{Null hypothesis} \quad H_0 : p = 0.3873$$

   $$\text{Alternative hypothesis} \quad H_1 : p > 0.3873$$

   That is, the null hypothesis posits that the proportion of voters supporting the NDP has not changed since 1999; the alternative hypothesis is that the proportion supporting the NDP has increased since 1999.

2. **Test statistic**. The appropriate test statistic is $\hat{p}$, the proportion of those polled who express support for the NDP.

3. **Distribution of test statistic**. Since the sample size of $n = 800$ is large,
$$\hat{p} \text{ is Nor} \left(p, \sqrt{\frac{pq}{n}} \right).$$
To check that n is large, check to see whether n exceeds 5 divided by the smaller of p or $q = 1 - p$. For determining this, use $\hat{p} = 0.42$ for the estimate of p and $5/0.42 = 11.9 < 800$, so the sample size is large and $\hat{p}$ can be assumed to have a normal distribution.

4. **Critical region**. The question asks for an $\alpha = 0.10$ level of significance and the alternative hypothesis is one-directional, so the critical region for rejecting H_0 is at the extreme right end of the normal distribution. For a B area of $\alpha = 0.10$, the Z-value is 1.28. The region of rejection for the null hypothesis is all Z-values greater than 1.28.
$$\text{Region of rejection of } H_0 : \ Z > 1.28$$
$$\text{Area of nonrejection of } H_0 : \ Z \leq 1.28$$

5. **Conclusion**. The final step involved in the hypothesis test is to determine whether the sample proportion $\hat{p} = 0.42$ is in the critical region. This is accomplished by obtaining the Z-value associated with $\hat{p}$, that is,
$$Z = \frac{\hat{p} - p}{\sqrt{\frac{pq}{n}}}.$$
The hypothesized proportion of NDP supporters is $p = 0.3873$ so this value and the corresponding $q = 1 - p = 0.6127$ can be used to provide an estimate of pq in $\sqrt{pq/n}$, the standard deviation of $\hat{p}$.
$$\begin{aligned} Z &= \frac{\hat{p} - p}{\sqrt{\frac{pq}{n}}} \\ &= \frac{0.0327}{\sqrt{\frac{0.3873 \times 0.6127}{800}}} \end{aligned}$$

$$\begin{aligned} &= \frac{0.0327}{\sqrt{\frac{0.2373}{800}}} \\ &= \frac{0.0327}{\sqrt{0.000297}} \\ &= \frac{0.0327}{0.0172} \\ &= 1.8986 > 1.28 \end{aligned}$$

As a result, the Z-value differs from $Z = 0$ enough to be in the critical region. This means that $\hat{p} = 0.42$ differs enough from $p = 0.3873$ to reject the null hypothesis. The null hypothesis is rejected and the alternative hypothesis, that support for the NDP has increased, is accepted at the 0.10 level of significance.

2. Test for support for the Saskatchewan Party

Let p be the true proportion of Saskatchewan voters who supported the Saskatchewan Party just before the November 5, 2003 election. The hypothesis test is as follows.

1. **Hypotheses**. The question is whether support for the Saskatchewan Party has changed. In this case the null hypothesis is no change in support for the Saskatchewan Party while the alternative hypothesis is that there has been a change. The hypotheses are:

   $$\text{Null hypothesis} \quad H_0 : p = 0.3961$$

   $$\text{Alternative hypothesis} \quad H_1 : p \neq 0.3961$$

   That is, the null hypothesis posits that the proportion of voters supporting the Saskatchewan Party has not changed since 1999; the alternative hypothesis is that the proportion has changed since 1999.

2. **Test statistic**. The appropriate test statistic is $\hat{p}$, the proportion of those polled who express support for the Saskatchewan Party.

3. **Distribution of test statistic**. Since the sample size of $n = 800$ is large,

   $$\hat{p} \text{ is Nor} \left(p, \sqrt{\frac{pq}{n}} \right).$$

To check that n is large, check to see whether n exceeds 5 divided by the smaller of p or $q = 1 - p$. For determining this, use $\hat{p} = 0.39$ for the estimate of p and $5/0.39 = 12.8 < 800$, so the sample size is large and $\hat{p}$ can be assumed to have a normal distribution.

4. **Critical region**. The question asks for an $\alpha = 0.10$ level of significance and the alternative hypothesis is two-directional, so the critical region for rejecting H_0 is the extreme 0.05 of the distribution at the left end of the distribution plus the extreme 0.05 of the distribution at the right end. For a B area of $\alpha = 0.05$, the Z-values are ± 1.645.

$$\text{Region of rejection of } H_0: \; Z < 1.645 \text{ or } Z > +1.645$$

$$\text{Area of nonrejection of } H_0: \; -1.645 \leq Z \leq +1.645.$$

5. **Conclusion**. The final step involved in the hypothesis test is to determine whether the sample proportion $\hat{p} = 0.39$ is in the critical region. This is accomplished by obtaining the Z-value associated with $\hat{p}$, that is,

$$Z = \frac{\hat{p} - p}{\sqrt{\frac{pq}{n}}}.$$

The hypothesized proportion of Saskatchewan Party supporters is $p = 0.3961$ so this value and the corresponding $q = 1 - p = 0.6039$ can be used to provide an estimate of pq in $\sqrt{pq/n}$, the standard deviation of $\hat{p}$.

$$\begin{aligned} Z &= \frac{\hat{p} - p}{\sqrt{\frac{pq}{n}}} \\ &= \frac{0.39 - 0.3961}{\sqrt{\frac{0.3961 \times 0.6039}{800}}} \\ &= \frac{0.0061}{\sqrt{\frac{0.2392}{800}}} \\ &= \frac{0.0061}{\sqrt{0.000299}} \end{aligned}$$

$$\begin{aligned} &= \frac{0.0061}{0.0172} \\ &= 0.3528 > -1.645 \text{ and } < +1.645. \end{aligned}$$

As a result, the Z-value is not different enough from $Z = 0$, or $\hat{p} = 0.39$ is not different enough from $p = 0.3961$, to reject the null hypothesis at the 0.01 level of significance.

3. Comments

From the CBC poll 800 respondents in late October 2003, there is initial evidence that support for the NDP increased by a few percentage points (from 38.73for the Saskatchewan party declined very slightly (from 39.61From the above tests, at the 0.10 level of significance, it can be concluded that support for the NDP increased and support for the Saskatchewan Party was unchanged.

At the time the poll was conducted, there was the possibility of Type I error in the conclusion concerning the NDP and the possibility of type II error in the conclusion about the Saskatchewan Party. Since the proportion of voters who would ultimately vote NDP was not known at the time of the CBC poll, there was the possibility that the poll sampled a set of voters who were more likely, than the population as a whole, to vote NDP. If there had been no change in support for the NDP, this could have resulted in type I error, rejecting the null hypothesis of no change in NDP support and concluding that support had increased. In fact, given the election results, support for the NDP had increased to 44.61election day, November 5, 2003. While there was the possibility of Type I error, such does not appear to have occurred here.

For the hypothesis test about support for the Saskatchewan Party, there was the possibility of type II error, that is, there may have been a change in support for the Saskatchewan Party, even though the CBC poll result was consistent with the conclusion of no change in support. There very likely was type II error, so that Saskatchewan Party support was not exactly equal to the 39.61the Saskatchewan Party received 39.61different than the 39.35it did not change very much, so that the consequence of type II error were minimal here.

The conclusions from the hypothesis tests turned out to be correct, as demonstrated by the results on election day. It may be that opinions shifted slightly between the time of the poll and election day, so the poll could not have been expected to be an accuarate prediction of popular vote on November 5. But the poll came very close to predicting the popular vote on election day.

Next day: Review and a short discussion of the chi square test for independence.